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SENSITIVITY OF A FEASIBLE SCHEME IN EDUCATIONAL INFORMATION RESOURCES SHARING SYSTEM

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Abstract. In the existing works, the P2P (peer-to-peer) educational information resources sharing system was characterized by a group of max-product fuzzy relation inequalities (FRIs). In fact, any solution of the max-product FRIs reflects a feasible scheme in the corresponding educational information resources sharing system. To embody the ability to resist external interference for a feasible scheme, we define the concept of sensitivity for a given solution in the relevant FRIs system. Moreover, we distinguish the positive sensitivity, the negative sensitivity and the sensitivity. Detailed resolution procedures are proposed for solving these sensitivities, with some numerical illustrative examples.

1. Introduction

Fuzzy relation equations (FREs) were the generalization of the classical relation (composed) equations. They were first proposed by E. Sanchez [28]. W. Pedrycz further discussed the generalized forms of FREs and their applications [23]. In the typical system of FREs, the composed operations used are *max* ($\vee$) and *min* ($\wedge$). Subsequently, it was found that the max-product composition is more powerful, in some cases, than the original max-min composition [2, 14, 23]. Several resolution methods have been proposed for solving the max-product FREs [20, 24, 31]. The structure representing the solution set to the max-min FREs is the same as that for the max-product FREs. In such an FREs system, there exists a unique maximum solution and finitely many minimal solutions when the system is consistent. Obtaining the maximum solution is always trivial. As a consequence, the challenge in solving the FREs system is deriving all the minimal solutions. B.-S. Shieh focused on this issue [29]. As pointed out in [12, 17], solving such an FREs system is equivalently a set covering problem, which is NP-hard. To reduce the calculation amount, a consistent FREs system was first reduced to an irreducible form, leaving its solution set unchanged [29]. The irreducible system was then converted into a covering problem and solved [29].

Since obtaining the whole solution set of the FREs system with max-product composition is a difficult task, searching for some specific solutions became another interesting research topic. The common approach was to construct the relevant optimization problem with the FREs constraint. The problem, which has a linear objective function and a max-product FREs constraint, was investigated in [15] for

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the first time. Since the feasible domain is non-convex when the constraint system has more than one minimal solution, such linear programming with a max-product FREs constraint is typically a non-convex optimization problem. In [15], the famous branch-and-bound method was adopted to search for the optimal solution(s) in a groundbreaking manner. In the following decades, the branch-and-bound method became the most commonly used approach for dealing with optimization problems with FREs constraints. P. Li and S.-C. Fang [13] proposed a new algorithm to deal with the same optimization problem presented in [11], adopting the concept of chained-set suite. Besides minimizing a linear objective function, nonlinear optimization problems with FREs constraints were also investigated. Evolutionary algorithms, including Genetic Algorithm (GA) [9] and Particle Swarm Optimization (PSO) [6], became universal approaches for solving such nonlinear fuzzy relation optimization problems. Some special nonlinear fuzzy relation optimization problems were also discussed and solved, either with a separable objective function [10] or a geometric one [30, 39].

J. Drewniak [5] first explored the fuzzy relation inequality (FRI). He introduced the formulae of a system with FRIs, compared to the system of FREs [5]. More introduction to the FRI was presented in [18]. The resolution methods for the FREs can be applied for FRIs, including solving the complete solution set or solving the relevant optimization problems. A.A. Molai investigated the minimization problems in which the constraint system was the FRIs with max-product composition [19, 21].

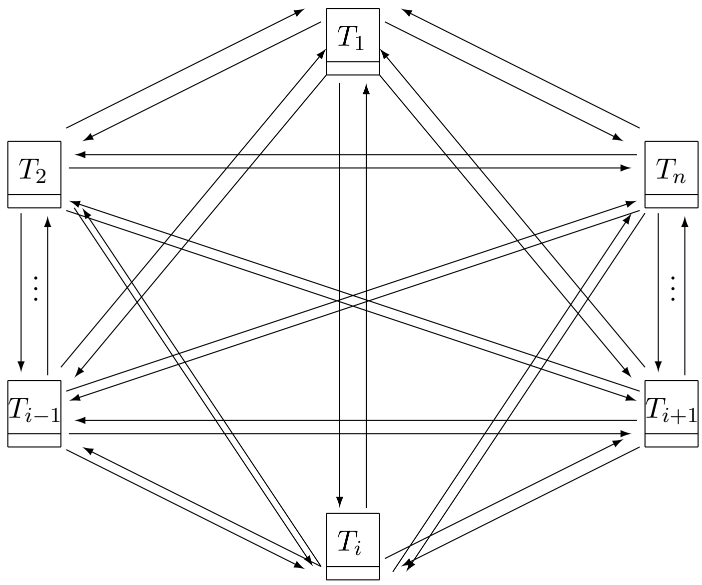

Fig. 1. P2P educational information resources sharing system.

