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SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS USING POWER SERIES METHOD AND SUMUDU TRANSFORM

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Abstract. An analytic function can be approximated by its corresponding power series. The advantage of the Sumudu transform is to turn differential equations into algebraic equations. Combining the power series method with the Sumudu transform, the approximation solutions of the fractional differential equations are studied. The efficient and accuracy results between approximated solutions and exact solutions are comfirmed with perfect consistency, and it can be affirmed that the methods are easy to implement with the smaller analytical error by the several numbercial examles.

1. Introduction

Throughout the several decades, many scholars have been investigating and debating the linear and nonlinear fractional differential equations in assorted fields, which engender in the physical sciences as well as in engineering, these kinds of equations play influential role to evolve mathematical tools to realize fractional modeling.

Several numerical, analytical methods have been proposed and are widely used for solving these partial equations. Admit for its performance in solving linear order, nonlinear partial differential equations, the interesting convert it was evidence in [9,10,16] .The homotopy analysis transform method was one of the more technicalities utilize in the solutions for the nonlinear factor [14] . By placing the solution in a rapid approximation series, HPM paired with the Sumudu transform tool improves the answer in a closed shape [19]. The theoretical formulation of initial value issues for fractional differential equations may be done in two methods [6]. In literature [4], the fractional-order power series technique for solving the nonlinear fractional-order partial differential equation was found to be relatively simple in implementation with an application of the direct power series method. In [7] used the adaptive single piecewise interpolation reproducing kernel method to solve the fractional partial differential equation. This improved method not only improves the calculation accuracy but also reduces the waste of time. In literature [12], the

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non-conformable double Laplace transform are introduced, studied and applied to solve some fractional partial differential equations involving the non-conformable fractional derivative. The study showed that this transform is effective and easy to apply to create an exact solution for types of fractional partial differential equations. In [17] combined the Elzaki transform method with the new homotopy perturbation method for the first time and solve initial value problems numerically and analytically, such as nonlinear fractional differential equations of various normal orders. They found that the initial conditions have a big impact on the equations result. They given three beginning value issues that were solved as precise or approximation solutions with high rigor to demonstrate the methods power and correctness. It was clear that solving nonlinear partial differential equations with the crossbred approach was the best alternative. In literature [20], the residual power series method was given for solving the approximate analytical solution of the fractional Rosenau-Haynam equations. The approximate solution of the equations could be obtained by using the $(n-1)\alpha$ times derivative of the residual function as 0. The results showed that the residual power series method was a more effective method for solving the fractional Rosenau-Haynam equation. In literature [5], according to variational theory, the Lagrange multiplier was calculated and the variational iteration method scheme was constructed to studied fractional predator-prey model.

Relatively recently, the Sumudu transform have been developed. In literature [18], the Sumudu transform method was used to solve the equations nonlinear portion. Some basic properties and theorems which help us to solve the governing problem using the suggested approach were revised. The benefit of this approach was that it solves the equations directly and reliably, without the prerequisite for perturbations or linearization or extensive computer labor. In [13] dealt with the series approximation of 2D and 3D convection-diffusion by Sumudu homotopy perturbation method and Elzaki homotopy perturbation method. The accuracy of the proposed schemes was confirmed with the aid of a graphical match between approximated results and exact results. The solution of a time-fractional vibration equation was obtained for the large membranes using powerful homotopy perturbation technique via Sumudu transform in reference [11]. In [3] offered straightforward computational advantages for approximate range-limited numerical solutions of certain ordinary, mixed, and partial linear differential and integro-differential equations. In [8] developed a method to obtain approximate solution of nonlinear system of partial differential equations with the help of Sumudu decomposition method.

The advantage of the Sumudu transform is to turn differential equations into algebraic equations. Inspired by these literatures, combining the power series method with the Sumudu transform, we will study the approximate solutions of the fractional differential equations. The advantages of this method is perfect consistency of comnining power series and Sumudu transform for obtainity exact approximate results. The several examplees show that its analytical errord are smaller.

2. Preliminaries

Definition 2.1 ([2])**.** The Caputo non-integer derivative operator of order μ with respect to t is defined as following

$$D^{\mu}h(x) = (\Gamma(n-\mu))^{-1}\int_0^x (x-t)^{-\mu+n-1}h^{(n)}(t)dt, \tag{2.1}$$

where $\mu > 0, x > 0, n \geq \mu > n-1, n \in N$.

The Caputo non-integer derivative operator is a linear operation:

$$D^{\mu}(\alpha f(x) + \beta g(x)) = \alpha D^{\mu} f(x) + \beta D^{\mu} f(x), \tag{2.2}$$

where α and β are constants. We have $D^{\mu}k = 0$ for the Caputo derivative, if k is constant,

$$D^{\mu}x^m = \begin{cases} \frac{\Gamma(m+1)}{\Gamma(m-\mu+1)}x^{m-\mu}, m \in N_0, m \geq \mu, \\ \\ \quad 0, \qquad m \in N_0, m < \mu, \end{cases}$$

where $N_0 = \{1, 2, 3, \dots\}$.

Definition 2.2 ([2])**.** For the variable x and coefficients $a_n, n = 1, 2, 3, \dots, \infty$, if $x \geq x_0$, the fractional power series about the point x_0 is defined as

$$\sum_{n=0}^{\infty} a_n(x-x_0)^{n\mu} = a_0 + a_1(x-x_0)^{\mu} + a_2(x-x_0)^{2\mu} + a_3(x-x_0)^{3\mu} + \cdots , \tag{2.3}$$

where $\mu > 0, x > 0, m \geq \mu > m-1, n \in N^{+}$.

