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CONTROLLABILITY OF TEMPERED CAPUTO FRACTIONAL DYNAMICAL SYSTEMS

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Abstract. In this article, we explored the controllability of fractional dynamical systems with tempered Caputo fractional derivative (FD). We develop the necessary and sufficient conditions for the solution representation of controllability of linear dynamical system involving tempered Caputo FD by utilizing the Grammian matrix. We use Schauder's fixed point theorem to establish sufficient conditions for the controllability of nonlinear fractional dynamical systems involving tempered Caputo FD. The theoretical results are validated through numerical example obtained using successive approximation techniques.

1. Introduction and preliminaries

Over the past three decades, *fractional calculus* has emerged as an advanced branch of mathematical analysis dealing with differentiation and integration of non-integer orders. This field provides a robust framework for modeling complex systems that exhibit nonlinear dynamics, such as those found in physics, engineering, and economics [5, 13, 28]. For instance, Baleanu et al. [12] developed a fractional model of Caputo-type to study the dynamics of the COVID-19 pandemic and to explore its fundamental behaviors.

In the domain of fractional derivatives (FDs), various types are employed to account for memory effects. Over the past decade, experts in dynamical systems theory have utilized a variety of FDs [31]. The most frequently applied FDs in literature are the Caputo and Riemann-Liouville derivatives, both characterized by a singular kernel. To address this limitation, the Caputo-Fabrizio FD was introduced, which features a nonsingular kernel and operates locally. Later, the Atangana-Baleanu FD was proposed as a novel derivative with both nonsingular and nonlocal properties.

Controllability is a key concept in control theory, signifying the ability to steer a dynamical system from any given initial state to a target final state using an appropriate set of controls. The controllability of nonlinear systems in finite-dimensional spaces has been extensively studied, often through the use of fixed-point theorems [9, 23]. Several researchers [10, 11, 14, 20, 21] have derived controllability results for both linear and nonlinear fractional dynamical systems, using tools such as the Gramian matrix and the rank condition. Recently, Vishnukumar et al. [32] investigated the controllability of fractional dynamical systems with a single delay in the

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control function, specifically using the Caputo fractional derivative. There is a new variation of fractional calculus called tempered fractional calculus, as a more flexible alternative with considerable promise for practical applications. A fractional derivative is a convolution with a power law. A tempered fractional derivative multiplies that power law kernel by an exponential factor.

However, to the best of our knowledge, there has been no research exploring the controllability of nonlinear fractional dynamical systems with tempered Caputo fractional derivatives. In this work, we aim to bridge this gap by analyzing the controllability of such systems using the Gramian matrix and Schauder's fixed-point theorem. In the context of fractional calculus, several studies have explored various integral equations and approximation methods, including the works [2–4, 6–8, 15, 17, 18, 26, 27, 29].

Consider the nonlinear fractional differential equation with tempered Caputo fractional derivatives:

(1.1)
$$\begin{cases} {}^{C}\mathrm{D}_a^{ꝗ,ƛ}x(ꞇ) = Az(tꞇ) + Bu(ꞇ) + f(ꞇ, z(ꞇ), u(ꞇ)), \quad ꞇ \in J = [a,b], \quad 0 < ꝗ < 1,\ ƛ \geq 0, \\ z(a) = z_a, \end{cases}$$

where ${}^{C}\mathrm{D}_a^{ꝗ,ƛ}(\cdot)$ represents the tempered Caputo fractional derivative of order $ꝗ$. In this equation, $z \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control function, A is an $n \times n$ matrix, B is an $n \times m$ matrix, and f is a continuous function mapping $J \times \mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}^n$.

In this section, we outline important definitions, lemmas, notations, and basic information needed to establish our main findings.