The FREs with max-min composition were used for describing the P2P educational information resources sharing system [34]. In such a system, there are n terminals (see Fig. 1), with notations $T_1, \ldots, T_n$. Each terminal is able to download its required file from any other terminal, based on peer-to-peer data transmission mechanism. The P2P educational information resources sharing system could be reduced into the following max-min FREs,

$$\bigvee_{j\in N}(a_{ij}\wedge x_j)=b_i, \forall i\in M, \tag{1.1}$$

where $N=\{1,\dots,n\}$ $M=\{1,\dots,m\}$. In the above system (1.1), the variables $\{x_1,\dots,x_n\}$ represent the quality levels, on which the terminals send out their local resources/files. The parameter a_{ij} represents the bandwidth, while b_i characterizes the download traffic requirement of the ith terminal. The minimization semi-latticized fuzzy relation geometric programming, employing system (1.1) as its constraint, was explored in [35]. Subsequently, single-variable term fuzzy relation geometric programming [37] and fuzzy relation geometric programming [38], also subject to system (1.1), were investigated respectively. When the download traffic requirement is no longer an exact value, but is requested to be no less than b_i, then the P2P sharing system should be reduced to the FRIs with max-min composition as [33]

$$\bigvee_{j\in N}(a_{ij}\wedge x_j)\geq b_i, \forall i\in M. \tag{1.2}$$

Furthermore, when the download traffic requirement is considered to be a range, e.g., between $\underline{c}_i$ and $\overline{c}_i$, then system (1.2) can be further written as [3,16]

$$\underline{c}_i\leq\bigvee_{j\in N}(a_{ij}\wedge x_j)\leq\overline{c}_i, \forall i\in M. \tag{1.3}$$

Characterized by the above max-min FREs system (1.1) or the max-min FRIs systems (1.2) and (1.3), the terminals in the P2P educational information resources sharing system are assumed to be linked through lines. However, when the terminals in the system are wirelessly linked, the above max-min system was no longer applicable. Instead of the max-min system, the P2P educational information resources sharing system should be reduced to the following FRIs with max-product composition [27,36],

$$\begin{cases}\underline{c}_1\leq a_{11}x_1\wedge a_{12}x_2\wedge\cdots\wedge a_{1n}x_n\leq\overline{c}_1,\\ \underline{c}_2\leq a_{21}x_1\wedge a_{22}x_2\wedge\cdots\wedge a_{2n}x_n\leq\overline{c}_2,\\ \vdots\\ \underline{c}_m\leq a_{m1}x_1\wedge a_{m2}x_2\wedge\cdots\wedge a_{mn}x_n\leq\overline{c}_m,\end{cases} \tag{1.4}$$

where $x=(x_1,x_2,\dots,x_n)^T$, $A=(a_{ij})\in[0,1]^{m\times n}$, $\underline{c}=(\underline{c}_1,\underline{c}_2,\dots,\underline{c}_m)^T$, $\overline{c}=(\overline{c}_1,\overline{c}_2,\dots,\overline{c}_m)^T$. That is to say, considering the wired connection, the educational information resources sharing system could be characterized by the max-min system (1.1), (1.2) or (1.3), while considering the wireless connection, it should be characterized by the max-product system (1.4). Different connection types lead to different composition. But what is similar is that any solution of the above fuzzy relation system indeed reflects indeed a feasible (flow control) scheme in the educational information resources sharing system.

In recent years, the max-product FRIs haven been applied in the wireless communication basic-station system [25,26]. Moreover, the authors considered the term-absent situation [25,26]. For the consistent system with max-product composition, the minimal solutions were discussed [40], while for the inconsistent system, the approximate solutions were defined and resolved [1, 32]. Regarding the max-product system, the composition could be further generalized to the max-t-norm one [7, 8]. Moreover, the max-product FREs were also extended to the bipolar ones [4, 22].

In all the references [3, 16, 27, 33–38], during the file transfer process, external interference was not taken into account. However, as is well known, the random external interferences are ubiquitous. Hence, in this work we aim to consider the random external interferences in an educational information resources sharing system. These random external disturbances will cause variation to the parameters $\{a_{ij} \mid i \in M, j \in N\}$. To reflect the extent to which a solution can withstand the parameters' changes, we define and investigate the concept of sensitivity of a given solution to system (1.4), which indeed represents a feasible scheme in the educational information resources sharing system.

The remainder is organized as follows. Sec. 2 introduces foundational concepts and properties of the max-product FRIs system (1.4). In Sec. 3, the concepts of positive sensitivity, negative sensitivity and sensitivity are defined, respectively. The main results are presented in Sec. 4. In this section detailed resolution procedures are designed for finding the positive sensitivity, negative sensitivity and sensitivity. Moreover, we propose an algorithm for computing the overall sensitivity of a given solution, in this section. To verify the effectiveness of our proposed resolution procedures, several examples are provided in Sec. 5. Results and discussion are presented in Sec. 6, while Sec. 7 provides a brief conclusion.

2. Max-product FRIs system (1.4)

This section provides some necessary properties of the max-product FRIs system (1.4).

The above system of inequalities (1.4) can be written as

$$\underline{c} \leq A \odot x \leq \overline{c}, \tag{2.1}$$

where $\odot$ represents the max-product operator and all the parameters and variables belong to the unit interval $[0, 1]$.

For convenience, two sets of indices M and N are denoted as follows:

$$M = \{1, 2, \dots, m\}, \quad N = \{1, 2, \dots, n\}.$$

Additionally, in the text, the solution set of the system of inequalities (2.1) is always denoted as

$$X(A, \underline{c}, \overline{c}) = \{x \in [0, 1]^n \mid \underline{c} \leq A \odot x \leq \overline{c}\}.$$

Definition 2.1. The system of inequalities (2.1) is called **consistent** if it has a solution, i.e., there exists an $x \in X(A, \underline{c}, \overline{c})$ such that $\underline{c} \leq A \odot x \leq \overline{c}$. Otherwise, (2.1) is called **inconsistent**.