Theorem 2.3 ([2])**.** *Let the radius of convergence for the function with fractional power series representation*

$$h(x) = \sum_{n=0}^{\infty} a_n x^{n\mu}, 0 \leq x \in R,$$

be greater than zero. Then, for $m \geq \mu > m-1, n \in N^{+}$, the following expression holds true:

$$D^{\mu}h(x) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\mu+1)}{\Gamma((n-1)\mu+1)} x^{(n-1)\mu}.$$

Note 1. If $h(x) = \sum_{n=0}^{\infty} a_n x^n,\ n \in N^{+}$. then $D^{\mu}h(x) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n+1-\mu)} x^{n-\mu}$.

Definition 2.4 ([13, 18])**.** The Sumudu transform is defined as follows:

$$S[f(t)] = G(p) = \frac{1}{p}\int_0^{\infty} e^{-\frac{t}{p}} f(t)dt, p \in (-\eta_1, \eta_2), \tag{2.4}$$

over the provided set of functions, $A = \{f(t)|\ there\ \ existes\ \ M, \eta_1, \eta_2 > 0, f(t) < Me^{\frac{t}{\eta_j}}, if\ \ t \in (-1)^j \times [0, \infty), j = 1, 2\ \}$.

Theorem 2.5 ([18]). *Let $f(t)$ and $g(t)$ be any two functions whose Sumudu transforms exist. Then, for arbitrary constant a and b, we have*

$$S[af(t)+bg(t)] = aS[f(t)] + bS[g(t)]. \tag{2.5}$$

Theorem 2.6 ([13,18]). *The function $f(t)=1,\ t^{\alpha}$, respectively, the Sumudu transform of $f(t)$ is $S[1]=1, S[t^{\alpha}]=\Gamma(\alpha+1)p^{\alpha}$.*

3. Illustrative example

Example 3.1. Consider the time fractional-order logistic model [5]:

$$D_t^{\alpha}u(t) = 0.2u(t) - 0.1u^2(t) + f(t), \tag{3.1}$$

where $\alpha = 1.5, f(t) = \Gamma(2.5) - 0.2t^{\alpha} + 0.1t^{2\alpha}$, with initial conditions $u(0)=0$. Exact solutions of the problem is $u(t) = t^{1.5}$.

Let us suppose the approximate solution for the given problem is

$$u(t) = \sum_{m=0}^{\infty} a_m t^{m\alpha}, 0 \le m \in Z. \tag{3.2}$$

Then we have

$$D_t^{\alpha}u(t) = \sum_{m=1}^{\infty} a_m \frac{\Gamma(m\alpha+1)}{\Gamma((m-1)\alpha+1)} t^{(m-1)\alpha}, \tag{3.3}$$

$$\begin{aligned} \sum_{m=1}^{\infty} a_m \frac{\Gamma(m\alpha+1)}{\Gamma((m-1)\alpha+1)} t^{(m-1)\alpha} &= 0.2\sum_{m=0}^{\infty} a_m t^{m\alpha} - 0.1\sum_{m=0}^{\infty} a_m t^{m\alpha} \sum_{m=0}^{\infty} a_m t^{m\alpha} \\ &\quad + \Gamma(2.5) - 0.2t^{\alpha} + 0.1t^{2\alpha} \\ &= 0.2\sum_{m=0}^{\infty} a_m t^{m\alpha} - 0.1\sum_{m=0}^{\infty}\sum_{j=0}^{m} a_j a_{m-j} t^{m\alpha} \\ &\quad + \Gamma(2.5) - 0.2t^{\alpha} + 0.1t^{2\alpha}. \end{aligned} \tag{3.4}$$

Using Sumudu transform on both sides of the above equations on t, we have

$$S[D_t^{\alpha}u(t)] = \sum_{m=1}^{\infty} a_m \Gamma(m\alpha+1) p^{(m-1)\alpha}, \tag{3.5}$$

$$\begin{aligned} &S\Big[0.2\sum_{m=0}^{\infty} a_m t^{m\alpha} - 0.1\sum_{m=0}^{\infty}\sum_{j=0}^{m} a_j a_{m-j} t^{m\alpha} + \Gamma(2.5) - 0.2t^{\alpha} + 0.1t^{2\alpha}\Big] \\ &= 0.2\sum_{m=0}^{\infty} a_m \Gamma(m\alpha+1) p^{m\alpha} - 0.1\sum_{m=0}^{\infty}\sum_{j=0}^{m} a_j a_{m-j} \Gamma(m\alpha+1) p^{m\alpha} \\ &\quad + \Gamma(2.5) - 0.2\Gamma(\alpha+1)p^{\alpha} + 0.1\Gamma(2\alpha+1)p^{2\alpha}. \end{aligned} \tag{3.6}$$

According to the above equation (3.1), we obtain

$$\begin{aligned}\sum_{m=1}^{\infty} a_m\Gamma(m\alpha+1)p^{(m-1)\alpha} = 0.2\sum_{m=0}^{\infty} a_m\Gamma(m\alpha+1)p^{m\alpha} \\ - 0.1\sum_{m=0}^{\infty}\sum_{j=0}^{m} a_j a_{m-j}\Gamma(m\alpha+1)p^{m\alpha} \\ + \Gamma(2.5) - 0.2\Gamma(\alpha+1)p^{\alpha} + 0.1\Gamma(2\alpha+1)p^{2\alpha}.\end{aligned} \tag{3.7}$$

