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FIXED POINT RESULTS FOR NONLINEAR RELATIONAL RATIONAL CONTRACTIONS ON DISLOCATED METRIC SPACES WITH APPLICATION

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Abstract. In this article, we use a binary relation to demonstrate the existence and uniqueness of fixed points in the context of dislocated metric space under the new generalized (ϕ, ψ)-rational contraction. Additionally, we present an example to demonstrate our recently validated findings. Lastly, we provide an application to fractional differential equation.

1. Introduction

The Banach contraction [7] theorem is indeed a fundamental result in mathematical analysis with broad applications across various disciplines. It states that a contraction mapping from a complete metric space to itself has a unique fixed point, and it has found extensive use in diverse areas such as functional analysis, optimization, differential equations, and dynamical systems. The richness and versatility of the Banach contraction theorem concept have led to a vast literature exploring on applications and extension, making it an essential tool in modern mathematics. The expansion of fixed-point theory into the realm of partially ordered metric spaces has indeed been a significant development in recent years, leading to fruitful applications in various areas of mathematics.

One early finding in this direction was made by Turinici [22] who explored fixed-point results in ordered metrizable uniform spaces, laying a foundation for further investigations in this field. Following this, Ran and Reurings [20] utilized fixed-point results in partially ordered metric spaces to solve matrix equations, showcasing the utility of this approach in linear algebra problems. Similarly, Nieto and Rodriguez-Lopez [19] utilized fixed-point results in partially ordered metric spaces to address partial differential equations with periodic boundary conditions, highlighting its relevance in mathematical physics.

In 2015, Alam et al. [2] made a significant contribution by establishing a generalization of the Banach contraction principle (BCP) using an amorphous binary relation instead of a partial order. This breakthrough opened up new avenues for research, leading to the proposal of various relation-theoretic results by several researchers [3, 15, 16]. Many subsequent works have focused on generalizing the BCP to encompass a broader range of contractive conditions, often based on one or more auxiliary functions. One notable version of this generalization was proposed by Dutta and Choudhury [10], which has since been further generalized and improved by numerous researchers [3], demonstrating the ongoing evolution and refinement of

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these concepts. Additionally, Chandok et al. [9] established results concerning the existence and uniqueness of fixed points for a certain rational type of contraction endowed with a partial order, contributing to the theoretical framework of fixed-point theory in partially ordered metric spaces.

Subsequently, Kumar et al. [17] extended the results of Chandok et al. [9] within the context of complete partial metric spaces, further expanding the applicability of these findings [3]. Hitzler [12] presented a noteworthy extension of the BCP by introducing dislocated metric spaces. It is worth noting that dislocated metric spaces (DMS) are sometimes also referred to as DMS (as observed by Amini-Harandi [5]). For further insights into DMS, interested readers can refer to works such as [13, 14].

In this article, we use a binary relation to demonstrate the existence and uniqueness of fixed points in the context of dislocated metric space under the new generalized (ϕ, ψ)-rational contraction. Additionally, we present an example to demonstrate our recently validated findings. Lastly, we provide an application to fractional differential equation.

Throughout this manuscript $\mathbb{N}_0$, $\mathbb{N}$, and $\mathbb{R}$, denote the set of whole numbers, natural numbers, and real numbers respectively.

Definition 1.1 ([11])**.** Let $\check{G}$ be a non-empty set. Then a mapping $p : \check{G} \times \check{G} \to \mathbb{R}^+$ is said to be a partial metric on $\check{G}$ if for all $\check{\mathfrak{S}}, \varkappa, \mathfrak{G} \in \check{G}$,

(i) $\check{\mathfrak{S}} = \varkappa \iff p(\check{\mathfrak{S}}, \check{\mathfrak{S}}) = p(\check{\mathfrak{S}}, \varkappa) = p(\varkappa, \varkappa)$,
(ii) $p(\check{\mathfrak{S}}, \check{\mathfrak{S}}) \leq p(\check{\mathfrak{S}}, \varkappa)$,
(iii) $p(\check{\mathfrak{S}}, \varkappa) = p(\varkappa, \check{\mathfrak{S}})$,
(iv) $p(\check{\mathfrak{S}}, \varkappa) \leq p(\check{\mathfrak{S}}, \mathfrak{G}) + p(\mathfrak{G}, \varkappa) - p(\mathfrak{G}, \mathfrak{G})$.

The pair $(\check{G}, p)$ is called a partial metric space.

Definition 1.2 ([12])**.** Let $\check{G}$ be a non-empty set. Then a mapping $\eth : \check{G} \times \check{G} \to \mathbb{R}^+$ is said to be dislocated on $\check{G}$ if for all $\check{\mathfrak{S}}, \varkappa \in \check{G}$

(i) $\eth(\check{\mathfrak{S}}, \varkappa) = 0 \implies \check{\mathfrak{S}} = \varkappa$,
(ii) $\eth(\check{\mathfrak{S}}, \varkappa) = \eth(\varkappa, \check{\mathfrak{S}})$,
(iii) $\eth(\check{\mathfrak{S}}, \varkappa) \leq \eth(\check{\mathfrak{S}}, \mathfrak{G}) + \eth(\mathfrak{G}, \varkappa)$.

The pair $(\check{G}, \eth)$ is called a dislocated (or metric like) space. Here it can be pointed out that all the requirement of a metric are met out except $\eth(\check{\mathfrak{S}}, \check{\mathfrak{S}})$ may be positive for $\check{\mathfrak{S}} \in \check{G}$.

Remark 1.3 ([1])**.** Every metric is a partial metric and every partial metric is a dislocated but converse implication is not true in general.

Definition 1.4 ([5])**.** Let $\{\check{\mathfrak{S}}_n\}$ be a sequence in a DMS $(\check{G}, \eth)$. Then we say that

- $\{\check{\mathfrak{S}}_n\}$ converges to a point $\check{\mathfrak{S}}$ in $\check{G}$ if and only if $\lim_{n\to\infty} \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}) = \eth(\check{\mathfrak{S}}, \check{\mathfrak{S}})$,
- $\{\check{\mathfrak{S}}_n\}$ is Cauchy in $\check{G}$ if and only if $\lim_{n,m\to\infty} \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_m)$ (finitely) exists,
- the DMS $(\check{G}, \eth)$ is complete if every Cauchy sequence $\{\check{\mathfrak{S}}_n\}$ in $\check{G}$ converges to a point $\check{\mathfrak{S}}$ in $\check{G}$ with respect to topology $\tau_\eth$ generated by $\eth$ (denote as $\check{\mathfrak{S}}_n \xrightarrow{\tau_\eth} \check{\mathfrak{S}}$) such that $\lim_{n,m\to\infty} \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_m) = \eth(\check{\mathfrak{S}}, \check{\mathfrak{S}}) = \lim_{n\to\infty} \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}})$.

Definition 1.5 ([11, 18])**.** Let $\mathcal{H}$ be a binary relation on $\check{G}$. Then $\check{\mathfrak{S}}, \varkappa \in \check{G}$,

(i) The inverse relation $\mathcal{H}^{-1} = \{(\check{\mathfrak{S}}, \varkappa) \in \check{G}^2 : (\varkappa, \check{\mathfrak{S}}) \in \mathcal{H}\}$ and symmetric closure $\mathcal{H}^s := \mathcal{H} \cup \mathcal{H}^{-1}$.