Definition 2.2. Let $\hat{x} \in X(A, \underline{c}, \overline{c})$. If for any $x \in X(A, \underline{c}, \overline{c})$, we always have $\hat{x} \leq x$, then $\hat{x}$ is called the **maximum solution** of (2.1). Moreover, $\check{x} \in X(A, \underline{c}, \overline{c})$ is called a **minimal solution** of (2.1), if for any $x' \in X(A, \underline{c}, \overline{c})$ with $x' \leq \check{x}$, it holds that $x' = \check{x}$.

Property 1 ([18])**.** Let $x \in [0,1]^n$. Then the necessary and sufficient conditions for $x \in X(A, \underline{c}, \overline{c})$ are the following two conditions:

(i) For any $i \in M$ and $j \in N$, there is $a_{ij}x_j \le \overline{c}_i$.
(ii) For any $i \in M$, there exists $j \in N$ such that $a_{ij}x_j \ge \underline{c}_i$.

We define the operator "@" as

$$a_{ij}@\overline{c}_i = \begin{cases} \frac{\overline{c}_i}{a_{ij}}, & \text{if } a_{ij} \ge \overline{c}_i, \\ 1, & \text{if } a_{ij} < \overline{c}_i. \end{cases} \tag{2.2}$$

Denote the vector $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) \in [0,1]^n$, where

$$\hat{x}_j = \min_{i \in M} (a_{ij}@\overline{c}_i), \quad j = 1, 2, \ldots, n. \tag{2.3}$$

For the vector $\hat{x}$, we obtain the following Property 2 and Theorem 2.3.

Property 2 ([18])**.** If $x \in X(A, \underline{c}, \overline{c})$, then $x \le \hat{x}$.

Theorem 2.3 ([18])**.** *The system of inequalities* (2.1) *is consistent if and only if* $\hat{x} \in X(A, \underline{c}, \overline{c})$.

Theorem 2.3 can be used to check whether the system of inequalities (2.1) is consistent. Moreover, from Property 2, we know that if the system of inequalities (2.1) is consistent, then the vector $\hat{x}$ is always the maximum solution.

3. Definition and uniqueness of the sensitivity of a given solution in the system of inequalities (2.1)

Let $y \in X(A, \underline{c}, \overline{c})$ be a solution of the system of inequalities (2.1). In this section, we will define the concepts of positive sensitivity, negative sensitivity, and sensitivity of the solution y, and explore some related properties that lay the necessary foundation for solving the sensitivity.

Let $A \in [0,1]^{m \times n}$ and let $d \in [0,1]$ be a real number. We denote

$$A^+ = A + d, \quad A^- = A - d.$$

Property 3. For any $y \in [0,1]^n$ and $d \in [0,1]$, the following inequality holds:

$$A^- \odot y \le A \odot y \le A^+ \odot y.$$

Proof. Omitted. □

Definition 3.1 (Positive Sensitivity)**.** Let $d^+ \in [0,1]$. The number d^+ is called the **positive sensitivity** of the given solution y (with respect to the coefficient matrix A), if it satisfies the following three conditions:

(i) $A + d^+ \in [0,1]^{m \times n}$,
(ii) $y \in X(A + d^+, \underline{c}, \overline{c})$,
(iii) If $d \in [0,1]$ and $d > d^+$, then either $A^+ \notin [0,1]^{m \times n}$ or $y \notin X(A + d, \underline{c}, \overline{c})$.

Definition 3.2 (Negative Sensitivity)**.** Let $d^- \in [0,1]$. The number d^- is called the **negative sensitivity** of the given solution y (with respect to the coefficient matrix A), if it satisfies the following three conditions:

(i) $A - d^- \in [0,1]^{m \times n}$,

(ii) $y \in X(A - d^{-}, \underline{c}, \overline{c})$,

(iii) If $d \in [0,1]$ and $d > d^{-}$, then either $A^{-} \notin [0,1]^{m\times n}$ or $y \notin X(A - d, \underline{c}, \overline{c})$.

Definition 3.3 (Sensitivity)**.** Let d^{+} and d^{-} be the positive and negative sensitivities of the given solution y (with respect to the coefficient matrix A). Then the **sensitivity** of the solution y is defined as $d = \min(d^{+}, d^{-})$.

Property 4 (Uniqueness)**.** Let $y \in X(A, \underline{c}, \overline{c})$ be a solution of the system of inequalities (2.1). If the positive sensitivity (negative sensitivity or overall sensitivity) of the given solution y exists, then the positive sensitivity (negative sensitivity or overall sensitivity) is unique.

By contradiction. Assume that y has two different positive sensitivities, denoted by d_1^{+} and d_2^{+}, and let $d_1^{+} < d_2^{+}$. Since d_2^{+} is the positive sensitivity of y, by Definition 3.1, we know $A + d_2^{+} \in [0,1]^{m\times n}$ and $y \in X(A + d_2^{+}, \underline{c}, \overline{c})$. Since d_1^{+} is also the positive sensitivity of y, and $d_1^{+} < d_2^{+}$, by Definition 3.1 (iii) we have either $A + d_2^{+} \notin [0,1]^{m\times n}$ or $y \notin X(A + d_2^{+}, \underline{c}, \overline{c})$. This leads to a contradiction. Therefore, the positive sensitivity of y is unique. Similarly, it can be proven that the negative sensitivity (or overall sensitivity) is also unique. This completes the proof ☐

4. Resolution methods for the sensitivity of a given solution in the system of inequalities (2.1)

In this section, we present methods for solving the positive and negative sensitivities of the solution y. From the positive and negative sensitivities of y, the sensitivity of y can also be derived.