Take $a_0 = u(0) = 0$. Comparing the coefficients at both sides, we get the following result according to equations (3.7):

$$a_1\Gamma(2\alpha+1) = 0.2a_0 - 0.1a_0a_0 + \Gamma(2.5), \tag{3.8}$$

$$a_2\Gamma(2\alpha+1) = 0.2a_1\Gamma(\alpha+1) - 0.1\cdot 2a_0a_1\Gamma(\alpha+1) - 0.2\Gamma(\alpha+1), \tag{3.9}$$

$$\begin{aligned}a_3\Gamma(3\alpha+1) = 0.2a_2\Gamma(2\alpha+1) - 0.1\cdot(2a_0a_2 + a_1a_1)\Gamma(2\alpha+1) \\ + 0.1\Gamma(2\alpha+1),\end{aligned} \tag{3.10}$$

$$\begin{aligned}a_m\Gamma(m\alpha+1) = 0.2a_{m-1}\Gamma((m-1)\alpha+1) \\ - 0.1\cdot\sum_{j=0}^{m-1} a_j a_{m-1-j}\Gamma((m-1)\alpha+1), \\ m = 4,5,6,7,\ldots\end{aligned} \tag{3.11}$$

So the following results are obtained

$$a_1 = \frac{\Gamma(2.5)}{\Gamma(\alpha+1)} = 1, a_2 = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}(0.2a_1 - 0.2) = 0,$$

$$a_3 = 0.1\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}(1-1) = 0,$$

$$a_m = \frac{\Gamma((m-1)\alpha+1)}{\Gamma(m\alpha+1)}\Big(0.2a_{m-1} - 0.1\cdot\sum_{j=0}^{m-1} a_j a_{m-1-j}\Big) = 0,$$

$$m = 4,5,6,7,\ldots.$$

We get the first four items of the function $u(t)$ as following:

$$u_1 = 0 + t^{1.5} + 0 + 0 = t^{1.5}.$$

Comparison plots of exact solutions, approximated solutions is shown in Figure 1.

Note 2. Taking $t = 0, 0.2, 0.4, 0.6, 0.8, 1$, respectively, the results in this paper compare with those in the literature [5] as following in Table 1 and in Table 2:

Example 3.2. Consider the succeeding nonlinear order structure [2]:

$$\begin{cases} D^{\mu}u(x,t) = 1 + v(x,t)\frac{du(x,t)}{dx} + u(x,t), \\ D^{\mu}v(x,t) = 1 - u(x,t)\frac{dv(x,t)}{dx} - v(x,t), \end{cases} \tag{3.12}$$

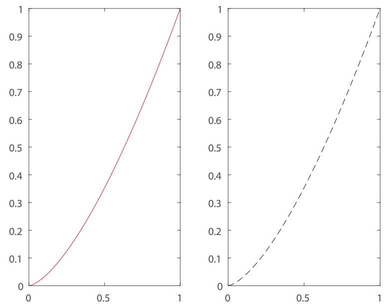

FIGURE 1. Comparison plots of exact solutions, approximated solutions.

TABLE 1. Numerical values for $u(t)$ by [5] for different values of $t = 0, 0.2, 0.4, 0.6, 0.8, 1$.

t	$\|u(t) - u(0)\|$	$\|u(t) - u_1(t)\|$
0.0	0.0000×10^0	0.0000×10^0
0.2	8.9443×10^{-2}	3.4629×10^{-4}
0.4	2.5298×10^{-1}	2.6503×10^{-3}
0.6	4.6476×10^{-1}	8.4205×10^{-3}
0.8	7.1554×10^{-1}	1.8488×10^{-2}
1	1.0000×10^0	3.2848×10^{-2}

TABLE 2. Numerical values for $u(t)$ in this paper for different values of $t = 0, 0.2, 0.4, 0.6, 0.8, 1$.

t	$\|u(t) - u(0)\|$	$\|u(t) - u_1(t)\|$
0.0	0.0000×10^0	0.0000×10^0
0.2	8.9443×10^{-2}	0.0000×10^0
0.4	2.5298×10^{-1}	0.0000×10^0
0.6	4.6476×10^{-1}	0.0000×10^0
0.8	7.1554×10^{-1}	0.0000×10^0
1	1.0000×10^0	0.0000×10^0

with initial conditions $u(x, 0) = e^{-x}, v(x, 0) = e^x$. Exact solutions of the problem is $u(x, t) = e^{t-x}, v(x, t) = e^{x-t}$ when $\mu = 1$.

Let us suppose the approximate solution for the given problem is

$$u(x,t)=\sum_{m=0}^{\infty}a_m(x)t^{m\mu},v(x,t)=\sum_{m=0}^{\infty}b_m(x)t^{m\mu}. \tag{3.13}$$

Then we have

$$D_t^{\mu}u(x,t)=\sum_{m=1}^{\infty}a_m(x)\frac{\Gamma(m\mu+1)}{\Gamma((m-1)\mu+1)}t^{(m-1)\mu}, \tag{3.14}$$

$$D_t^{\mu}v(x,t)=\sum_{m=1}^{\infty}b_m(x)\frac{\Gamma(m\mu+1)}{\Gamma((m-1)\mu+1)}t^{(m-1)\mu}, \tag{3.15}$$

$$\begin{aligned}1+v(x,t)\frac{du(x,t)}{dx}+u(x,t)&=1+\sum_{m=0}^{\infty}b_m(x)t^{m\mu}\sum_{m=0}^{\infty}a'_m(x)t^{m\mu}\\&\quad+\sum_{m=0}^{\infty}a_m(x)t^{m\mu}\\&=1+\sum_{m=0}^{\infty}\sum_{j=0}^{m}b_j(x)a'_{m-j}(x)t^{m\mu}\\&\quad+\sum_{m=0}^{\infty}a_m(x)t^{m\mu},\end{aligned} \tag{3.16}$$

$$\begin{aligned}1-u(x,t)\frac{dv(x,t)}{dx}-v(x,t)&=1-\sum_{m=0}^{\infty}a_m(x)t^{m\mu}\sum_{m=0}^{\infty}b'_m(x)t^{m\mu}\\&\quad-\sum_{m=0}^{\infty}b_m(x)t^{m\mu}\\&=1-\sum_{m=0}^{\infty}\sum_{j=0}^{m}a_j(x)b'_{m-j}(x)t^{m\mu}\\&\quad-\sum_{m=0}^{\infty}b_m(x)t^{m\mu}.\end{aligned} \tag{3.17}$$