(ii) $\check{\mathfrak{S}}$ and $\varkappa$ are $\mathcal{H}$-comparative if either $(\check{\mathfrak{S}}, \varkappa) \in \mathcal{H}$ or $(\varkappa, \check{\mathfrak{S}}) \in \mathcal{H}$. We denote it by $[\check{\mathfrak{S}}, \varkappa] \in \mathcal{H}$.

(iii) A sequence $\check{\mathfrak{S}}_n \subset \check{G}$ is called $\mathcal{H}$- preserving if $(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1}) \in \mathcal{H} \ \forall \ n \in N_0$.

Motivated by Alam and Imdad [2], Ahmadullah et al. [1] define relation-theoretic variants of completeness and continuity in DMS.

Definition 1.6 ([1])**.** Let $(\check{G}, \eth)$ be a DMS equipped with a binary relation $\mathcal{H}$. We say that $(\check{G}, \eth)$ is $\mathcal{H}$-complete if every $\mathcal{H}$-preserving Cauchy sequence $\{\check{\mathfrak{S}}_n\} \in \check{G}$, there is some $\check{\mathfrak{S}} \in \check{G}$ such that

$$\lim_{n\to\infty} \eth(\check{\mathfrak{S}}_n, \varkappa_n) = \eth(\check{\mathfrak{S}}, \check{\mathfrak{S}}) = \lim_{n\to\infty} (\check{\mathfrak{S}}_n, \check{\mathfrak{S}}).$$

Recall that the limit of a convergent sequence in DMS need not be unique.

Definition 1.7 ([1])**.** Let $(\check{G}, \eth)$ be a DMS equipped with a binary relation $\mathcal{H}$. Then a mapping $\mathfrak{I} : \check{G} \to \check{G}$ is said to be

- sequentially-continuous at $\check{\mathfrak{S}}$ if for any sequence $\{\check{\mathfrak{S}}_n\}$ with $\check{\mathfrak{S}}_n \xrightarrow{\tau_\eth} \check{\mathfrak{S}}$, we have $\mathfrak{I}(\check{\mathfrak{S}}_n) \xrightarrow{\tau_\eth} \mathfrak{I}(\check{\mathfrak{S}})$. As usual, $\mathfrak{I}$ is said to be a sequentially-continuous if it is a sequentially-continuous at each point of $\check{G}$.
- $\mathcal{H}$-sequentially-continuous at $\check{\mathfrak{S}}$ if for any $\mathcal{H}$-preserving sequence $\{\check{\mathfrak{S}}_n\}$ with $\check{\mathfrak{S}}_n \xrightarrow{\tau_\eth} \check{\mathfrak{S}}$, we have $\mathfrak{I}(\check{\mathfrak{S}}_n) \xrightarrow{\tau_\eth} \mathfrak{I}(\check{\mathfrak{S}})$. As usual, $\mathfrak{I}$ is said to be a $\mathcal{H}$-sequentially-continuous if it is an $\mathcal{H}$ - sequentially - continuous at each point of $\check{G}$.

Definition 1.8 ([2])**.** Let $(\check{G}, \eth)$ be a DMS. A binary relation $\mathcal{H}$ defined on $\check{G}$ is called $\eth$"-self-closed if whenever $\{\check{\mathfrak{S}}_n\}$ is an $\mathcal{H}$-preserving sequence and $\check{\mathfrak{S}}_n \xrightarrow{\tau_\eth} \check{\mathfrak{S}}$ then there exists a subsequence $\{\check{\mathfrak{S}}_{n_k}\}$ of $\{\check{\mathfrak{S}}_n\}$ with $[\check{\mathfrak{S}}_{n_k}, \check{\mathfrak{S}}] \in \mathcal{H}$ for all $k \in \mathbb{N}_0$.

Definition 1.9 ([2])**.** Let $\check{G}$ be a nonempty set and $\mathfrak{I}$ a self-mapping on $\check{G}$. A binary relation $\mathcal{H}$ defined on $\check{G}$ is called $\mathfrak{I}$-closed if for any $\check{\mathfrak{S}}, \varkappa \in \check{G}$

$$(\check{\mathfrak{S}}, \varkappa) \in \mathcal{H} \implies (\mathfrak{I}\check{\mathfrak{S}}, \mathfrak{I}\varkappa) \in \mathcal{H}.$$

Proposition 1.10 ([2])**.** *Let $\check{G}$, $\mathfrak{I}$ and $\mathcal{H}$ be same as in Definition* 1.9*. $\mathcal{H}^s$ must also be $\mathfrak{I}$-closed if $\mathcal{H}$ is $\mathfrak{I}$-closed.*

Definition 1.11 ([21])**.** Let $\check{G}$ be a nonempty set and $\mathcal{H}$ a binary relation on $\check{G}$. A subset Y of $\check{G}$ is called $\mathcal{H}$-directed if for each $\check{\mathfrak{S}}, \varkappa \in Y$, there exists $\mathfrak{G} \in \check{G}$ such that $(\check{\mathfrak{S}}, \mathfrak{G}) \in \mathcal{H}$ and $(\varkappa, \mathfrak{G}) \in \mathcal{H}$.

Definition 1.12 ([6])**.** Given $N \in \mathbb{N}_0$, $N \geq 2$, a binary relation $\mathcal{H}$ defined on a non-empty set $\check{G}$ is called N-transitive if for any $\check{\mathfrak{S}}_0, \check{\mathfrak{S}}_1, \check{\mathfrak{S}}_2, \cdots, \check{\mathfrak{S}}_N \in \check{G}$

$$(\check{\mathfrak{S}}_{i-1}, \check{\mathfrak{S}}_i) \in \mathcal{H} \ for \ each \ i(1 \leq i \leq N) \implies (\check{\mathfrak{S}}_0, \check{\mathfrak{S}}_N) \in \mathcal{H}.$$

Notice that notion of 2-transitivity coincides with transitivity. Following Turinici [22], $\mathcal{H}$ is called finitely transitive if it is N-transitive for some $N \geq 2$.

Definition 1.13 ([6]). A binary relation $\mathcal{H}$ defined on a nonempty set $\check{G}$ is called locally finiltely transitive if for each denumerable subset E of $\check{G}$, there exists $N = N(E) \geq 2$, such that $\mathcal{H}|_E$ is N-transitive.

Definition 1.14 ([4]). Let $\check{G}$ be a nonempty set and $\mathfrak{I}$ a self mapping on $\check{G}$. A binary relation $\mathcal{H}$ on $\check{G}$ is called locally finitely $\mathfrak{I}$-transitive if for each denumerable subset E of $\mathfrak{I}(\check{G})$, there exists $N = N(E) \geq 2$, such that $\mathcal{H}|_E$ is N-transitive".

Lemma 1.15 ([8]). *Let $(\check{G}, \eth)$ be a metric space and $\{\check{\mathfrak{S}}_n\}$ a sequence in $\check{G}$. If $\{\check{\mathfrak{S}}_n\}$ is not a Cauchy sequence, then $\exists\ \epsilon > 0$ and subsequences $\{\check{\mathfrak{S}}_{n_\varsigma}\}$ & $\{\check{\mathfrak{S}}_{m_\varsigma}\}$ of $\{\check{\mathfrak{S}}_n\}$ such that*

(i) $\varsigma \leq m_\varsigma < n_\varsigma\ \forall\ \varsigma \in \mathbb{N}$,
(ii) $\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{n_\varsigma}) \geq \epsilon$,
(iii) $\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{p_\varsigma}) \geq \epsilon,\ \forall\ p_\varsigma \in \{m_{\varsigma+1}, m_{\varsigma+2}, \ldots, n_{\varsigma-2}, n_{\varsigma-1}\}$.