Definition 1.1 ([24])**.** Let $z(ꞇ)$ be a real-valued, piecewise continuous function that is integrable over (a,b). For $ꝗ > 0$ and $ƛ \geq 0$, the tempered Riemann-Liouville (R-L) fractional integral of order $ꝗ$ is expressed as:

$$I_a^{ꝗ;ƛ}z(ꞇ) = e^{-ƛꞇ}I_a^{ꝗ}(e^{ƛꞇ}z(ꞇ)) = \frac{1}{\Gamma(ꝗ)}\int_a^{ꞇ}(ꞇ-ɼ)^{ꝗ-1}e^{-ƛ(ꞇ-ɼ)}z(ɼ)\, dɼ,$$

where $I_a^{ꝗ}z(ꞇ)$ represents the R-L fractional integral:

$$I_a^{ꝗ}z(ꞇ) = \frac{1}{\Gamma(ꝗ)}\int_a^{ꞇ}(ꞇ-ɼ)^{ꝗ-1}w(ɼ)\, dɼ.$$

Definition 1.2 ([24])**.** For $n-1 < ꝗ < n$, where $n \in \mathbb{N}$, and $ƛ \geq 0$, the tempered R-L fractional derivative (FD) of order $ꝗ$ is defined as:

$$D_a^{ꝗ;ƛ}z(ꞇ) = e^{-ƛt}D_a^{ꝗ}(e^{ƛt}z(ꞇ)) = \frac{e^{-ƛt}}{\Gamma(n-ꝗ)}\frac{d^n}{dt^n}\int_a^{ꞇ}(ꞇ-ɼ)^{n-ꝗ-1}e^{ƛt}w(ɼ)\, dɼ,$$

where $D_a^{ꝗ}z(ꞇ)$ denotes the R-L FD:

$$D_a^{ꝗ}(e^{ƛꞇ}z(ꞇ)) = \frac{d^n}{dꞇ^n}\left(I_a^{n-ꝗ}(e^{ƛꞇ}z(ꞇ))\right) = \frac{1}{\Gamma(n-ꝗ)}\frac{d^n}{dꞇ^n}\int_a^{ꞇ}(ꞇ-ɼ)^{n-ꝗ-1}e^{ƛꞇ}z(ɼ)\, dɼ,$$

for $z \in C[a,b]$.

Definition 1.3. For $n-1<\mathfrak{q}<n$, where $n\in\mathbb{N}$ and $\lambda\geq 0$, the tempered Caputo or λ-tempered FD of order $\mathfrak{q}$ is given by:

$$^{C}D_{a}^{\mathfrak{q};\lambda}z(\mathfrak{t})=e^{-\lambda\mathfrak{t}}\ {}^{C}D_{a}^{\mathfrak{q}}(e^{\lambda\mathfrak{t}}z(\mathfrak{t}))=\frac{e^{-\lambda\mathfrak{t}}}{\Gamma(n-\mathfrak{q})}\int_{a}^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{r})^{n-\mathfrak{q}-1}\frac{d^{n}}{ds^{n}}(e^{\lambda t}w(\mathfrak{r}))\,d\mathfrak{r},$$

where ${}^{C}D_{a}^{\mathfrak{q}}z(\mathfrak{t})$ denotes the Caputo FD:

$$^{C}D_{a}^{\mathfrak{q}}(e^{\lambda\mathfrak{t}}z(\mathfrak{t}))=\frac{1}{\Gamma(n-\mathfrak{q})}\int_{a}^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{r})^{n-\mathfrak{q}-1}\frac{d^{n}}{ds^{n}}(e^{\lambda\mathfrak{t}}w(\mathfrak{r}))\,d\mathfrak{r},$$

for $u\in \mathrm{C}^{n}[a,b]$.

Remark 1.4. For $\lambda=0$, the definition of the tempered Caputo FD reduces to the well-known Caputo FD.

Definition 1.5. The Laplace transform (LT) of a real function g, defined for all real numbers $\lambda\geq 0$, is given by:

$$F(\mathfrak{t})=\mathcal{L}\{g(\mathfrak{t})\}=\int_{0}^{\infty}e^{-\lambda\mathfrak{t}}g(\mathfrak{t})\,d\mathfrak{t},\quad \mathfrak{t}\in\mathbb{C}. \tag{1.2}$$

For the integral (1.2) to exist, the function $f(\mathfrak{t})$ must be of exponential order q, meaning there exist positive constants M and T such that $e^{-q\lambda}|f(\mathfrak{t})|\leq M$ for all $\lambda\geq T$.