Using Sumudu transform on both sides of the above equations on t, we have

$$S[D_t^{\mu}u(x,t)]=\sum_{m=1}^{\infty}a_m(x)\Gamma(m\mu+1)p^{(m-1)\mu}, \tag{3.18}$$

$$D_t^{\mu}v(x,t)=\sum_{m=1}^{\infty}b_m(x)\Gamma(m\mu+1)p^{(m-1)\mu}, \tag{3.19}$$

$$S\Big[1+v(x,t)\frac{du(x,t)}{dx}+u(x,t)\Big]=1+\sum_{m=0}^{\infty}\sum_{j=0}^{m}b_j(x)a'_{m-j}(x)\Gamma(m\mu+1)p^{m\mu}+\sum_{m=0}^{\infty}a_m(x)\Gamma(m\mu+1)p^{m\mu}, \tag{3.20}$$

$$S\Big[1-u(x,t)\frac{dv(x,t)}{dx}-v(x,t)\Big]=1-\sum_{m=0}^{\infty}\sum_{j=0}^{m}a_j(x)b'_{m-j}(x)\Gamma(m\mu+1)p^{m\mu}-\sum_{m=0}^{\infty}b_m(x)\Gamma(m\mu+1)p^{m\mu}. \tag{3.21}$$

Take $a_0(x)=u(x,0)=e^{-x}, b_0(x)=v(x,0)=e^{x}$. Comparing the coefficients at both sides, we get the following result according to equations (3.12):

$$a_1(x)\Gamma(\mu+1)=1+b_0(x)a'_0(x)+a_0(x)=1+e^{x}(-e^{-x})+e^{-x}=e^{-x}, \tag{3.22}$$

$$b_1(x)\Gamma(\mu+1)=1-a_0(x)b'_0(x)-a_0(x)=1-e^{-x}e^{x}-e^{x}=-e^{x}, \tag{3.23}$$

$$a_{m+1}(x)\Gamma((m+1)\mu+1)=\sum_{j=0}^{m}b_j(x)a'_{m-j}(x)\Gamma(m\mu+1)+a_m(x)\Gamma(m\mu+1), \tag{3.24}$$

$$b_{m+1}(x)\Gamma((m+1)\mu+1)=-\sum_{j=0}^{m}a_j(x)b'_{m-j}(x)\Gamma(m\mu+1)-b_m(x)\Gamma(m\mu+1), \quad m=1,2,3,4,\ldots \tag{3.25}$$

That is

$$a_1(x)=\frac{1}{\Gamma(\mu+1)}e^{-x}, b_1(x)=-\frac{1}{\Gamma(\mu+1)}e^{x}, \tag{3.26}$$

$$a_{m+1}(x)=\frac{\Gamma(m\mu+1)}{\Gamma((m+1)\mu+1)}\Big\{\sum_{j=0}^{m}b_j(x)a'_{m-j}(x)+a_m(x)\Big\}, \tag{3.27}$$

$$b_{m+1}(x)=-\frac{\Gamma(m\mu+1)}{\Gamma((m+1)\mu+1)}\Big\{\sum_{j=0}^{m}a_j(x)b'_{m-j}(x)+b_m(x)\Big\}, \quad m=1,2,3,4,\ldots. \tag{3.28}$$

We can obtain the following results according to the above recursive relationship:

$$a_0(x)=e^{-x} \tag{3.29}$$

$$b_0(x)=e^{x}, \tag{3.30}$$

$$a_1(x)=\frac{1}{\Gamma(\mu+1)}e^{-x} \tag{3.31}$$

$$b_1(x) = -\frac{1}{\Gamma(\mu+1)}e^{x}, \tag{3.32}$$

$$a_2(x) = \frac{1}{\Gamma(2\mu+1)}e^{-x} \tag{3.33}$$

$$b_2(x) = \frac{1}{\Gamma(2\mu+1)}e^{x}, \tag{3.34}$$

$$a_3(x) = \frac{1}{\Gamma(3\mu+1)}\Big[e^{-x} - \Big(2 - \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)\Gamma(\mu+1)}\Big)\Big], \tag{3.35}$$

$$b_3(x) = \frac{1}{\Gamma(3\mu+1)}\Big[-e^{x} - \Big(2 - \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)\Gamma(\mu+1)}\Big)\Big], \tag{3.36}$$

$$a_4(x) = \frac{1}{\Gamma(4\mu+1)}\Big[\Big(3 - \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)\Gamma(\mu+1)}\Big)e^{-x} - \Big(2 - \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)\Gamma(\mu+1)}\Big)\Big], \tag{3.37}$$

$$b_4(x) = \frac{1}{\Gamma(4\mu+1)}\Big[\Big(3 - \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)\Gamma(\mu+1)}\Big)e^{x} + \Big(2 - \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)\Gamma(\mu+1)}\Big)\Big], \tag{3.38}$$

$$\vdots$$

In a similar way, we can get the coefficients $a_m(x), b_m(x)$ of equations (3.13) and find, respectively:

$$\begin{aligned} u(x,t) = {} & e^{-x} + \frac{t^{\mu}}{\Gamma(\mu+1)}e^{-x} + \frac{t^{2\mu}}{\Gamma(2\mu+1)}e^{-x} \\ & + \frac{t^{3\mu}}{\Gamma(3\mu+1)}\Big[e^{-x} - \Big(2 - \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)\Gamma(\mu+1)}\Big)\Big] \\ & + \frac{t^{4\mu}}{\Gamma(4\mu+1)}\Big[\Big(3 - \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)\Gamma(\mu+1)}\Big)e^{-x} \\ & - \Big(2 - \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)\Gamma(\mu+1)}\Big)\Big] + \cdots \end{aligned} \tag{3.39}$$