Additionally, if $\lim_{n\to\infty} \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1}) = 0$, *then*

(iv) $\lim_{\varsigma\to\infty} \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{n_\varsigma+p}) = \epsilon\ \forall\ p \in \mathbb{N}_0$.

Lemma 1.16 ([6]). *Let $\check{G}$ be a non empty set, $\mathcal{H}$ a binary relation on $\check{G}$ and $\{\check{\mathfrak{S}}_n\}$ is a $\mathcal{H}$-preserving sequence in $\check{G}$". If $\mathcal{H}$ is a N-transitive on $Y := \{\check{\mathfrak{S}}_n : n \in \mathbb{N}_0\}$ for some natural number $N \geq 2$, then*

$$(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1+r(N-1)}) \in \mathcal{H},\ \forall\ n, r \in \mathbb{N}_0.$$

2. A new class of (Φ, Ψ)-contraction

Let Φ denote the class of the functions $\phi : [0, \infty) \to [0, \infty)$ which satisfy the following assumptions:

Φ_1 : ϕ is right continuous;
Φ_2 : ϕ is monotonic increasing and $\phi(0) = 0$.

Let Ψ denote the class of the functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following assumptions:

Ψ_1 : $\psi(\mathfrak{t}) > 0, \mathfrak{t} > 0$;
Ψ_2 :$\lim_{\mathfrak{t}\to r} \inf \psi(r) > 0,\ \forall\ r > 0$.

Remark 2.1 ([3]). Axiom Ψ_1 is equivalent to the following:
Ψ_1' : If $\exists\ \mathfrak{t} \in [0, \infty)$ such that $\psi(\mathfrak{t}) = 0,$ *then* $\mathfrak{t} = 0$.

Proposition 2.2 ([3]). *If there exists a pair of auxiliary functions $\phi, \psi : [0, \infty) \to [0, \infty)$, which satisfies axioms Φ_2 and Ψ_1 such that $\forall\ s \in [0, \infty)$ and $t \in (0, \infty)$,*

$$\phi(\mathfrak{s}) \leq \phi(\mathfrak{t}) - \psi(\mathfrak{t}),\ \textit{then}\ \mathfrak{s} < \mathfrak{t}.$$

Proposition 2.3. *Let $(\check{G}, \eth)$ be a DMS and $\mathfrak{I}$ a self mapping on $\check{G}$. If $\exists$ an auxiliary functions $\phi, \psi : [0, \infty) \to [0, \infty)$, which satisfies axioms Φ_2 and Ψ_1 respectively, such that $\mathfrak{I}$ is a (ϕ, ψ)-contraction of Alam et al. [3], then $\mathfrak{I}$ is contractive and hence is continuous.*

Given a binary relation $\mathcal{H}$ and a self-mapping $\mathfrak{I}$ on a non-empty set $\check{G}$, we use the following notations:

(i) $F(\mathfrak{I}) :=$ the set of all fixed points of $\mathfrak{I}$,

(ii) $\gamma(\check{\mathfrak{S}},\varkappa,\mathcal{H})$: the class of all paths in $\mathcal{H}$ from $\check{\mathfrak{S}}$ to $\varkappa$,

(iii)

$$\begin{aligned} M(\check{\mathfrak{S}},\varkappa) &= \max\Big\{\eth(\check{\mathfrak{S}},\varkappa),\eth(\check{\mathfrak{S}},\Im\check{\mathfrak{S}}),\eth(\varkappa,\Im\varkappa),\frac{\eth(\check{\mathfrak{S}},\Im\varkappa)+\eth(\varkappa,\Im\check{\mathfrak{S}})}{2}, \\ &\quad \frac{\eth(\varkappa,\Im\varkappa)[1+\eth(\check{\mathfrak{S}},\Im\check{\mathfrak{S}})]}{1+\eth(\check{\mathfrak{S}},\varkappa)},\frac{\eth(\check{\mathfrak{S}},\Im\check{\mathfrak{S}})[1+\eth(\check{\mathfrak{S}},\Im\check{\mathfrak{S}})]}{1+\eth(\check{\mathfrak{S}},\varkappa)}\Big\}. \end{aligned}$$

(iv)

$$\begin{aligned} N(\check{\mathfrak{S}},\varkappa) &= \max\Big\{\eth(\check{\mathfrak{S}},\varkappa),\eth(\check{\mathfrak{S}},\Im\check{\mathfrak{S}}),\eth(\varkappa,\Im\varkappa), \\ &\quad \frac{\eth(\varkappa,\Im\varkappa)[1+\eth(\check{\mathfrak{S}},\Im\check{\mathfrak{S}})]}{1+\eth(\check{\mathfrak{S}},\varkappa)},\frac{\eth(\check{\mathfrak{S}},\Im\check{\mathfrak{S}})[1+\eth(\check{\mathfrak{S}},\Im\check{\mathfrak{S}})]}{1+\eth(\check{\mathfrak{S}},\varkappa)}\Big\}. \end{aligned}$$

Remark 2.4. Observe that $N(\check{\mathfrak{S}},\varkappa)\le M(\check{\mathfrak{S}},\varkappa)$ $(\forall\check{\mathfrak{S}},\varkappa\in\check{G})$.

Now, we define the new generalized (ϕ,ψ)-rational contraction as follows:

Definition 2.5. Let $(\check{G},\eth)$ be complete DMS and $\Im:\check{G}\to\check{G}$ be a self mapping, then $\Im$ is said to satisfy new generalized (ϕ,ψ)-rational contraction if for $\phi\in\Phi$ and $\psi\in\Psi$ such that

$$\phi(\eth(\Im\check{\mathfrak{S}},\Im\varkappa))\le\phi(M(\check{\mathfrak{S}},\varkappa))-\psi(N(\check{\mathfrak{S}},\varkappa))\ \forall\ \check{\mathfrak{S}},\varkappa\in\check{G}\ \textit{with}\ (\check{\mathfrak{S}},\varkappa)\in\mathcal{H}.$$

Proposition 2.6. *Given a DMS $(\check{G},\eth)$ equipped with a binary relation $\mathcal{H}$, a mapping $\Im:\check{G}\to\check{G}$ and an auxiliary function $\phi\in\Phi$ and $\psi\in\Psi$, the following contractivity conditions are equivalent:*

(a) $\phi(\eth(\Im\check{\mathfrak{S}},\Im\varkappa))\le\phi(M(\check{\mathfrak{S}},\varkappa))-\psi(N(\check{\mathfrak{S}},\varkappa))\ \forall\ \check{\mathfrak{S}},\varkappa\in\check{G}$ *with* $(\check{\mathfrak{S}},\varkappa)\in\mathcal{H}$.

(b) $\phi(\eth(\Im\check{\mathfrak{S}},\Im\varkappa))\le\phi(M(\check{\mathfrak{S}},\varkappa))-\psi(N(\check{\mathfrak{S}},\varkappa))\ \forall\ \check{\mathfrak{S}},\varkappa\in\check{G}$ *with* $[\check{\mathfrak{S}},\varkappa]\in\mathcal{H}$.