Definition 1.6 ([28])**.** The convolution of two functions, $g_1(\mathfrak{t})$ and $g_2(\mathfrak{t})$, is defined as:

$$g_1(\mathfrak{t})*g_2(\mathfrak{t})=\int_{0}^{\mathfrak{t}}g_2(\mathfrak{t}-\mathfrak{r})g_1(\mathfrak{r})\,d\mathfrak{r}.$$

Definition 1.7 ([28])**.** If $F(\mathfrak{t})$ and $G(\mathfrak{t})$ are the Laplace transforms of the functions $g_1(\mathfrak{t})$ and $g_2(\mathfrak{t})$, respectively, then:

$$\mathcal{L}\{g_1*g_2\}=F(\mathfrak{t})G(\mathfrak{t}).$$

Lemma 1.8 ([22])**.** *For $\mathfrak{t}>0$ and $\mathfrak{t}>\lambda$:*

$$\mathcal{L}\{e^{\lambda t}\}(\mathfrak{t})=\frac{e^{\lambda a}}{\mathfrak{t}-\lambda}.$$

Lemma 1.9 ([25])**.** *The Laplace transform of the tempered Caputo FD is given by:*

$$\mathcal{L}\left\{{}^{C}\mathrm{D}_{a}^{\mathfrak{q},\lambda}z(t)\right\}(\mathfrak{t})=(\mathfrak{t}+\lambda)^{\mathfrak{q}}\mathcal{L}[z(t)](\mathfrak{t})-e^{-\lambda a}\sum_{k=0}^{n-1}(\mathfrak{t}+\lambda)^{\mathfrak{q}-k-1}w_{\lambda}^{[k]}(a), \tag{1.3}$$

where $k=0,1,2,\ldots,n-1$ and $w_{\lambda}^{[n]}(\mathfrak{t})=\left(\frac{d}{d\mathfrak{t}}\right)^{n}\left(e^{\lambda t}z(\mathfrak{t})\right)$.

Definition 1.10 ([28])**.** The two-parameter Mittag-Leffler (M-L) function is defined as:

$$\mathrm{E}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(n\alpha+\beta)} \tag{1.4}$$

for all $Re(\alpha), Re(\beta) > 0$ and $z \in \mathbb{C}$. The M-L function for a matrix $A_{m\times m}$ is given by:

$$\mathrm{E}_{\alpha,\beta}(A) = \sum_{n=0}^{\infty} \frac{A^n}{\Gamma(n\alpha+\beta)}. \tag{1.5}$$

Lemma 1.11 ([22])**.** *Let $Re(\alpha) > 0$ and $\left|\frac{\kappa}{\lambda^\alpha}\right| < 1$. Then:*

$$\mathcal{L}\left[\mathrm{E}_\alpha\left(\kappa(t-a)^\alpha\right)\right] = \frac{\lambda^{\alpha-1}}{\lambda^\alpha - \kappa},$$

and:

$$\mathcal{L}\left[(t-a)^{\beta-1}\,\mathrm{E}_{\alpha,\beta}\left(\kappa(t-a)^\alpha\right)\right] = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha-\kappa}.$$

2. Controllability of linear systems

Consider the linear fractional differential system with the tempered Caputo fractional derivative (FD):

$$\begin{cases} {}^{C}\mathrm{D}_a^{\mathfrak{q},\lambda} z(t) = Az(t) + Bu(t), \ t \in \mathrm{J}, \ 0 < \mathfrak{q} < 1, \ \lambda \geq 0, \\ z(a) = z_a, \end{cases} \tag{2.1}$$

where ${}^{C}\mathrm{D}_a^{\mathfrak{q},\lambda}(\cdot)$ represents the tempered Caputo FD of order $\mathfrak{q}$. Here, $z \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control input, A is an $n\times n$ matrix, and B is an $n \times m$ matrix.

Lemma 2.1. *The solution to equation* (2.1) *is given by:*

$$z(t) = e^{-\lambda t}\mathrm{E}_{\mathfrak{q},1}\left(\mathrm{A}(t-a)^{\mathfrak{q}}\right) z_a + \int_a^t (t-r)^{\mathfrak{q}-1} e^{-\lambda(t-r)} \mathrm{E}_{\mathfrak{q},\mathfrak{q}}\left(\mathrm{A}(t-r)^{\mathfrak{q}}\right) \mathrm{B}u(r)\, dr. \tag{2.2}$$