$$\begin{aligned} v(x,t) = {} & e^{x} + \frac{(-1)t^{\mu}}{\Gamma(\mu+1)}e^{x} + \frac{(-1)^2 t^{2\mu}}{\Gamma(2\mu+1)}e^{x} \\ & + \frac{(-1)^3 t^{3\mu}}{\Gamma(3\mu+1)}\Big[e^{x} + \Big(2 - \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)\Gamma(\mu+1)}\Big)\Big] \\ & + \frac{(-1)^4 t^{4\mu}}{\Gamma(4\mu+1)}\Big[\Big(3 - \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)\Gamma(\mu+1)}\Big)e^{x} \\ & + \Big(2 - \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)\Gamma(\mu+1)}\Big)\Big] + \cdots . \end{aligned} \tag{3.40}$$

TABLE 3. Numerical values for $u(x,t)$ in [2] for different values of t and μ when $x = 0.2$.

t	$\mu = 1$	$\mu = 0.8$	$\mu = 0.6$	$\mu = 0.4$	$\mu = 0.2$
0.25	1.051264145	1.177800483	1.376647053	1.690057985	2.190896007
0.5	1.349626476	1.552782962	1.825654773	2.178405687	2.598177788
0.75	1.731407662	1.995745862	2.305360890	2.636753607	2.913559185
1	2.217395789	2.524361162	2.833631770	3.091314344	3.183670371

If $\mu = 1$, the equations (3.39) and (3.40) will be turned into:

$$\begin{aligned} u(x,t) = {} & e^{-x} + \frac{t}{\Gamma(1+1)}e^{-x} + \frac{t^2}{\Gamma(2+1)}e^{-x} + \frac{t^3}{\Gamma(3+1)}e^{-x} \\ & + \frac{t^4}{\Gamma(4+1)}e^{-x} + \cdots = e^{t-x}, \end{aligned} \tag{3.41}$$

$$\begin{aligned} v(x,t) = {} & e^{x} + \frac{(-1)t}{\Gamma(1+1)}e^{x} + \frac{(-1)^2t^2}{\Gamma(2+1)}e^{x} + \frac{(-1)^3t^3}{\Gamma(3+1)}e^{x} \\ & + \frac{(-1)^4t^4}{\Gamma(4+1)}e^{x} + \cdots = e^{x-t}. \end{aligned} \tag{3.42}$$

When $\mu = 1$, $u(x,t)$ and $v(x,t)$ comparison with the exact solution, as shown in Figure 2 and Figure 3.

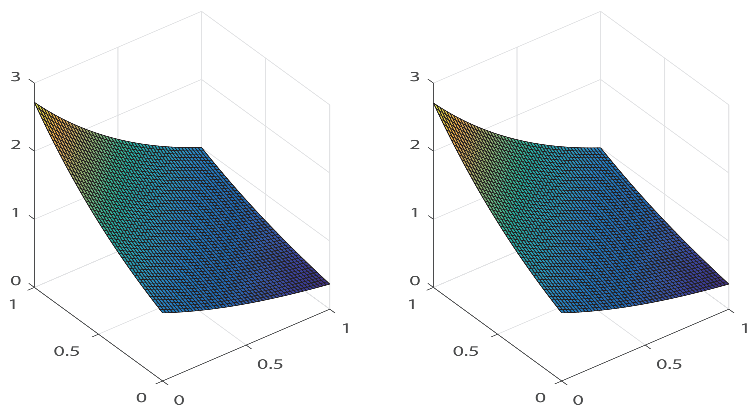

FIGURE 2. Comparison plots of exact solutions, approximated solutions for $\mu = 1$ of $u(x,t)$.

Note 3. Taking $x = 0.2$ as an example, the results in this paper compare with those in the literature [2] as following in Table 3, Table 4:

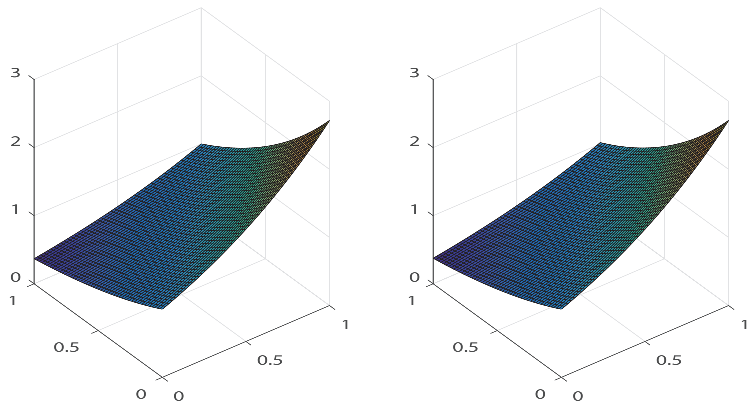

FIGURE 3. Comparison plots of exact solutions, approximated solutions for $\mu = 1$ of $v(x,t)$.

TABLE 4. Numerical values for $u(x,t)$ in [2] for different values of t and μ when $x = 0.2$.

t	$\mu = 1$	$\mu = 0.8$	$\mu = 0.6$	$\mu = 0.4$	$\mu = 0.2$
0.25	1.051264146	1.177800483	1.376647052	1.690057984	2.190896005
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0.75	1.731407662	1.995745861	2.305360889	2.636753609	2.913559185
1	2.217395790	2.524361162	2.833631770	3.091314345	3.183670371

We may get that the values $u(0.2, 0.25) = 1.051271096$ and $u(0.2, 1) = 2.225540928$ of the function $u(x,t) = e^{t-x}$, respectively. By comparison, the results in this paper are better, more effective and higher accurate than those in the literature [2] since the function $u(x,t) = e^{t-x}$ is increasing on t .