3. Main result

Theorem 3.1. *Let $(\check{G},\eth)$ be a dislocated metric space and $\mathcal{H}$ a binary relation on $\check{G}$. Let $\Im$: $\check{G}\to\check{G}$ be a self-mapping satisfying the following conditions.*

(i) *$\exists$ a subset $Y\subseteq\check{G}$ with $\Im\check{G}\subseteq Y$ such that $(Y,\eth)$ is $\mathcal{H}$-complete,*
(ii) *$\exists$ $\check{\mathfrak{S}}_0$ such that $(\check{\mathfrak{S}}_0,\Im\check{\mathfrak{S}}_0)\in\mathcal{H}$,*
(iii) *$\mathcal{H}$ is $\Im$-closed,*
(iv) *either $\Im$ is $\mathcal{H}$ - sequentially -continuous or $\mathcal{H}|_Y$ is $\eth$-self-closed,*
(v) *$\Im$ satisfy the new generalized (ϕ,ψ)- rational contraction.*

Then $\Im$ has a fixed point.

Proof. By condition *(ii)*, $\exists$ $\check{\mathfrak{S}}_0\in\check{G}$ such that $(\check{\mathfrak{S}}_0,\Im\check{\mathfrak{S}}_0)\in\mathcal{H}$. Now we define the sequence of Picard iterates $\check{\mathfrak{S}}_{n+1}=\Im\check{\mathfrak{S}}_n$. If $\Im\check{\mathfrak{S}}_0=\check{\mathfrak{S}}_0$ then nothing to prove.
If $\Im\check{\mathfrak{S}}_0\ne\check{\mathfrak{S}}_0$ then by the condition *(iii)*, we get

$$(\Im\check{\mathfrak{S}}_0,\Im^2\check{\mathfrak{S}}_0),(\Im^2\check{\mathfrak{S}}_0,\Im^3\check{\mathfrak{S}}_0),(\Im^3\check{\mathfrak{S}}_0,\Im^4\check{\mathfrak{S}}_0)\cdots(\Im^n\check{\mathfrak{S}}_0,\Im^{n+1}\check{\mathfrak{S}}_0)\cdots\in\mathcal{H}. \tag{3.1}$$

As $(\check{\mathfrak{S}}_n,\check{\mathfrak{S}}_{n+1})\in\mathcal{H}\ \forall\ n\in\mathbb{N}_0$, i.e., $\{\check{\mathfrak{S}}_n\}$ is $\mathcal{H}$-preserving sequence.
Denote $\eth_n:=\eth(\check{\mathfrak{S}}_n,\check{\mathfrak{S}}_{n+1})$. If $\exists$ $n_0\in\mathbb{N}_0$ such that $\eth_{n_0}=0$, then by (3.1), we

conclude that $\check{\mathfrak{S}}_{n_0} = \check{\mathfrak{S}}_{n_0+1} = \mathfrak{J}(\check{\mathfrak{S}}_{n_0})$ so that $\check{\mathfrak{S}}_{n_0}$ is a fixed point of $\mathfrak{J}$. Otherwise, we have $\eth_n > 0\ \forall\ n \in \mathbb{N}_0$. Applying condition (v), we get

$$\phi(\eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1})) = \phi(\eth(\mathfrak{J}\check{\mathfrak{S}}_{n-1}, \mathfrak{J}\check{\mathfrak{S}}_n)) \leq \phi(M(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)) - \psi(N(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)) \tag{3.2}$$

where

$$\begin{aligned}
&M(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)\\
&= \max\Big\{\eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n), \eth(\check{\mathfrak{S}}_{n-1}, \mathfrak{J}\check{\mathfrak{S}}_{n-1}), \eth(\check{\mathfrak{S}}_n, \mathfrak{J}\check{\mathfrak{S}}_n)\frac{\eth(\check{\mathfrak{S}}_{n-1}, \mathfrak{J}\check{\mathfrak{S}}_n) + \eth(\check{\mathfrak{S}}_n, \mathfrak{J}\check{\mathfrak{S}}_{n-1})}{2},\\
&\quad \frac{\eth(\check{\mathfrak{S}}_n, \mathfrak{J}\check{\mathfrak{S}}_n)[1 + \eth(\check{\mathfrak{S}}_{n-1}, \mathfrak{J}\check{\mathfrak{S}}_{n-1})]}{1 + \eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)}, \frac{\eth(\check{\mathfrak{S}}_{n-1}, \mathfrak{J}\check{\mathfrak{S}}_{n-1})[1 + \eth(\check{\mathfrak{S}}_{n-1}, \mathfrak{J}\check{\mathfrak{S}}_{n-1})]}{1 + \eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)}\Big\}\\
&= \max\Big\{\eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n), \eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n), \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1}), \frac{\eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_{n+1}) + \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_n)}{2},\\
&\quad \frac{\eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1})[1 + \eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)]}{1 + \eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)}, \frac{\eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)[1 + \eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)]}{1 + \eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)}\Big\}\\
&= \max\Big\{\eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n), \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1}), \frac{\eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_{n+1}) + \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_n)}{2}\Big\}\\
&= \max\Big\{\eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n), \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1})\Big\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&N(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)\\
&= \max\Big\{\eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n), \eth(\check{\mathfrak{S}}_{n-1}, \mathfrak{J}\check{\mathfrak{S}}_{n-1}), \eth(\check{\mathfrak{S}}_n, \mathfrak{J}\check{\mathfrak{S}}_n),\\
&\quad \frac{\eth(\check{\mathfrak{S}}_n, \mathfrak{J}\check{\mathfrak{S}}_n)[1 + \eth(\check{\mathfrak{S}}_{n-1}, \mathfrak{J}\check{\mathfrak{S}}_{n-1})]}{1 + \eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)}, \frac{\eth(\check{\mathfrak{S}}_{n-1}, \mathfrak{J}\check{\mathfrak{S}}_{n-1})[1 + \eth(\check{\mathfrak{S}}_{n-1}, \mathfrak{J}\check{\mathfrak{S}}_{n-1})]}{1 + \eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)},\Big\}\\
&= \max\Big\{\eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n), \eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n), \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1}),\\
&\quad \frac{\eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1})[1 + \eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)]}{1 + \eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)}, \frac{\eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)[1 + \eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)]}{1 + \eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n)}\Big\}\\
&= \max\Big\{\eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n), \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1})\Big\}.
\end{aligned}$$

Therefore (3.2) can be rewritten as

$$\begin{aligned}
\phi(\eth_n) &= \phi(\eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1}))\\
&\leq \phi(\max(\eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n), \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1})))\\
&\quad - \psi(\max(\eth(\check{\mathfrak{S}}_{n-1}, \check{\mathfrak{S}}_n), \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1})))\\
&\leq \phi(\max(\eth_{n-1}, \eth_n)) - \psi(\max(\eth_{n-1}, \eth_n)).
\end{aligned} \tag{3.3}$$

Now if $\eth_n > \eth_{n-1}$ then by equation (3.3), we get

$$\phi(\eth_n) \leq \phi(\eth_n) - \psi(\eth_n).$$

By properties of Φ, we get $\psi(\eth_n) \leq 0$, which is a contradiction to condition Ψ_2.