Proof. Taking the Laplace transform (LT) of both sides of (2.1) and using Lemma 1.9 for $n = 1$, we get:

$$(t+\lambda)^{\mathfrak{q}}\mathcal{L}[z(t)](t) - e^{-\lambda a}(t+\lambda)^{\mathfrak{q}-1} z_a = \mathrm{A}\mathcal{L}[z(t)] + \mathrm{B}\mathcal{L}[u(t)].$$

Solving for $\mathcal{L}[z(t)]$ gives:

$$\mathcal{L}[z(t)] = \frac{e^{-\lambda a}(t+\lambda)^{\mathfrak{q}-1}}{(t+\lambda)^{\mathfrak{q}} I - \mathrm{A}} z_a + \frac{\mathrm{B}}{(t+\lambda)^{\mathfrak{q}} I - \mathrm{A}} \mathcal{L}[u(t)].$$

Taking the inverse LT and applying Lemmas 1.8 and 1.11, we get:

$$\begin{aligned} z(t) &= e^{-\lambda t}\mathrm{E}_{\mathfrak{q},1}\left(\mathrm{A}(t-a)^{\mathfrak{q}}\right) x_a + \mathcal{L}^{-1}\left[\frac{\mathrm{B}}{(t+\lambda)^{\mathfrak{q}} I - \mathrm{A}}\right] \mathcal{L}^{-1}[\mathcal{L}[u(t)]] \\ &= e^{-\lambda t}\mathrm{E}_{\mathfrak{q},1}\left(\mathrm{A}(t-a)^{\mathfrak{q}}\right) z_a + \int_a^t (t-r)^{\mathfrak{q}-1} e^{-\lambda(t-r)} \mathrm{E}_{\mathfrak{q},\mathfrak{q}}\left(\mathrm{A}(t-r)^{\mathfrak{q}}\right) \mathrm{B}u(r)\, dr. \end{aligned}$$

□

Definition 2.2. The system (2.1) is controllable on J if, for any $z_a, z_b \in \mathbb{R}^n$, there exists a control $u(\cdot) \in \mathrm{L}^2(\mathrm{J}, \mathbb{R}^m)$ such that the solution of (2.1) satisfies $z(a) = z_a$ and $z(b) = z_b$.

Theorem 2.3. *The system* (2.1) *is controllable on* J *if and only if the* $n \times n$ *Grammian matrix*

$$G = \int_a^b \mathrm{E}_{ɋ,ɋ}\left(\mathrm{A}(b-ɼ)^{ɋ}\right) \mathrm{BB}^* \mathrm{E}_{ɋ,ɋ}\left(\mathrm{A}^*(b-ɼ)^{ɋ}\right)\, dɼ \tag{2.3}$$

is positive definite.

Proof. Assume that G is positive definite. Then it is non-singular, and thus its inverse exists. Define the control function:

$$\begin{aligned} u(ɼ) = {} & \left[(b-ɼ)^{ɋ-1} e^{-ƛ(b-ɼ)}\right]^{-1} \mathrm{B}^* \mathrm{E}_{ɋ,ɋ}\left(\mathrm{A}^*(b-ɼ)^{ɋ}\right) \\ & \times G^{-1}\left[z_b - e^{-ƛb}\mathrm{E}_{ɋ,1}\left(\mathrm{A}(b-a)^{ɋ}\right) z_a\right]. \end{aligned} \tag{2.4}$$

Using (2.3) and (2.4) in (2.2) at $t = b$, we have:

$$\begin{aligned} z(b) &= e^{-ƛb}\mathrm{E}_{ɋ,1}\left(\mathrm{A}(b-a)^{ɋ}\right) z_a + \int_a^b (b-ɼ)^{ɋ-1} e^{-ƛ(b-ɼ)} \mathrm{E}_{ɋ,ɋ}\left(\mathrm{A}(b-ɼ)^{ɋ}\right) \\ &\quad \times \mathrm{B}\left[(b-ɼ)^{ɋ-1} e^{-ƛ(b-ɼ)}\right]^{-1} \mathrm{B}^* \mathrm{E}_{ɋ,ɋ}\left(\mathrm{A}^*(b-ɼ)^{ɋ}\right) \\ &\quad \times G^{-1}\left[z_b - e^{-ƛb}\mathrm{E}_{ɋ,1}\left(\mathrm{A}(b-a)^{ɋ}\right) z_a\right]\, dɼ \\ &= e^{-ƛb}\mathrm{E}_{ɋ,1}\left(\mathrm{A}(b-a)^{ɋ}\right) z_a + \int_a^b \mathrm{E}_{ɋ,ɋ}\left(\mathrm{A}(b-ɼ)^{ɋ}\right) \mathrm{BB}^* \mathrm{E}_{ɋ,ɋ}\left(\mathrm{A}^*(b-ɼ)^{ɋ}\right)\, dɼ \\ &\quad \times G^{-1}\left[z_b - e^{-ƛb}\mathrm{E}_{ɋ,1}\left(\mathrm{A}(b-a)^{ɋ}\right) z_a\right] \\ &= e^{-ƛb}\mathrm{E}_{ɋ,1}\left(\mathrm{A}(b-a)^{ɋ}\right) z_a + GG^{-1}\left[z_b - e^{-ƛb}\mathrm{E}_{ɋ,1}\left(\mathrm{A}(b-a)^{ɋ}\right) z_a\right] \\ &= z_b. \end{aligned}$$

Thus, the system (2.1) is controllable on J.
On the other hand, if G is not positive definite, then there exists a $z \neq 0$ such that:

$$z^* G z = 0,$$

i.e.,

$$z^* \int_a^b \mathrm{E}_{ɋ,ɋ}\left(\mathrm{A}(b-ɼ)^{ɋ}\right) \mathrm{BB}^* \mathrm{E}_{ɋ,ɋ}\left(\mathrm{A}^*(b-ɼ)^{ɋ}\right)\, dɼ z = 0.$$

This implies that, for all $ɼ \in \mathrm{J}$:

$$z^* \mathrm{E}_{ɋ,ɋ}\left(\mathrm{A}(b-ɼ)^{ɋ}\right) \mathrm{B} = 0.$$

Let $z_a = \left[e^{-ƛt}\mathrm{E}_{ɋ,1}\left(\mathrm{A}(t-a)^{ɋ}\right)\right]^{-1} z$. Since the system (2.1) is controllable, there exists a control $u(ɼ)$ such that $z(a) = z_a$ and $z(b) = 0$. Then, from (2.2), we have:

$$\begin{aligned} z(b) &= e^{-ƛb}\mathrm{E}_{ɋ,1}\left(\mathrm{A}(b-a)^{ɋ}\right) z_a + \int_a^b (b-ɼ)^{ɋ-1} e^{-ƛ(b-ɼ)} \mathrm{E}_{ɋ,ɋ}\left(\mathrm{A}(b-ɼ)^{ɋ}\right) \mathrm{B}u(ɼ)\, dɼ \\ 0 &= z + \int_a^b (b-ɼ)^{ɋ-1} e^{-ƛ(b-ɼ)} z^* \mathrm{E}_{ɋ,ɋ}\left(\mathrm{A}(b-ɼ)^{ɋ}\right) \mathrm{B}u(ɼ)\, dɼ, \end{aligned}$$

leading to $z^*z = 0$, which contradicts $z \neq 0$. Therefore, G must be positive definite. $\square$

3. Controllability of nonlinear systems

Let $\mathrm{Y} = \mathrm{C}_n(\mathrm{J}) \times \mathrm{C}_m(\mathrm{J})$, where $\mathrm{C}_n(\mathrm{J})$ is the Banach space of continuous $\mathbb{R}^n$-valued functions on J. Hence, Y forms a Banach space with the norm $\| (z,u) \| = \| z \| + \| u \|$, where $\| z \| = \sup\{z(t) : t \in \mathrm{J}\}$ and $\| u \| = \sup\{u(t) : t \in \mathrm{J}\}$.