Example 3.3. Consider the following fractional Rosenau-Haynam differential equation [20] :

$$\begin{aligned} D_t^{\alpha} u(x,t) &= u(x,t)D_{xxx}u(x,t) + u(x,t)D_x u(x,t) \\ &+ 3D_x u(x,t)D_{xx}u(x,t), t > 0, \end{aligned} \tag{3.43}$$

where $0 < \alpha < 1$ is fractional-order constant, t is time, x is Spatial coordinate, $D_{xxx}u(x,t)$ denotes third derivative for $u(x,t)$ on x , with initial conditions $u(x,0) = -\frac{8}{3}c\cos^2\frac{x}{4} = -\frac{4}{3}c\cos\frac{x}{2} - \frac{4}{3}c$. The analytical solutions of the problem is given by $u(x,t) = -\frac{8}{3}c\cos^2(\frac{1}{4}(x-ct)) = -\frac{4}{3}c\cos(\frac{1}{2}(x-ct)) - \frac{4}{3}c$ when $\alpha = 1$.

Let us suppose the approximate solution for the given problem is

$$u(x,t) = \sum_{m=0}^{\infty} a_m(x)t^{m\alpha}. \tag{3.44}$$

Then we have

$$D_t^\alpha u(x,t) = \sum_{m=1}^{\infty} a_m(x)\frac{\Gamma(m\alpha+1)}{\Gamma((m-1)\alpha+1)}t^{(m-1)\alpha}, \tag{3.45}$$

$$D_x u(x,t) = \sum_{m=0}^{\infty} a'_m(x)t^{m\alpha}, \tag{3.46}$$

$$D_{xx} u(x,t) = \sum_{m=0}^{\infty} a''_m(x)t^{m\alpha}, \tag{3.47}$$

$$D_{xxx} u(x,t) = \sum_{m=0}^{\infty} a'''_m(x)t^{m\alpha}, \tag{3.48}$$

$$\begin{aligned}\sum_{m=1}^{\infty} a_m(x)\frac{\Gamma(m\alpha+1)}{\Gamma((m-1)\alpha+1)}t^{(m-1)\alpha} &= \sum_{m=0}^{\infty}\sum_{j=0}^{m} a_j(x)a'''_{m-j}(x)t^{m\alpha}\\ &+ \sum_{m=0}^{\infty}\sum_{j=0}^{m} a_j(x)a'_{m-j}(x)t^{m\alpha}\\ &+ 3\sum_{m=0}^{\infty}\sum_{j=0}^{m} a'_j(x)a''_{m-j}(x)t^{m\alpha}.\end{aligned} \tag{3.49}$$

Using Sumudu transform on both sides of the above equations on t, we have

$$\begin{aligned}\sum_{m=1}^{\infty} a_m(x)\Gamma(m\alpha+1)p^{(m-1)\alpha} &= \sum_{m=0}^{\infty}\sum_{j=0}^{m} a_j(x)a'''_{m-j}(x)\Gamma(m\alpha+1)p^{m\alpha}\\ &+ \sum_{m=0}^{\infty}\sum_{j=0}^{m} a_j(x)a'_{m-j}(x)\Gamma(m\alpha+1)p^{m\alpha}\\ &+ 3\sum_{m=0}^{\infty}\sum_{j=0}^{m} a'_j(x)a''_{m-j}(x)\Gamma(m\alpha+1)p^{m\alpha}.\end{aligned} \tag{3.50}$$

That is

$$\begin{aligned}a_{m+1}(x)\Gamma((m+1)\alpha+1) &= \sum_{j=0}^{m} a_j(x)a'''_{m-j}(x)\Gamma(m\alpha+1)\\ &+ \sum_{j=0}^{m} a_j(x)a'_{m-j}(x)\Gamma(m\alpha+1)\\ &+ 3\sum_{j=0}^{m} a'_j(x)a''_{m-j}(x)\Gamma(m\alpha+1).\end{aligned} \tag{3.51}$$

Take $a_0(x) = u(x,0) = -\frac{4}{3}c\cos\frac{x}{2} - \frac{4}{3}c$. Comparing the coefficients at both sides by the above recursive relationship, we get the same results with the literature [20]:

$$a_1(x) = \frac{1}{\Gamma(\alpha+1)}\Big(-\frac{2}{3}c^2\Big)\sin\frac{x}{2}, \tag{3.52}$$

$$a_2(x) = \frac{1}{\Gamma(2\alpha+1)}\Big(\frac{1}{3}c^3\Big)\cos\frac{x}{2}, \tag{3.53}$$

$$a_3(x) = \frac{1}{\Gamma(3\alpha+1)}\Big(\frac{1}{6}c^4\Big)\sin\frac{x}{2}, \ldots \tag{3.54}$$

So the first four terms of the approximate solution of the equation are

$$\begin{aligned} u(x,t) = {} & \Big(-\frac{4}{3}c\cos\frac{x}{2} - \frac{4}{3}c\Big) + t^{\alpha}\frac{1}{\Gamma(\alpha+1)}\Big(-\frac{2}{3}c^2\Big)\sin\frac{x}{2} \\ & + t^{2\alpha}\frac{1}{\Gamma(2\alpha+1)}\Big(\frac{1}{3}c^3\Big)\cos\frac{x}{2} + t^{3\alpha}\frac{1}{\Gamma(3\alpha+1)}\Big(\frac{1}{6}c^4\Big)\sin\frac{x}{2} + \cdots . \end{aligned} \tag{3.55}$$

Take $c = 1, \alpha = 1$, comparison chart of approximate and actual solutions as shown in Figure 4.