Hence we have $\eth_n < \eth_{n-1}$ which amount to say that $\eth_n$ is decreasing sequence. Then equation (3.3) yields that

$$\phi(\eth_n) \leq \phi(\eth_{n-1}) - \psi(\eth_{n-1}). \tag{3.4}$$

In view of Proposition 2.2 and (3.4) gives rise

$$\eth_n < \eth_{n-1} \ \forall n \in \mathbb{N}_0,$$

which yields that the sequence $\{\eth_n\}$ is a decreasing sequence of +ve real numbers. Since it is bounded below by 0 (as a lower bound), there is an element $\kappa \geq 0$ such that

$$\lim_{n\to\infty} \eth_n = \kappa. \tag{3.5}$$

Now, we claim that $\kappa = 0$. On contrary suppose that $\kappa > 0$. Taking upper limit in Eqn. (3.4), we obtain

$$\begin{aligned} \lim_{n\to\infty} \sup\ \phi(\eth_n) \leq&\ \lim_{n\to\infty} \sup\ \phi(\eth_{n-1}) + \lim_{n\to\infty} \sup\ [-\psi(\eth_{n-1})] \\ \leq&\ \lim_{n\to\infty} \sup\ \phi(\eth_{n-1}) - \lim_{n\to\infty} \inf\ \psi(\eth_{n-1}). \end{aligned} \tag{3.6}$$

Using (3.5) and (3.6) reduces to

$$\phi(\kappa) \leq \phi(\kappa) - \lim_{n\to\infty} \inf \psi(\eth_{n-1}),$$

implying thereby

$$\lim_{\eth_n\to\kappa} \inf \psi(\eth_{n-1}) = \lim_{n\to\infty} \inf \psi(\eth_{n-1}) \leq 0,$$

which contradicts the property of Ψ_2. Therefore we have

$$\lim_{n\to\infty} \eth_n = \lim_{n\to\infty} \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1}) = 0. \tag{3.7}$$

Now we show that $\{\check{\mathfrak{S}}_n\}$ to be Cauchy sequence. Let on contrary that $\{\check{\mathfrak{S}}_n\}$ is not a Cauchy sequence. Therefore, by Lemma 1.15, $\exists\ \epsilon > 0$ and subsequences $\{\check{\mathfrak{S}}_{n_\varsigma}\}$ & $\{\check{\mathfrak{S}}_{m_\varsigma}\}$ of $\{\check{\mathfrak{S}}_n\}$ such that $\varsigma \leq m_\varsigma < n_\varsigma$, $\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{n_\varsigma}) \geq \epsilon$ and $\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{p_\varsigma}) < \epsilon$ where $p_\varsigma \in \{m_\varsigma + 1, m_\varsigma + 2, \ldots, n_\varsigma - 2, n_\varsigma - 1\}$. Further, using (3.6) and Lemma 1.15, we have

$$\lim_{\varsigma\to\infty} \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{n_\varsigma+p}) = \epsilon \ \forall p \in \mathbb{N}_0. \tag{3.8}$$

As $\{\check{\mathfrak{S}}_n\}$ is $\mathcal{H}$-preserving and $\{\check{\mathfrak{S}}_n\} \subset \mathfrak{I}(\check{G})$ (owing to (3.1)) and hence the range $E := \{\check{\mathfrak{S}}_n : n \in \mathbb{N}_0\}$ (of the sequence $\{\check{\mathfrak{S}}_n\}$), is a denumerable subset of $\mathfrak{I}(\check{G})$, By locally finitely $\mathfrak{I}$-transitivity of $\mathcal{H}$, $\exists$ a natural number $\mathcal{N} = \mathcal{N}(E) \geq 2$, such that $\mathcal{H}|_E$ is $\mathcal{N}$-transitive. As $m_\varsigma < n_\varsigma$ and $\mathcal{N} - 1 > 0$, using Division Algorithm we have

$$\begin{aligned} n_\varsigma - m_\varsigma &= (\mathcal{N} - 1)(\alpha_\varsigma - 1) + (\mathcal{N} - \beta_\varsigma) \\ &\alpha_\varsigma - 1 \geq 0, 0 \leq \mathcal{N} - \beta_\varsigma < \mathcal{N} - 1 \\ \iff &\begin{cases} n_\varsigma + \beta_\varsigma = m_\varsigma + 1 + (\mathcal{N} - 1)\alpha_\varsigma \\ \alpha_\varsigma \geq 1, 1 < \beta_\varsigma \leq \mathcal{N}. \end{cases} \end{aligned}$$

Here above α_ς & β_ς are natural numbers such that β_ς can assume positive integral value in interval $(1, \mathcal{N}]$. Hence, we can choose subsequences $\{\check{\mathfrak{S}}_{n_\varsigma}\}$ & $\{\check{\mathfrak{S}}_{m_\varsigma}\}$ of $\{\check{\mathfrak{S}}_n\}$

satisfying Eqn. (3.8) such that β_ς remains constant say β, which is independent of ς. Write

$$m'_\varsigma = n_\varsigma + \beta = m_\varsigma + 1 + (\mathcal{N}-1)\alpha_\varsigma \tag{3.9}$$

where $\beta(1 < \beta \leq \mathcal{N})$ is constant. Owing to (3.8) & (3.9), we get

$$\lim_{n\to\infty} \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma}) = \lim_{n\to\infty} \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{n_\varsigma}) + \beta = \epsilon. \tag{3.10}$$

Using triangular inequality, we have

$$\begin{aligned} \eth(\check{\mathfrak{S}}_{m_\varsigma+1}, \check{\mathfrak{S}}_{m'_\varsigma+1}) &\leq \eth(\check{\mathfrak{S}}_{m_\varsigma+1}, \check{\mathfrak{S}}_{m_\varsigma}) + \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma}) + \eth(\check{\mathfrak{S}}_{m'_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma+1}) \\ \textit{and } \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma}) &\leq \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m_\varsigma+1}) + \eth(\check{\mathfrak{S}}_{m_\varsigma+1}, \check{\mathfrak{S}}_{m'_\varsigma+1}) + \eth(\check{\mathfrak{S}}_{m'_\varsigma+1}, \check{\mathfrak{S}}_{m'_\varsigma}) \end{aligned}$$

therefore, we have

$$\begin{aligned} &\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma}) - \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m_\varsigma+1}) - \eth(\check{\mathfrak{S}}_{m'_\varsigma+1}, \check{\mathfrak{S}}_{m'_\varsigma}) \\ &\qquad \leq \eth(\check{\mathfrak{S}}_{m_\varsigma+1}, \check{\mathfrak{S}}_{m'_\varsigma+1}) \leq \eth(\check{\mathfrak{S}}_{m_\varsigma+1}, \check{\mathfrak{S}}_{m_\varsigma}) + \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma}) + \eth(\check{\mathfrak{S}}_{m'_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma+1}) \end{aligned}$$

which on letting $\varsigma \to \infty$ and using (3.7) and (3.10), gives rise

$$\lim_{n\to\infty} \eth(\check{\mathfrak{S}}_{m_\varsigma+1}, \check{\mathfrak{S}}_{m'_\varsigma+1}) = \epsilon. \tag{3.11}$$

In view of (3.9) and Lemma 1.16, we have $\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma}) \in \mathcal{H}$. By using Eqn. (3.1) and condition (v), we have

$$\begin{aligned} \phi(\eth(\check{\mathfrak{S}}_{m_\varsigma+1}, \check{\mathfrak{S}}_{m'_\varsigma+1})) &= \phi(\eth(\Im\check{\mathfrak{S}}_{m_\varsigma}, \Im\check{\mathfrak{S}}_{m'_\varsigma})) \\ &\leq \phi(M(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma})) - \psi(N(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma})). \end{aligned}$$