For any $(y, v) \in \mathrm{Y}$, the system (1.1) becomes:

$$\begin{cases} {}^{C}\mathrm{D}_a^{q,\lambda} z(t) = Az(t) + Bu(t) + f(t, y(t), v(t)), \quad t \in \mathrm{J}, \\ z(a) = z_a. \end{cases} \tag{3.1}$$

Lemma 3.1. *For a given control $u(t) \in \mathrm{L}^2(\mathrm{J}, \mathbb{R}^m)$, the solution to system* (3.1) *is given by:*

$$\begin{aligned} z(t) = {} & e^{-\lambda t}\mathrm{E}_{q,1}\left(\mathrm{A}(t-a)^{q}\right) z_a + \int_a^t (t-r)^{q-1} e^{-\lambda(t-r)} \mathrm{E}_{q,q}\left(\mathrm{A}(t-r)^{q}\right) \mathrm{B}u(r)\, dr \\ & + \int_a^t (t-r)^{q-1} e^{-\lambda(t-r)} \mathrm{E}_{q,q}\left(\mathrm{A}(t-r)^{q}\right) f(r, y(r), v(r))\, dr. \end{aligned} \tag{3.2}$$

Proof. The proof follows similarly to Lemma 2.1. $\square$

Theorem 3.2. *The nonlinear system* (1.1) *is controllable on* J *if the function g satisfies $\lim_{|p|\to\infty} \frac{|g(t,p)|}{|p|} = 0$ uniformly for $t \in \mathrm{J}$, where $|p| = |y| + |v|$, and if the associated linear system* (2.1) *is controllable on* J.

Proof. Define the operator $L : \mathrm{Y} \to \mathrm{Y}$ by $L(y, v) = (z, u)$, where:

$$\begin{aligned} u(t) = {} & \left[(b-t)^{q-1} e^{-\lambda(b-t)}\right]^{-1} \mathrm{B}^*\mathrm{E}_{q,q}\left(\mathrm{A}^*(b-t)^{q}\right) G^{-1} \\ & \times \Bigg[z_b - e^{-\lambda(b)}\mathrm{E}_{q,1}\left(\mathrm{A}(b-a)^{q}\right) z_a \\ & \qquad - \int_a^b (b-r)^{q-1} e^{-\lambda(b-r)} \mathrm{E}_{q,q}\left(\mathrm{A}(b-r)^{q}\right) f(r, y(r), v(r))\, dr \Bigg], \end{aligned}$$

and

$$\begin{aligned} z(t) = {} & e^{-\lambda t}\mathrm{E}_{q,1}\left(\mathrm{A}(t-a)^{q}\right) z_a + \int_a^t (t-r)^{q-1} e^{-\lambda(t-r)} \mathrm{E}_{q,q}\left(\mathrm{A}(t-r)^{q}\right) \mathrm{B}u(r)\, dr \\ & + \int_a^t (t-r)^{q-1} e^{-\lambda(t-r)} \mathrm{E}_{q,q}\left(\mathrm{A}(t-r)^{q}\right) f(r, y(r), v(r))\, dr. \end{aligned}$$

Now, choose constants:

$$\begin{aligned} & \hat{a_1} = \|(b-r)^{q-1} e^{-\lambda(b-r)}\|, \quad \hat{a_2} = \|\mathrm{E}_{q,q}\left(\mathrm{A}(b-r)^{q}\right)\|, \\ & \hat{a} = \sup\left\{1, \hat{a_1}\hat{a_2}\|\mathrm{B}^*\|(b-a)\right\}, \quad \hat{b_1} = \|e^{-\lambda b}\mathrm{E}_{q,1}\left(\mathrm{A}(b-a)^{q}\right) z_a\|, \\ & \hat{c_1} = 4\left[\hat{a_2}^2\|\mathrm{B}^*\|G^{-1}(b-a)\right], \quad \hat{c_2} = 4\left[\hat{a_1}\hat{a_2}(b-a)\right], \quad \hat{d} = \max\{\hat{d_1}, \hat{d_2}\}. \end{aligned}$$

Given $|z(t)|$ and $|u(t)|$, one can now choose $\hat{r} > 0$ such that $\| q \| \leq \hat{r}$. Let $\mathrm{X}(\hat{r}) = \left\{(z,u) : \| z \| \leq \frac{\hat{r}}{2}, \| u \| \leq \frac{\hat{r}}{2}\right\}$ be a convex, closed, and bounded subset of Y.