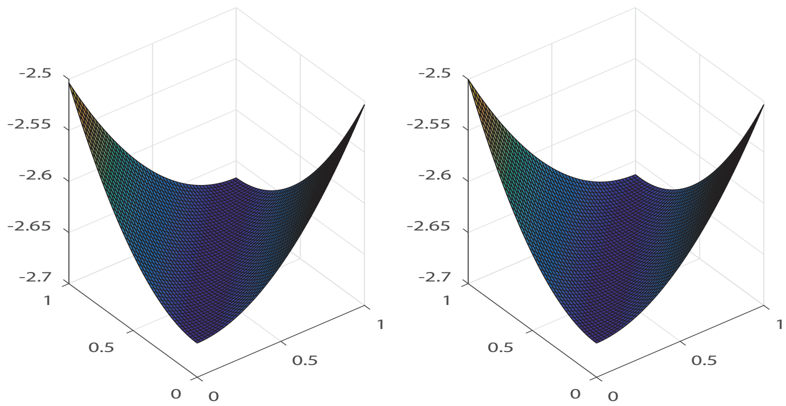

FIGURE 4. Comparison plots of exact solutions, approximated solutions for $c = 1$ and $\alpha = 1$.

Example 3.4. Consider the following one-dimensional Fractional Burger's equation [15]:

$$D_t^{\alpha}u(x,t) + D_x\Big(\frac{u^2}{2}\Big) = vD_{xx}u + g(x,t)\ ,(x,t) \in (x,t)\times(0,T), \tag{3.56}$$

with the initial condition $u(x,0)$, where $0 < \alpha < 0$ is Caputo fractional derivative, t and x are time and space parameters, respectively. $v > 0$is the viscosity constant, $g(x,t) = (\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \pi t^4\sin(\pi x) + v\pi^2t^2)\cos(\pi x)$. The exact solution is $u(x,t) = t^2\cos(\pi x)$.

Let us suppose the approximate solution for the problem is

$$u(x,t)\sum_{m=0}^{\infty} a_m(x)t^m. \tag{3.57}$$

Then we have

$$D_t^{\alpha} u(x,t) = \sum_{m=1}^{\infty} a_m(x)\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}t^{m-\alpha}, \tag{3.58}$$

$$D_x\Big(\frac{u^2}{2}\Big) = \frac{1}{2}\sum_{m=0}^{\infty}\sum_{j=0}^{m}(a_j'(x)a_{m-j}(x) + a_j(x)a_{m-j}'(x))t^m, \tag{3.59}$$

$$D_{xx}u = \sum_{m=0}^{\infty} a_m''(x)t^m. \tag{3.60}$$

$$\begin{aligned}\sum_{m=1}^{\infty} a_m(x)&\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}t^{m-\alpha}\\ &+\frac{1}{2}\sum_{m=0}^{\infty}\sum_{j=0}^{m}(a_j'(x)a_{m-j}(x) + a_j(x)a_{m-j}'(x))t^m\\ &= v\sum_{m=0}^{\infty} a_m''(x)t^m + g(x,t).\end{aligned} \tag{3.61}$$

Using Sumudu transform on both sides of the above equations on t, we obtain have

$$\begin{aligned}\sum_{m=1}^{\infty} &a_m(x)\Gamma(m+1)p^{m-\alpha}\\ &+\frac{1}{2}\sum_{m=0}^{\infty}\sum_{j=0}^{m}(a_j'(x)a_{m-j}(x) + a_j(x)a_{m-j}'(x))\Gamma(m+1)p^m\\ &= v\sum_{m=0}^{\infty} a_m''(x)\Gamma(m+1)p^m + (2p^{2-\alpha} - \pi\Gamma(4+1)p^4\sin(\pi x)\\ &\quad + v\pi^3\Gamma(3)p^2)\cos(\pi x).\end{aligned} \tag{3.62}$$

Take $a_0(x) = u(x,0) = 0$, comparing the coefficients at both sides by the above recursive relationship, we get the results:

$$a_0(x) = 0, a_1(x) = 0, a_2(x) = \cos(\pi x), a_m(x) = 0,\ m = 3,4,5,\dots. \tag{3.63}$$

So the approximate solution of this equation is

$$u(x,t) = t^2\cos(\pi x). \tag{3.64}$$

The solution is the same with the exact solution.

Example 3.5. Consider the following time-space-fractional nonlinear KdV-Burgers equation [1]:

$$D_t^{\alpha}u(x,t)+u(x,t)D_x^{\beta}u(x,t)+D_{xx}u(x,t)+D_{xxx}u(x,t)=0, \tag{3.65}$$

where $D_{xxx}u(x,t)$ denotes third derivative for $u(x,t)$ on x , $0<\alpha\leq 0,\ 0<\beta\leq 1, x,t>0$, subject to initial conditions $u(x,0)=x$.