Let $M_\varsigma = M(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma})$ and $N_\varsigma = N(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma})$ then

$$\phi(\eth(\check{\mathfrak{S}}_{m_\varsigma+1}, \check{\mathfrak{S}}_{m'_\varsigma+1})) \leq \phi(M_\varsigma) - \psi(N_\varsigma). \tag{3.12}$$

So,

$$\begin{aligned} &M_\varsigma = M(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma}) \\ &= \max\Big\{\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma}), \eth(\check{\mathfrak{S}}_{m_\varsigma}, \Im\check{\mathfrak{S}}_{m_\varsigma}), \\ &\quad \eth(\check{\mathfrak{S}}_{m'_\varsigma}, \Im\check{\mathfrak{S}}_{m'_\varsigma}), \frac{\eth(\check{\mathfrak{S}}_{m_\varsigma}, \Im\check{\mathfrak{S}}_{m'_\varsigma}) + \eth(\check{\mathfrak{S}}_{m'_\varsigma}, \Im\check{\mathfrak{S}}_{m_\varsigma})}{2}, \\ &\quad \frac{\eth(\check{\mathfrak{S}}_{m'_\varsigma}, \Im\check{\mathfrak{S}}_{m'_\varsigma})[1 + \eth(\check{\mathfrak{S}}_{m_\varsigma}, \Im\check{\mathfrak{S}}_{m_\varsigma})]}{1 + \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma})}, \frac{\eth(\check{\mathfrak{S}}_{m_\varsigma}, \Im\check{\mathfrak{S}}_{m_\varsigma})[1 + \eth(\check{\mathfrak{S}}_{m_\varsigma}, \Im\check{\mathfrak{S}}_{m_\varsigma})]}{1 + \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma})}\Big\} \\ &= \max\Big\{\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma}), \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m_\varsigma+1}), \\ &\quad \eth(\check{\mathfrak{S}}_{m'_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma+1}) \frac{\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma+1}) + \eth(\check{\mathfrak{S}}_{m'_\varsigma}, \check{\mathfrak{S}}_{m_\varsigma+1})}{2}, \\ &\quad \frac{\eth(\check{\mathfrak{S}}_{m'_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma+1})[1 + \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m_\varsigma+1})]}{1 + \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma})}, \frac{\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m_\varsigma+1})[1 + \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m_\varsigma+1})]}{1 + \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma})}\Big\}. \end{aligned}$$

Then taking limit $\varsigma \to \infty$ and using (3.7), (3.10), (3.11) we get

$$M_\varsigma = \max\,\{\epsilon, 0, 0, \epsilon, 0, 0\} = \epsilon. \tag{3.13}$$

Analogously,

$$\begin{aligned}
N_\varsigma &= N(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma})\\
&= \max\Big\{\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma}), \eth(\check{\mathfrak{S}}_{m_\varsigma}, \mathfrak{I}\check{\mathfrak{S}}_{m_\varsigma}), \eth(\check{\mathfrak{S}}_{m'_\varsigma}, \mathfrak{I}\check{\mathfrak{S}}_{m'_\varsigma}),\\
&\quad \frac{\eth(\check{\mathfrak{S}}_{m'_\varsigma}, \mathfrak{I}\check{\mathfrak{S}}_{m'_\varsigma})[1+\eth(\check{\mathfrak{S}}_{m_\varsigma}, \mathfrak{I}\check{\mathfrak{S}}_{m_\varsigma})]}{1+\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma})}, \frac{\eth(\check{\mathfrak{S}}_{m_\varsigma}, \mathfrak{I}\check{\mathfrak{S}}_{m_\varsigma})[1+\eth(\check{\mathfrak{S}}_{m_\varsigma}, \mathfrak{I}\check{\mathfrak{S}}_{m_\varsigma})]}{1+\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma})}\Big\}\\
&= \max\Big\{\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma}), \eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m_\varsigma+1}), \eth(\check{\mathfrak{S}}_{m'_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma+1}),\\
&\quad \frac{\eth(\check{\mathfrak{S}}_{m'_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma+1})[1+\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m_\varsigma+1})]}{1+\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma})}, \frac{\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m_\varsigma+1})[1+\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m_\varsigma+1})]}{1+\eth(\check{\mathfrak{S}}_{m_\varsigma}, \check{\mathfrak{S}}_{m'_\varsigma})}\Big\}.
\end{aligned}$$

Then taking limit $\varsigma \to \infty$ and using (3.7), (3.10) and (3.11) we get

$$N_\varsigma = \max\{\epsilon, 0, 0, 0, 0\} = \epsilon. \tag{3.14}$$

Taking upper limit in Eqn. (3.11), we get

$$\lim_{\varsigma\to\infty}\sup \phi(\eth(\check{\mathfrak{S}}_{m_\varsigma+1}, \check{\mathfrak{S}}_{m'_\varsigma+1})) \leq \lim_{\varsigma\to\infty}\sup \phi(M_\varsigma) + \lim_{\varsigma\to\infty}\sup[-\psi(N_\varsigma)],$$

which on using (3.10), (3.11), (3.13) & (3.14) becomes

$$\phi(\epsilon) \leq \phi(\epsilon) - \lim_{\varsigma\to\infty}\inf \psi(\epsilon)$$

yielding thereby

$$\lim_{\varsigma\to\infty}\inf \psi(\epsilon) = \lim_{\varsigma\to\infty}\inf \psi(\epsilon) \leq 0,$$

which contradicts to the property Ψ_2. It follows that $\{\check{\mathfrak{S}}_n\}$ is an $\mathcal{H}$-preserving Cauchy in Y. By $\mathcal{H}$-completeness of $(Y, \eth)$, there is $\varkappa \in Y$ such that the sequence $\{\check{\mathfrak{S}}_n\}$ converges to $\varkappa$ with respect to topology $\tau_\eth$ generated by $\eth$ i.e.,

$$\lim_{n\to\infty} \eth(\check{\mathfrak{S}}_n, \varkappa) = \eth(\varkappa, \varkappa) = \lim_{n\to\infty} \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1}) = 0. \tag{3.15}$$

Firstly suppose that $\mathfrak{I}$ is $\mathcal{H}$-sequentially- continuous. Then $\check{\mathfrak{S}}_{n+1} = \mathfrak{I}\check{\mathfrak{S}}_n \xrightarrow{\tau_\eth} \mathfrak{I}\varkappa$, so that

$$\begin{aligned}
\lim_{n\to\infty} \eth(\check{\mathfrak{S}}_{n+1}, \mathfrak{I}\varkappa) &= \lim_{n\to\infty} \eth(\mathfrak{I}\check{\mathfrak{S}}_n, \mathfrak{I}\varkappa)\\
&= \eth(\mathfrak{I}\varkappa, \mathfrak{I}\varkappa)\\
&= \lim_{n\to\infty} \eth(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}_{n+1}) = 0.
\end{aligned} \tag{3.16}$$

On using triangular inequality, (3.15) and (3.16), we have $\eth(\varkappa, \mathfrak{I}\varkappa) = 0$, so that $\varkappa$ is a fixed point of $\mathfrak{I}$.