4.1. **Resolution of the positive sensitivity.** For any $i \in M$ and $j \in N$, when $y_j = 0$, we stipulate $\frac{\overline{c}_i}{y_j} = 1$. Thus, we can define the following notations:

$$d_1^{+} = \min_{i\in M, j\in N} \{1 - a_{ij}\}, \tag{4.1}$$

$$d_2^{+} = \min_{i\in M, j\in N} \left\{\frac{\overline{c}_i}{y_j} - a_{ij}\right\}, \tag{4.2}$$

and

$$d^{+} = \min(d_1^{+}, d_2^{+}). \tag{4.3}$$

Proposition 4.1. *Let $y \in X(A, \underline{c}, \overline{c})$ be a solution of the system* (2.1)*. For any $i \in M$ and $j \in N$, we always have $\frac{\overline{c}_i}{y_j} - a_{ij} \geq 0$.*

Proof. Since $y \in X(A, \underline{c}, \overline{c})$, we have

$$0 \leq y_j \leq 1, \quad \forall j \in N, \tag{4.4}$$

and

$$\underline{c}_i \leq a_{i1}y_1 \vee a_{i2}y_2 \vee \cdots \vee a_{in}y_n \leq \overline{c}_i, \quad \forall i \in M. \tag{4.5}$$

Thus,

$$a_{ij}y_j \leq \overline{c}_i, \quad \forall i \in M, j \in N. \tag{4.6}$$

If $y_j \neq 0$, then $0 < y_j \leq 1$. From the above, we obtain $a_{ij} \leq \frac{\overline{c}_i}{y_j}$, which implies:

$$\frac{\overline{c}_i}{y_j} - a_{ij} \geq 0, \quad \forall i \in M, j \in N. \tag{4.7}$$

If $y_j = 0$, by stipulation, $\frac{\overline{c}_i}{y_j} = 1$. Therefore, $a_{ij} \leq 1 = \frac{\overline{c}_i}{y_j}$, i.e.,

$$\frac{\overline{c}_i}{y_j} - a_{ij} \geq 0, \quad \forall i \in M, j \in N. \tag{4.8}$$

Thus, $\frac{\overline{c}_i}{y_j} - a_{ij} \geq 0$ holds in all cases. This completes the proof. □

Theorem 4.2. *Let d^+ be defined by formulas (4.1))-(4.3). Then d^+ is the positive sensitivity of the given solution y (with respect to the coefficient matrix A).*

Proof. (i) From the assumption $a_{ij} \in [0,1]$, for any $i \in M$ and $j \in N$, we know $1 - a_{ij} \in [0,1]$. Thus, we have

$$d_1^+ = \min_{i \in M, j \in N} \{1 - a_{ij}\} \in [0,1].$$

By Proposition 4.1, we know

$$\frac{\overline{c}_i}{y_j} - a_{ij} \geq 0, \quad \forall i \in M, j \in N. \tag{4.9}$$

Therefore, $d_2^+ = \min_{i \in M, j \in N} \left\{ \frac{\overline{c}_i}{y_j} - a_{ij} \right\} \geq 0$. Considering that $0 \leq d_1^+ \leq 1$, we have

$$0 \leq d^+ = \min(d_1^+, d_2^+) \leq d_1^+ \leq 1.$$

(ii) Since $y \in X(A, \underline{c}, \overline{c})$, it is clear that $A \odot y \geq \underline{c}$. From $d^+ \geq 0$ and Property 3, we obtain

$$(A + d^+) \odot y \geq A \odot y \geq \underline{c}. \tag{4.10}$$

On the other hand, from the definition of d_2^+, we know

$$d_2^+ \leq \frac{\overline{c}_i}{y_j} - a_{ij}, \quad \forall i \in M, j \in N. \tag{4.11}$$

Thus

$$d_2^+ + a_{ij} \leq \frac{\overline{c}_i}{y_j}, \quad \forall i \in M, j \in N, \tag{4.12}$$

i.e.,

$$(d_2^+ + a_{ij}) y_j \leq \overline{c}_i, \quad \forall i \in M, j \in N. \tag{4.13}$$

This means

$$(A + d_2^+) \odot y \leq \overline{c}. \tag{4.14}$$

Since $d^+ = \min(d_1^+, d_2^+)$, from formula (4.14) and Property 3, we get

$$(A + d^+) \odot y \leq (A + d_2^+) \odot y \leq \overline{c}. \tag{4.15}$$

From (4.10) and (4.15), we conclude that $y \in X(A + d^+, \underline{c}, \overline{c})$.

(iii) If $d \in [0,1]$ and $d > d^+$. Then by $d^+ = \min(d_1^+, d_2^+)$, we have either $d > d_1^+$ or $d > d_2^+$.