From the Arzelà-Ascoli theorem, $L : \mathrm{X}(\hat{ɼ}) \to \mathrm{X}(\hat{ɼ})$ is compact and continuous. By Schauder's fixed point theorem, there exists a $(y, v) \in \mathrm{X}(\hat{ɼ})$ such that $L(y, v) = (z, u)$. Hence, $z(ʈ)$ is the solution to system (1.1), and:

$$\begin{aligned}
z(b) = e^{-ƛb}\mathrm{E}_{ɋ,1}\left(\mathrm{A}(b-a)^{ɋ}\right) z_a + \int_a^b \mathrm{E}_{ɋ,ɋ}\left(\mathrm{A}(b-ɼ)^{ɋ}\right)\mathrm{B} \\
\times \left[(b-s)^{ɋ-1}e^{-ƛ(b-ʈ)}\right]^{-1}\mathrm{B}^*\mathrm{E}_{ɋ,ɋ}\left(\mathrm{A}^*(b-ʈ)^{ɋ}\right) \\
\times G^{-1}\left[z_b - e^{-ƛb}\mathrm{E}_{ɋ,1}\left(\mathrm{A}(b-a)^{ɋ}\right) z_a\right] dɼ \\
+ \int_a^b (b-ɼ)^{ɋ-1}e^{-ƛ(b-ɼ)}\mathrm{E}_{ɋ,ɋ}\left(\mathrm{A}(b-ɼ)^{ɋ}\right) f(ɼ, y(ɼ), v(ɼ))\, dɼ.
\end{aligned}$$

Thus, $z(b) = z_b$, and system (1.1) is controllable on J. □

4. Numerical examples

Example 4.1. Consider the following nonlinear tempered Caputo fractional differential control system:

(4.1)
$$\begin{cases} {}^{C}\mathrm{D}_{0^+}^{\frac{1}{2},1} z(ʈ) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} z(ʈ) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(ʈ) + \begin{bmatrix} \sqrt{z_1^2(ʈ)+2} \\ 0 \end{bmatrix}, \quad ʈ \in [0,2], \\ z(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{cases}$$

Comparing (4.1) with (1.1), we get $ɋ = \frac{1}{2}, ƛ = 1,\ A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, a = 0, b = 2, z_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $f(ʈ, x(ʈ)) = \begin{bmatrix} \sqrt{z_1^2(ʈ)+2} \\ 0 \end{bmatrix}$ and $z(ʈ) = \begin{bmatrix} z_1(ʈ) \\ z_2(ʈ) \end{bmatrix}$. Let us take $z(2) = \begin{bmatrix} z_1(2) \\ z_2(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The M-L matrix function for the given matrix A is

$$\mathrm{E}_{\frac{1}{2},\frac{1}{2}}(At) = \begin{bmatrix} \mathrm{E}_{\frac{1}{2},\frac{1}{2}}(-ʈ) & 0 \\ 0 & \mathrm{E}_{\frac{1}{2},\frac{1}{2}}(ʈ) \end{bmatrix}.$$

The controllability Gramian matrix

$$\begin{aligned}
G[0,2] &= \int_0^2 \mathrm{E}_{\frac{1}{2},\frac{1}{2}}\left(\mathrm{A}(2-ɼ)^{\frac{1}{2}}\right)\mathrm{BB}^*\mathrm{E}_{\frac{1}{2},\frac{1}{2}}\left(\mathrm{A}^*(2-ɼ)^{\frac{1}{2}}\right) dɼ \\
&= \int_0^2 \begin{bmatrix} \mathrm{E}^2_{\frac{1}{2},\frac{1}{2}}(-(2-ɼ)^{\frac{1}{2}}) & \mathrm{E}_{\frac{1}{2},\frac{1}{2}}((2-ɼ)^{\frac{1}{2}})\mathrm{E}_{\frac{1}{2},\frac{1}{2}}(-(2-ɼ)^{\frac{1}{2}}) \\ \mathrm{E}_{\frac{1}{2},\frac{1}{2}}((2-ɼ)^{\frac{1}{2}})\mathrm{E}_{\frac{1}{2},\frac{1}{2}}(-(2-ɼ)^{\frac{1}{2}}) & \mathrm{E}^2_{\frac{1}{2},\frac{1}{2}}((2-ɼ)^{\frac{1}{2}}) \end{bmatrix} dr \\
&= \begin{bmatrix} 0.0665 & 1.7359 \\ 1.7359 & 168.1998 \end{bmatrix},
\end{aligned}$$

is positive definite. Therefore, the linear system corresponding to (4.1) is controllable on $[0,2]$. Further, $\lim_{|(z,u)|\to\infty} \frac{|f(ʈ,z,u)|}{|(z,u)|} = 0$ uniformly on $[0,1]$. The system