Alternately, if $\mathcal{H}|_Y$ is $\eth$-self closed. As $\{\check{\mathfrak{S}}_n\}$ is an $\mathcal{H}$-preserving sequence in Y and $\check{\mathfrak{S}}_n \xrightarrow{\tau_\eth} \varkappa$, there is a subsequence $\{\check{\mathfrak{S}}_{n_\varsigma}\}$ of $\{\check{\mathfrak{S}}_n\}$ with $[\check{\mathfrak{S}}_{n_\varsigma}, \varkappa] \in \mathcal{H}\ \forall\ \varsigma \in \mathbb{N}_0$. In view of condition (v) and Proposition 2.6, we have

$$\begin{aligned}
\phi(\eth(\check{\mathfrak{S}}_{n_\varsigma+1}, \mathfrak{I}\varkappa)) &= \phi(\eth(\mathfrak{I}\check{\mathfrak{S}}_{n_\varsigma}, \mathfrak{I}\varkappa))\\
&\leq \phi(M(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa)) - \psi(M(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa)).
\end{aligned} \tag{3.17}$$

Let $M_{n_\varsigma} = M(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa)$ and $N_{n_\varsigma} = N(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa)$ then

$$\phi(\eth(\check{\mathfrak{S}}_{n_\varsigma+1}, \mathfrak{I}\varkappa)) = \phi(M_{n_\varsigma}) - \psi(N_{n_\varsigma}). \tag{3.18}$$

So,

$$\begin{aligned} M_{n_\varsigma} &= M(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa) \\ &= \max\Big\{\eth(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa), \eth(\check{\mathfrak{S}}_{n_\varsigma}, \mathfrak{I}\check{\mathfrak{S}}_{n_\varsigma}), \eth(\varkappa, \mathfrak{I}\varkappa), \frac{\eth(\check{\mathfrak{S}}_{n_\varsigma}, \mathfrak{I}\varkappa) + \eth(\varkappa, \mathfrak{I}\check{\mathfrak{S}}_{n_\varsigma})}{2}, \\ &\quad \frac{\eth(\varkappa, \mathfrak{I}\varkappa)[1 + \eth(\check{\mathfrak{S}}_{n_\varsigma}, \mathfrak{I}\check{\mathfrak{S}}_{n_\varsigma})]}{1 + \eth(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa)}, \frac{\eth(\check{\mathfrak{S}}_{n_\varsigma}, \mathfrak{I}\check{\mathfrak{S}}_{n_\varsigma})[1 + \eth(\check{\mathfrak{S}}_{n_\varsigma}, \mathfrak{I}\check{\mathfrak{S}}_{n_\varsigma})]}{1 + \eth(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa)}\Big\} \\ &= \max\Big\{\eth(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa), \eth(\check{\mathfrak{S}}_{n_\varsigma}, \check{\mathfrak{S}}_{n_\varsigma+1}), \eth(\varkappa, \mathfrak{I}\varkappa), \frac{\eth(\check{\mathfrak{S}}_{n_\varsigma}, \mathfrak{I}\varkappa) + \eth(\varkappa, \check{\mathfrak{S}}_{n_\varsigma+1})}{2}, \\ &\quad \frac{\eth(\varkappa, \mathfrak{I}\varkappa)[1 + \eth(\check{\mathfrak{S}}_{n_\varsigma}, \check{\mathfrak{S}}_{n_\varsigma+1})]}{1 + \eth(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa)}, \frac{\eth(\check{\mathfrak{S}}_{n_\varsigma}, \check{\mathfrak{S}}_{n_\varsigma+1})[1 + \eth(\check{\mathfrak{S}}_{n_\varsigma}, \check{\mathfrak{S}}_{n_\varsigma+1})]}{1 + \eth(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa)}\Big\}. \end{aligned}$$

Taking limit $\varsigma \to \infty$ and using $\check{\mathfrak{S}}_{n_\varsigma} \xrightarrow{\tau_\eth} \varkappa$, we obtain

$$\lim_{k\to\infty} M_{n_\varsigma} = \max\Big\{0, 0, \eth(\varkappa, \mathfrak{I}\varkappa), \frac{\eth(\varkappa, \mathfrak{I}\varkappa)}{2}, \eth(\varkappa, \mathfrak{I}\varkappa), 0\Big\} = \eth(\varkappa, \mathfrak{I}\varkappa). \tag{3.19}$$

Similarly,

$$\begin{aligned} N_{n_\varsigma} &= N(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa) \\ &= \max\Big\{\eth(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa), \eth(\check{\mathfrak{S}}_{n_\varsigma}, \mathfrak{I}\check{\mathfrak{S}}_{n_\varsigma}), \eth(\varkappa, \mathfrak{I}\varkappa), \\ &\quad \frac{\eth(\varkappa, \mathfrak{I}\varkappa)[1 + \eth(\check{\mathfrak{S}}_{n_\varsigma}, \mathfrak{I}\check{\mathfrak{S}}_{n_\varsigma})]}{1 + \eth(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa)}, \frac{\eth(\check{\mathfrak{S}}_{n_\varsigma}, \mathfrak{I}\check{\mathfrak{S}}_{n_\varsigma})[1 + \eth(\check{\mathfrak{S}}_{n_\varsigma}, \mathfrak{I}\check{\mathfrak{S}}_{n_\varsigma})]}{1 + \eth(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa)}\Big\} \\ &= \max\Big\{\eth(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa), \eth(\check{\mathfrak{S}}_{n_\varsigma}, \check{\mathfrak{S}}_{n_\varsigma+1}), \eth(\varkappa, \mathfrak{I}\varkappa), \\ &\quad \frac{\eth(\varkappa, \mathfrak{I}\varkappa)[1 + \eth(\check{\mathfrak{S}}_{n_\varsigma}, \check{\mathfrak{S}}_{n_\varsigma+1})]}{1 + \eth(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa)}, \frac{\eth(\check{\mathfrak{S}}_{n_\varsigma}, \check{\mathfrak{S}}_{n_\varsigma+1})[1 + \eth(\check{\mathfrak{S}}_{n_\varsigma}, \check{\mathfrak{S}}_{n_\varsigma+1})]}{1 + \eth(\check{\mathfrak{S}}_{n_\varsigma}, \varkappa)}\Big\} \end{aligned}$$

Taking limit $\varsigma \to \infty$ and using $\check{\mathfrak{S}}_{n_\varsigma} \xrightarrow{\tau_\eth} \varkappa$, we obtain

$$\lim_{\varsigma\to\infty} N_{n_\varsigma} = \max\big\{0, 0, \eth(\varkappa, \mathfrak{I}\varkappa), \eth(\varkappa, \mathfrak{I}\varkappa), 0\big\} = \eth(\varkappa, \mathfrak{I}\varkappa). \tag{3.20}$$

Taking upper limit in equation (3.18), we get

$$\lim_{\varsigma\to\infty} \sup \phi(\eth(\check{\mathfrak{S}}_{n_\varsigma}, \mathfrak{I}\varkappa)) \le \lim_{\varsigma\to\infty} \sup \phi(M_{n_\varsigma}) + \lim_{\varsigma\to\infty} \sup[-\psi(N_{n_\varsigma})],$$

which on using (3.19), (3.20) becomes

$$\phi(\eth(\varkappa, \mathfrak{I}\varkappa)) \le \phi(\eth(\varkappa, \mathfrak{I}\varkappa)) - \lim_{\varsigma\to\infty} \inf \psi(\eth(\varkappa, \mathfrak{I}\varkappa)),$$

yielding thereby

$$\lim_{\varsigma\to\infty} \inf \psi(\eth(\varkappa, \mathfrak{I}\varkappa)) = \lim_{\varsigma\to\infty} \inf \psi(\eth(\varkappa, \mathfrak{I}\varkappa)) \le 0,$$

which contradicts to the property Ψ_2. Hence, $\varkappa = \mathfrak{I}\varkappa$, that is $\varkappa$ is a fixed point of $\mathfrak{I}$. □