Case 1. If $d > d_1^+ = \min_{i\in M, j\in N}\{1 - a_{ij}\}$, then there exist $i_1 \in M$ and $j_1 \in N$ such that

$$d > d_1^+ = 1 - a_{i_1j_1}, \tag{4.16}$$

i.e.,

$$d + a_{i_1j_1} > 1. \tag{4.17}$$

This indicates $A + d \notin [0,1]^{m\times n}$.

Case 2. If $d > d_2^+ = \min_{i\in M, j\in N}\left\{\frac{\overline{c}_i}{y_j} - a_{ij}\right\}$, then there exist $i_2 \in M$ and $j_2 \in N$ such that

$$d > d_2^+ = \frac{\overline{c}_{i_2}}{y_{j_2}} - a_{i_2j_2}, \tag{4.18}$$

i.e.,

$$a_{i_2j_2} + d > \frac{\overline{c}_{i_2}}{y_{j_2}}. \tag{4.19}$$

Thus

$$\bigvee_{j\in J}(a_{i_2j} + d)y_j \geq (a_{i_2j_2} + d)y_{j_2} > \overline{c}_{i_2}. \tag{4.20}$$

This indicates $y \notin X(A + d, \underline{c}, \overline{c})$.

Following Definition 2.1 and the conditions (i), (ii) and (iii) verified above, we know that d^+ is the positive sensitivity of the solution y. This completes the proof. □

4.2. **Resolution of the negative sensitivity.** For any $i \in M$ and $j \in N$, when $y_j = 0$, we stipulate $\frac{\underline{c}_i}{y_j} = 0$. Thus, we can define the following sets of indices

$$d_1^- = \min_{i\in M, j\in N}\{a_{ij}\}, \tag{4.21}$$

$$d_2^- = \min_{i\in M}\max_{j\in J_i^-}\left\{a_{ij} - \frac{\underline{c}_i}{y_j}\right\}, \tag{4.22}$$

where

$$J_i^- = \{j \in N \mid a_{ij}y_j \geq \underline{c}_i\}. \tag{4.23}$$

We further denote

$$d^- = \min(d_1^-, d_2^-). \tag{4.24}$$

Theorem 4.3. *Let d^- be defined by formulas (4.21)-(4.23). Then d^- is the negative sensitivity of the given solution y (with respect to the coefficient matrix A).*

Proof. (i) From the assumption $a_{ij} \in [0,1]$, for any $i \in M$ and $j \in N$, we have

$$d_1^- = \min_{i\in M, j\in N}\{a_{ij}\} \in [0,1]. \tag{4.25}$$

Take any $i \in M$ and $j \in J_i^-$. From the definition of the index set J_i^-, we know that

$$a_{ij}y_j \geq \underline{c}_i. \tag{4.26}$$

If $y_j = 0$, then $a_{ij} - \frac{\underline{c}_i}{y_j} = a_{ij} \geq 0$. If $y_j > 0$, then from inequality (4.26), we have $a_{ij} - \frac{\underline{c}_i}{y_j} \geq 0$. Therefore, we obtain

$$a_{ij} - \frac{\underline{c}_i}{y_j} \geq 0, \quad \forall i \in M, j \in J_i^-. \tag{4.27}$$

As a result $d_2^- = \min_{i\in M} \max_{j\in J_i^-} \left\{a_{ij} - \frac{\underline{c}_i}{y_j}\right\} \geq 0$. Combining with equation (4.25), we get $0 \leq \min(d_1^-, d_2^-) \leq d_1^- \leq 1$. Thus $d^- = \min(d_1^-, d_2^-) \in [0,1]$.

In addition, since $d^- = \min(d_1^-, d_2^-) \leq d_1^- = \min_{i'\in M, j'\in N} \left\{a_{i'j'}\right\} \leq a_{ij}$, we have

$$a_{ij} - d^- \geq 0, \quad \forall i \in M, j \in N. \tag{4.28}$$

Note that $d^- \in [0,1]$. We have

$$a_{ij} - d^- \leq a_{ij} \leq 1, \quad \forall i \in M, j \in N. \tag{4.29}$$

The formulae (4.28) and (4.29) contribute to

$$0 \leq a_{ij} - d^- \leq 1, \quad \forall i \in M, j \in N. \tag{4.30}$$

This indicates $A - d^- \in [0,1]^{m\times n}$.

(ii) Since $y \in X(A, \underline{c}, \overline{c})$, it is clear that $A \odot y \leq \overline{c}$. Considering $d^- \in [0,1]$, by Property 3 we have

$$(A - d^-) \odot y \leq A \odot y \leq \overline{c}. \tag{4.31}$$

On the other hand, since $d_2^- = \min_{i\in M} \max_{j\in J_i^-} \left\{a_{ij} - \frac{\underline{c}_i}{y_j}\right\}$, we have

$$d_2^- \leq \max_{j\in J_i^-} \left\{a_{ij} - \frac{\underline{c}_i}{y_j}\right\}, \forall i \in M. \tag{4.32}$$

For each $i \in M$, there exists $j_i \in J_i^-$ such that

$$a_{ij_i} - \frac{\underline{c}_i}{y_{j_i}} = \max_{j\in J_i^-} \left\{a_{ij} - \frac{\underline{c}_i}{y_j}\right\}. \tag{4.33}$$