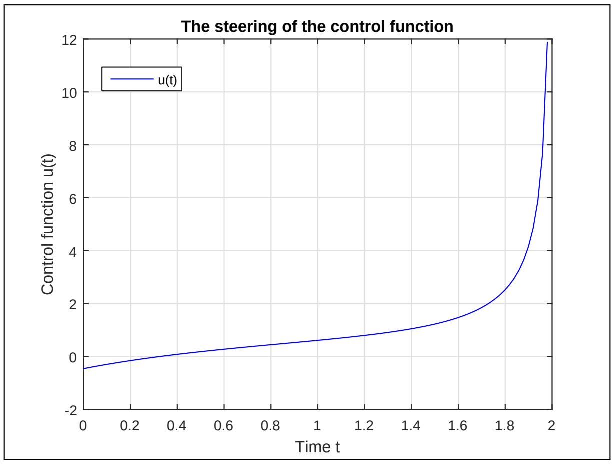

FIGURE 1. The trajectory of $u(t)$ of the system (4.1)on $[0, 2]$

(4.1) is controllable on $[0, 2]$ by theorem 3.2. The controlled trajectories of the system (4.1) steering from the initial state $z(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to a desired state $z(2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ during $[0, 2]$ can be approximated from the following algorithm

$$\begin{aligned} u^n(t) = & \left[(2-t)^{-0.50} e^{\frac{1}{2}(2-t)}\right] B^* E_{\frac{1}{2},\frac{1}{2}} \left(A^*(2-t)^{0.50}\right) G^{-1}[0,2] \\ & \times [z(2) - e^{-1} E_{0.50,1}\left(A(2)^{0.50}\right) z_0 - \int_0^2 (2-r)^{-0.50} e^{-\frac{1}{2}(2-r)} \\ & E_{0.50,0.50}\left(A(2-r)^{0.50}\right) f(r, y^n(r), v(r)) dr] \end{aligned}$$

$$\begin{aligned} z^{n+1}(t) = & e^{\frac{-s}{2}} E_{0.50,1}\left(A(t)^{0.50}\right) w_0 \\ & + \int_0^t (t-r)^{-0.50} e^{\frac{-(t-r)}{2}} E_{0.50,0.50}\left(A(t-r)^{0.50}\right) \\ & (Bu^n(r) + f(r, y^n(r), v(r))) \, dr \end{aligned}$$

with $y^0(t) = y_0$, where $n \in \mathbb{N}$. Using MATLAB, the controlled trajectories and steering control $u(t)$ are computed and are depicted in Fig.1 and Fig.2.

5. CONCLUSION

In this article, we studied the controllability of fractional dynamical system involving tempered Caputo FD. This study of controllability of Caputo FD gives the controllability results for many possible FDs, in particular Caputo FD. Here, we have used controllability Grammian matrix and Schauder's fixed point technique to establish sufficient conditions for controllability of fractional dynamical system. The numerical example is presented to illustrate the main results.

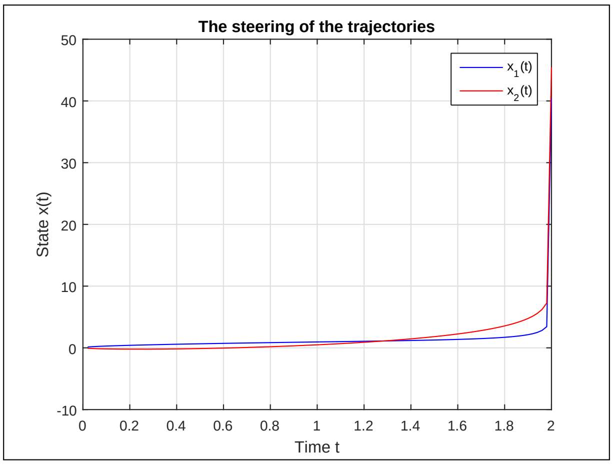

FIGURE 2. The trajectory of the system (4.1) steers from $\begin{bmatrix}0\\0\end{bmatrix}$ to the final state $\begin{bmatrix}1\\2\end{bmatrix}$ during the interval $[0, 2]$.

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