Let us suppose the approximate solution for the given problem is

$$u(x,t)=\sum_{m=0}^{\infty}a_m(x)t^{m\alpha}. \tag{3.66}$$

Then we have

$$D_t^{\alpha}u(x,t)=\sum_{m=1}^{\infty}a_m(x)\frac{\Gamma(m\alpha+1)}{\Gamma((m-1)\alpha+1)}t^{(m-1)\alpha}, \tag{3.67}$$

$$D_x^{\beta}u(x,t)=\sum_{m=0}^{\infty}D_x^{\beta}a_m(x)t^{m\alpha}, \tag{3.68}$$

$$D_{xx}u(x,t)=\sum_{m=0}^{\infty}a''_m(x)t^{m\alpha}, \tag{3.69}$$

$$D_{xxx}u(x,t)=\sum_{m=0}^{\infty}a'''_m(x)t^{m\alpha}, \tag{3.70}$$

$$\begin{aligned}&\sum_{m=1}^{\infty}a_m(x)\frac{\Gamma(m\alpha+1)}{\Gamma((m-1)\alpha+1)}t^{(m-1)\alpha}+\sum_{m=0}^{\infty}\sum_{j=0}^{m}a_j(x)D_x^{\beta}a_{m-j}(x)t^{m\alpha}\\&\qquad+\sum_{m=0}^{\infty}a''_m(x)t^{m\alpha}+\sum_{m=0}^{\infty}a'''_m(x)t^{m\alpha}=0.\end{aligned} \tag{3.71}$$

Using Sumudu transform on both sides of the above equations on t, we have

$$\begin{aligned}&\sum_{m=1}^{\infty}a_m(x)\Gamma(m\alpha+1)p^{(m-1)\alpha}+\sum_{m=0}^{\infty}\sum_{j=0}^{m}a_j(x)D_x^{\beta}a_{m-j}(x)\Gamma(m\alpha+1)p^{m\alpha}\\&\qquad+\sum_{m=0}^{\infty}a''_m(x)\Gamma(m\alpha+1)p^{m\alpha}+\sum_{m=0}^{\infty}a'''_m(x)\Gamma(m\alpha+1)p^{m\alpha}=0.\end{aligned} \tag{3.72}$$

The recursive relationship is obtained as following

$$\begin{aligned}&a_{m+1}(x)\Gamma((m+1)\alpha+1)\\&\quad+\Gamma(m\alpha+1)\Big(\sum_{j=0}^{m}a_j(x)D_x^{\beta}a_{m-j}(x)+a''_m(x)+a'''_m(x)\Big)=0,\\&\quad m=0,1,2,3,\ldots.\end{aligned} \tag{3.73}$$

Take $a_0(x) = u(x,0) = x$. Comparing the coefficients at both sides by the above recursive relationship, we get the following results:

$$a_1(x) = -\frac{1}{\Gamma(\alpha+1)\Gamma(2-\beta)}x^{2-\beta}, \tag{3.74}$$

$$\begin{aligned} a_2(x) = \frac{1}{\Gamma(2\alpha+1)\Gamma(2-\beta)}\Big[\Big(\frac{\Gamma(3-\beta)}{\Gamma(3-2\beta)} + \frac{1}{\Gamma(2-\beta)}\Big)x^{3-2\beta} \\ + (2-\beta)(1-\beta)x^{-\beta} - (2-\beta)(1-\beta)\beta x^{-\beta-1}\Big], \end{aligned} \tag{3.75}$$

$$\begin{aligned} a_3(x) = & \frac{-1}{\Gamma(3\alpha+1)}\Big[\frac{1}{\Gamma(2-\beta)}\Big(\frac{\Gamma(3-\beta)}{\Gamma(3-2\beta)} + \frac{1}{\Gamma(2-\beta)}\Big)\Big(\frac{\Gamma(4-2\beta)}{\Gamma(4-3\beta)} + \frac{1}{\Gamma(2-\beta)}\Big) \\ & + \frac{1}{\Gamma(2-\beta)\Gamma(2-\beta)}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+1)}\frac{\Gamma(3-\beta)}{\Gamma(3-2\beta)}\Big]x^{4-3\beta} \\ & - \frac{1}{\Gamma(3\alpha+1)}\frac{1-\beta)}{\Gamma(2-\beta)}\Big[\frac{2-\beta}{\Gamma(2-\beta)} + 2(3-2\beta)\Big(\frac{\Gamma(3-\beta)}{\Gamma(3-2\beta)} + \frac{1}{\Gamma(2-\beta)}\Big)\Big]x^{1-2\beta} \\ & - \frac{1}{\Gamma(3\alpha+1)}\frac{1-\beta)}{\Gamma(2-\beta)}\Big[(3-2\beta)(2-2\beta)(1-2\beta)\Big(\frac{\Gamma(3-\beta)}{\Gamma(3-2\beta)} + \frac{1}{\Gamma(2-\beta)}\Big) \\ & - \frac{(2-\beta)(1-\beta)\beta}{\Gamma(2-\beta)}\Big]x^{-2\beta} \\ & - \frac{1}{\Gamma(3\alpha+1)}\frac{(2-\beta)(1-\beta)\beta(\beta+1)}{\Gamma(2-\beta)}x^{-\beta-2} \\ & + \frac{1}{\Gamma(3\alpha+1)}\frac{2(2-\beta)(1-\beta)\beta(\beta+1)(\beta+2)}{\Gamma(2-\beta)}x^{-\beta-3} \\ & - \frac{1}{\Gamma(3\alpha+1)}\frac{(2-\beta)(1-\beta)\beta(\beta+1)(\beta+2)(\beta+3)}{\Gamma(2-\beta)}x^{-\beta-4} \\ & \cdots\cdots\cdot \end{aligned} \tag{3.76}$$

So the first four terms of the approximate solution of the equation are

$$u(x,t) = a_0(x) + a_1(x)t^{\alpha} + a_2(x)t^{2\alpha} + a_3(x)t^{3\alpha} + \cdots. \tag{3.77}$$

The result is the same with [1], but the coefficients is easier to obtain.

4. Conclusion

In this research, in combination with the Sumudu transform and the power series, the problem of solving the approximate solutions of the fractional differential equations are turned into algebraic equations, and the coefficients of the power series are obtained by comparing the degree of the power series. Comparing the approximate solutions with the exact solutins, the results have been comfirmed with perfect consistency. At the same time,the method mentioned in this paper also provide some reference values for solving the approximate solutions of the related differential equations and the integral equations.

Acknowledgments

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