Thus

$$d_2^- \leq \max_{j\in J_i^-} \left\{a_{ij} - \frac{\underline{c}_i}{y_j}\right\} = a_{ij_i} - \frac{\underline{c}_i}{y_{j_i}}, \forall i \in M, \tag{4.34}$$

i.e.,

$$(a_{ij_i} - d_2^-) y_{j_i} \geq \underline{c}_i, \forall i \in M. \tag{4.35}$$

Hence

$$(a_{i1} - d_2^-) y_1 \vee \cdots \vee (a_{in} - d_2^-) y_n \geq (a_{ij_i} - d_2^-) y_{j_i} \geq \underline{c}_i, \forall i \in M. \tag{4.36}$$

i.e.,

$$(A - d_2^-) \odot y \geq \underline{c}. \tag{4.37}$$

Considering $d^- = \min(d_1^-, d_2^-) \leq d_2^-$, we have

$$A - d^- \geq A - d_2^-. \tag{4.38}$$

Thus by Property 3,

$$(A - d^-) \odot y \geq (A - d_2^-) \odot y \geq \underline{c}. \tag{4.39}$$

Considering the formulae (4.31) and (4.39), it is clear $y \in X(A - d^-, \underline{c}, \overline{c})$.

(iii) If $d \in [0, 1]$ and $d > d^-$, then from the definition of d^-, we know that either $d > d_1^-$ or $d > d_2^-$ holds.

Case 1. If $d > d_1^- = \min_{i \in M, j \in N} \{a_{ij}\}$, then there exist $i_1 \in M$ and $j_1 \in N$ such that $d_1^- = a_{i_1 j_1}$. Thus $d > a_{i_1 j_1}$, i.e.,

$$a_{i_1 j_1} - d < 0. \tag{4.40}$$

Obviously $a_{i_1 j_1} - d$ is an element in the matrix $A - d$. Thus

$$A - d = (a_{ij} - d) \notin [0, 1]^{m \times n}. \tag{4.41}$$

Case 2. If $d > d_2^- = \min_{i \in M} \max_{j \in J_i^-} \{a_{ij} - \frac{\underline{c}_i}{y_j}\}$, then there exists $i_2 \in M$ such that

$$d_2^- = \max_{j \in J_{i_2}^-} \left\{ a_{i_2 j} - \frac{\underline{c}_{i_2}}{y_j} \right\}. \tag{4.42}$$

Take arbitrarily $j' \in J$. If $j' \notin J_{i_2}^-$, then by $J_{i_2}^- = \{j \in N \mid a_{i_2 j} y_j \geq \underline{c}_{i_2}\}$, we have $a_{i_2 j'} y_{j'} < \underline{c}_{i_2}$. Thus

$$(a_{i_2 j'} - d) y_{j'} \leq a_{i_2 j'} y_{j'} < \underline{c}_{i_2}, \quad \forall j' \notin J_{i_2}^-. \tag{4.43}$$

If $j' \in J_{i_2}^-$, then by (4.42) we have $d > d_2^- = \max_{j \in J_{i_2}^-} \{a_{i_2 j} - \frac{\underline{c}_{i_2}}{y_j}\} \geq a_{i_2 j'} - \frac{\underline{c}_{i_2}}{y_{j'}}$. Thus

$$(a_{i_2 j'} - d) y_{j'} < \underline{c}_{i_2}, \quad \forall j' \in J_{i_2}^-. \tag{4.44}$$

The formulae (4.43) and (4.44) contribute to

$$(a_{i_2 j'} - d) y_{j'} < \underline{c}_{i_2}, \quad \forall j' \in J.$$

This indicates $(a_{i_2 1} - d) y_1 (a_{i_2 2} - d) y_2 \vee \cdots \vee (a_{i_2 n} - d) y_n < \underline{c}_{i_2}$. Thus $y \notin X(A - d, \underline{c}, \overline{c})$.

Thus, by Definition 2.2 and the cases (i), (ii) and (iii) above, d^- is the negative sensitivity of the solution y. This completes the proof. □

4.3. **Resolution algorithm for the sensitivity of y.** Given the solution y, the above Subsections 4.1 and 4.2 provide some resolution formulae to calculate the positive sensitivity and negative sensitivity of y, respectively. According to Definition 3.3, the overall sensitivity of y could be further obtained. By summarizing the resolution formulae, we further develop an algorithm for computing the sensitivity of y as follows.

Algorithm I. for computing the sensitivity of the given solution y

Step 1. Compute d_1^+ according to Eq. (4.1).
Step 2. Compute d_2^+ according to Eq. (4.2).
Step 3. Compute the positive sensitivity by Eq. (4.3), i.e., $d^+ = \min\{d_1^+, d_2^+\}$.
Step 4. Compute d_1^- according to Eq. (4.21).
Step 5. Compute the index sets $\{J_i^- \mid i \in M\}$ according to Eq. (4.23).
Step 6. Compute d_2^- according to Eq. (4.22).
Step 7. Compute the negative sensitivity by Eq. (4.24), i.e., $d^- = \min\{d_1^-, d_2^-\}$.
Step 8. Compute the sensitivity of y by $d = \min\{d^+, d^-\}$, according to Definition 3.3.

5. Numerical examples

In this section we provide two numerical examples for verifying the resolution method proposed in the previous section.

Example 5.1. In the following system of fuzzy relational inequalities with max-product composition (5.1), for a given solution y, we will use the method described in Section 4 to compute the positive sensitivity of y. The considered system is expressed by

$$\underline{c} \leq A \odot x \leq \overline{c}, \tag{5.1}$$

in which the matrix A is given as

$$A = \begin{pmatrix} 0.4 & 0.5 & 0.1 & 0.4 & 0.3 & 0.7 \\ 0.9 & 0.4 & 0.3 & 0.2 & 0.5 & 0.3 \\ 0.4 & 0.2 & 0.3 & 0.1 & 0.8 & 0.3 \\ 0.2 & 0.1 & 0.3 & 0.4 & 0.7 & 0.2 \\ 0.6 & 0.4 & 0.5 & 0.4 & 0.7 & 0.6 \\ 0.7 & 0.5 & 0.6 & 0.3 & 0.2 & 0.3 \end{pmatrix},$$

while the vectors $\underline{c}$ and $\overline{c}$ are given as

$$\underline{c} = (0.42, 0.52, 0.25, 0.36, 0.28, 0.36),$$

$$\overline{c} = (0.53, 0.82, 0.71, 0.58, 0.52, 0.58).$$

The given solution is $y = (0.65, 0.85, 0.73, 0.74, 0.65, 0.80)$.

Solution: According to (2.2) and (2.3), we calculate the maximum solution as

$$\hat{x} = (0.757, 0.911, 0.888, 0.829, 0.757, 0.829).$$

After simple calculations, we verify that $\underline{c} \leq A \odot \hat{x} \leq \overline{c}$. Therefore, by Theorem 2.3, the system of inequalities (5.1) is consistent. Next, using the method described in the previous section, we compute the positive sensitivity of the given solution y with respect to A.

Computation of d_1^+: For $i = 1$, we calculate:

$$\min_{j\in N}\{1 - a_{1j}\} = (1-0.5) \wedge (1-0.1) \wedge (1-0.4) \wedge (1-0.3) \wedge (1-0.7) \wedge (1-0.4) = 0.3.$$

Similarly, for $i = 2, 3, 4, 5, 6$, we get $\min_{j\in N}\{1 - a_{2j}\} = 0.1$, $\min_{j\in N}\{1 - a_{3j}\} = 0.2$, $\min_{j\in N}\{1 - a_{4j}\} = 0.3$, $\min_{j\in N}\{1 - a_{5j}\} = 0.3$, $\min_{j\in N}\{1 - a_{6j}\} = 0.3$. As a result,

$$d_1^+ = \min_{i\in M, j\in N}\{1 - a_{ij}\} = \min\{0.3, 0.1, 0.2, 0.3, 0.3, 0.3\} = 0.1. \tag{5.2}$$

Computation of d_2^+: For $i = 1$, we calculate:

$$\begin{aligned}\min_{j\in N}\left\{\frac{\overline{c}_1}{y_j} - a_{1j}\right\} &= \left(\frac{0.53}{0.65} - 0.5\right) \wedge \left(\frac{0.53}{0.85} - 0.1\right) \wedge \left(\frac{0.53}{0.73} - 0.4\right) \wedge \left(\frac{0.53}{0.74} - 0.3\right) \\ &\quad \wedge \left(\frac{0.53}{0.65} - 0.7\right) \wedge \left(\frac{0.53}{0.80} - 0.4\right) \\ &= 0.315 \wedge 0.524 \wedge 0.326 \wedge 0.416 \wedge 0.116 \wedge 0.263 \\ &= 0.116.\end{aligned}$$

In the similar way, we find $\min_{j\in N}\{\frac{\overline{c}_2}{y_j} - a_{2j}\} = 0.362$, $\min_{j\in N}\{\frac{\overline{c}_3}{y_j} - a_{3j}\} = 0.292$, $\min_{j\in N}\{\frac{\overline{c}_4}{y_j} - a_{4j}\} = 0.192$, $\min_{j\in N}\{\frac{\overline{c}_5}{y_j} - a_{5j}\} = 0.063$, $\min_{j\in N}\{\frac{\overline{c}_6}{y_j} - a_{6j}\} = 0.182$. As a result,

$$\begin{aligned} d_2^+ &= \min_{i\in M, j\in N}\left\{\frac{\overline{c}_i}{y_j} - a_{ij}\right\} \\ &= \min\{0.116, 0.362, 0.292, 0.192, 0.063, 0.182\} = 0.063. \end{aligned} \tag{5.3}$$

Therefore, from formulas (4.3), (5.2), and (5.3), we conclude that the positive sensitivity of the solution $y = (0.65, 0.85, 0.73, 0.74, 0.65, 0.80)$ is

$$d^+ = \min(d_1^+, d_2^+) = \min(0.1, 0.063) = 0.063.$$

Example 5.2. We continue with the same system of inequalities (5.1) and the given solution y. For the given solution y, we further compute the negative sensitivity and overall sensitivity using the method from Section 4.

Solution: For $i = 1$, since

$$a_{11}y_1 = 0.325 < 0.42 = c_1, \quad a_{12}y_2 = 0.085 < 0.42 = c_1, \quad a_{13}y_3 = 0.292 < 0.42 = c_1,$$

$$a_{14}y_4 = 0.222 < 0.42 = c_1, \quad a_{15}y_5 = 0.455 > 0.42 = c_1, \quad a_{16}y_6 = 0.32 < 0.42 = c_1.$$

Thus, from formula (4.23), we have $J_1^- = \{5\}$. In the similar way we find

$$J_2^- = \{1\}, \quad J_3^- = \{1, 5\}, \quad J_4^- = \{5\}, \quad J_5^- = \{1, 2, 3, 4, 5, 6\}, \quad J_6^- = \{1, 2, 3\}$$

Computation of d_1^-: From formula (4.21), we calculate:

$$d_1^- = \min_{i\in M, j\in N}\{a_{ij}\} = 0.1 \tag{5.4}$$

Computation of d_2^-: For $i = 1$, we have

$$\max_{j\in J_1^-}\left\{a_{1j} - \frac{c_1}{y_j}\right\} = 0.7 - \frac{0.42}{0.65} = 0.7 - 0.646 = 0.054.$$

For $i = 2$, we have

$$\max_{j\in J_2^-}\left\{a_{2j} - \frac{c_2}{y_j}\right\} = 0.9 - \frac{0.52}{0.65} = 0.9 - 0.8 = 0.1.$$

For $i = 3$, we have

$$\max_{j\in J_3^-}\left\{a_{3j} - \frac{c_3}{y_j}\right\} = \left(0.4 - \frac{0.25}{0.65}\right) \vee \left(0.8 - \frac{0.25}{0.65}\right) = 0.015 \vee 0.415 = 0.415.$$

For $i = 4$, we have

$$\max_{j\in J_4^-}\left\{a_{4j} - \frac{c_4}{y_j}\right\} = 0.7 - \frac{0.36}{0.65} = 0.7 - 0.554 = 0.146.$$

For $i = 5$, we have

$$\begin{aligned} \max_{j\in J_5^-}\left\{a_{5j} - \frac{c_5}{y_j}\right\} &= \left(0.6 - \frac{0.28}{0.65}\right) \vee \left(0.4 - \frac{0.28}{0.85}\right) \vee \left(0.5 - \frac{0.28}{0.73}\right) \\ &\quad \vee \left(0.4 - \frac{0.28}{0.74}\right) \vee \left(0.7 - \frac{0.28}{0.65}\right) \vee \left(0.6 - \frac{0.28}{0.80}\right) \\ &= 0.269. \end{aligned}$$

Table 1. Operations costed in each step in Algorithm I

Step i	Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 7	Step 8
operation	mn	$2mn$	1	mn	$2mn$	$2mn$	1	1

For $i = 6$, we have

$$\max_{j\in J_6^-}\left\{a_{6j} - \frac{c_6}{y_j}\right\} = \left(0.7 - \frac{0.36}{0.65}\right) \vee \left(0.5 - \frac{0.36}{0.85}\right) \vee \left(0.6 - \frac{0.36}{0.73}\right) = 0.146.$$

As result,

$$\begin{aligned} d_2^- &= \min_{i\in M}\max_{j\in J_i^-}\left\{a_{ij} - \frac{c_i}{y_j}\right\} \\ &= \min\{0.054, 0.1, 0.415, 0.146, 0.269, 0.146\} \\ &= 0.054. \end{aligned} \tag{5.5}$$

Therefore, from formulas (4.24), (5.4), and (5.5), we have $d^- = \min(d_1^-, d_2^-) = 0.054$. Thus, the negative sensitivity of the solution $y = (0.65, 0.85, 0.73, 0.74, 0.65, 0.80)$ is 0.054.

Note that it has been calculated in Example 5.1 that $d^+ = 0.063$. Hence the sensitivity of y is

$$d(y) = \min(d^+, d^-) = \min(0.063, 0.054) = 0.054.$$

6. Results and discussion

- Results for our defined sensitivity

From the previous study on the sensitivity of a given solution, the following results can be directly derived.

(i) When system (1.4) is consistent with a given solution y, then the sensitivity of y always exists. Moreover, the sensitivity should be unique, when it exists.

(ii) The sensitivity of a given solution could be obtained by some resolution formulae. Moreover, the resolution formulae can be carried out by a detailed algorithm.

- Further discussion on the resolution algorithm for the sensitivity

In Section 4, we have developed Algorithm I for calculating the sensitivity of a given solution. It can be seen in Examples 5.1 and 5.2 that our proposed Algorithm I is feasible. Using Algorithm I, one is able to compute the sensitivity.

Moreover, we further find that Algorithm I is efficient, considering its computation complexity. In fact, the operations required in each step in Algorithm I can be directly computed, as shown in Table 1. As a result, all these 8 steps will cost

$$mn + 2mn + 1 + mn + 2mn + 2mn + 1 + 1 = 8mn + 3$$

operations, in the worst case. Hence the computation complexity of Algorithm I is $\mathcal{O}(mn)$. That is to say, Algorithm I has a polynomial computation complexity.

7. Conclusion

The random external interference will cause the perturbations to the educational information resources sharing system. In the existing works, such a system has been described by the max-product FRIs system (1.4). A feasible scheme is indeed a solution to (1.4). The perturbations caused by the random external interference lead directly to the variation in the parameters $\{a_{ij} \mid i \in M, j \in N\}$. To embody such parameter variations, we define and study three types of sensitivities, for a given solution of system (1.4), in this work. Our main contribution is to design some effective resolution approaches for these sensitivities. Algorithm I is designed, step by step, for computing the sensitivity of a given solution. Our proposed Algorithm I is verified to be feasible and efficient, through some numerical examples and its computation complexity. Its computation complexity is polynomial.

In the future, the concept of sensitivity will be extended to more types of fuzzy relation systems.

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