Theorem 3.2. *In addition of Theorem 3.1, If $\mathfrak{I}(\check{G})$ is $\mathcal{H}^s$-directed, then $\mathfrak{I}$ admits unique fixed point.*

Proof. Let $\check{\mathfrak{S}}, \varkappa$ are two fixed points, i.e., $\mathfrak{I}\check{\mathfrak{S}} = \check{\mathfrak{S}}$ & $\mathfrak{I}\varkappa = \varkappa$ then two cases arise.
Case I: If $(\check{\mathfrak{S}}, \varkappa) \in \mathcal{H}$ then

$$\phi(\eth(\check{\mathfrak{S}}, \varkappa)) = \phi(\eth(\mathfrak{I}\check{\mathfrak{S}}, \mathfrak{I}\varkappa)) \leq \phi(M(\check{\mathfrak{S}}, \varkappa)) - \psi(N(\check{\mathfrak{S}}, \varkappa)). \tag{3.21}$$

So, then

$$\begin{aligned} M(\check{\mathfrak{S}}, \varkappa) &= \max\Big\{\eth(\check{\mathfrak{S}}, \varkappa), \eth(\check{\mathfrak{S}}, \mathfrak{I}\check{\mathfrak{S}}), \eth(\varkappa, \mathfrak{I}\varkappa), \frac{\eth(\check{\mathfrak{S}}, \mathfrak{I}\varkappa) + \eth(\varkappa, \mathfrak{I}\check{\mathfrak{S}})}{2}, \\ &\quad \frac{\eth(\varkappa, \mathfrak{I}\varkappa)[1 + \eth(\check{\mathfrak{S}}, \mathfrak{I}\check{\mathfrak{S}})]}{1 + \eth(\check{\mathfrak{S}}, \varkappa)}, \frac{\eth(\check{\mathfrak{S}}, \mathfrak{I}\check{\mathfrak{S}})[1 + \eth(\check{\mathfrak{S}}, \mathfrak{I}\check{\mathfrak{S}})]}{1 + \eth(\check{\mathfrak{S}}, \varkappa)}\Big\} \\ &= \max\Big\{\eth(\check{\mathfrak{S}}, \varkappa), \eth(\check{\mathfrak{S}}, \check{\mathfrak{S}}), \eth(\varkappa, \varkappa), \frac{\eth(\check{\mathfrak{S}}, \varkappa) + \eth(\varkappa, \check{\mathfrak{S}})}{2}, \\ &\quad \frac{\eth(\varkappa, \varkappa)[1 + \eth(\check{\mathfrak{S}}, \check{\mathfrak{S}})]}{1 + \eth(\check{\mathfrak{S}}, \varkappa)}, \frac{\eth(\check{\mathfrak{S}}, \check{\mathfrak{S}})[1 + \eth(\check{\mathfrak{S}}, \check{\mathfrak{S}})]}{1 + \eth(\check{\mathfrak{S}}, \varkappa)}\Big\} \\ &= \max\Big\{\eth(\check{\mathfrak{S}}, \varkappa), 0, 0, \eth(\check{\mathfrak{S}}, \varkappa), 0, 0, \Big\} = \eth(\check{\mathfrak{S}}, \varkappa). \end{aligned}$$

Similarly, we get $N(\check{\mathfrak{S}}, \varkappa) = \eth(\check{\mathfrak{S}}, \varkappa)$. Then from (3.21) can be written as

$$\begin{aligned} \phi(\eth(\check{\mathfrak{S}}, \varkappa)) = \phi(\eth(\mathfrak{I}\check{\mathfrak{S}}, \mathfrak{I}\varkappa)) &\leq \phi(M(\check{\mathfrak{S}}, \varkappa)) - \psi(N(\check{\mathfrak{S}}, \varkappa)) \\ &\leq \phi(\eth(\check{\mathfrak{S}}, \varkappa)) - \psi(\eth(\check{\mathfrak{S}}, \varkappa)), \end{aligned}$$

yielding there by $\psi(\eth(\check{\mathfrak{S}}, \varkappa)) < 0$, which is a contradiction to the condition Ψ_2. Hence $\check{\mathfrak{S}} = \varkappa$.
Case II: If $(\check{\mathfrak{S}}, \varkappa) \notin \mathcal{H}$ then by $\mathfrak{I}(\check{G})$ is $\mathcal{H}^s$-directed then $\exists\ z \in \check{G}$ such that $(\check{\mathfrak{S}}, \mathfrak{G}) \in \mathcal{H}$ and $(\mathfrak{G}, \varkappa) \in \mathcal{H}$. Since $\mathcal{H}$ is $\mathfrak{I}$-closed $\mathfrak{I}^n\mathfrak{G}$ will be related to $\mathfrak{I}^n\check{\mathfrak{S}}$ i.e., $(\mathfrak{I}^n\mathfrak{G}, \mathfrak{I}^n\check{\mathfrak{S}} = \check{\mathfrak{S}}) \in R$ for any $n \in \mathbb{N}_0$. Then by condition (v) of Theorem 3.1, for any $n \in \mathbb{N}_0$, we have

$$\begin{aligned} \phi(\eth(\mathfrak{I}^n\mathfrak{G}, \check{\mathfrak{S}})) &= \phi(\eth(\mathfrak{I}^n\mathfrak{G}, \mathfrak{I}^n\check{\mathfrak{S}})) \\ &\leq \phi(M(\mathfrak{I}^{n-1}\check{\mathfrak{S}}, \mathfrak{I}^{n-1}\check{\mathfrak{S}})) - \psi(N(\mathfrak{I}^{n-1}\check{\mathfrak{S}}, \mathfrak{I}^{n-1}\check{\mathfrak{S}})). \end{aligned} \tag{3.22}$$

Now,

$$\begin{aligned} M(\mathfrak{I}^{n-1}\mathfrak{G}, \mathfrak{I}^{n-1}\check{\mathfrak{S}}) &= \max\Big\{\eth(\mathfrak{I}^{n-1}\mathfrak{G}, \mathfrak{I}^{n-1}\check{\mathfrak{S}}), \eth(\mathfrak{I}^{n-1}\mathfrak{G}, \mathfrak{I}\mathfrak{I}^{n-1}\mathfrak{G}), \eth(\mathfrak{I}^{n-1}\check{\mathfrak{S}}, \mathfrak{I}\mathfrak{I}^{n-1}\check{\mathfrak{S}}) \\ &\quad \frac{\eth(\mathfrak{I}^{n-1}\mathfrak{G}, \mathfrak{I}\mathfrak{I}^{n-1}\check{\mathfrak{S}}) + \eth(\mathfrak{I}^{n-1}\check{\mathfrak{S}}, \mathfrak{I}\mathfrak{I}^{n-1}\mathfrak{G})}{2}, \\ &\quad \frac{\eth(\mathfrak{I}^{n-1}\check{\mathfrak{S}}, \mathfrak{I}\mathfrak{I}^{n-1}\check{\mathfrak{S}})[1 + \eth(\mathfrak{I}^{n-1}\mathfrak{G}, \mathfrak{I}\mathfrak{I}^{n-1}\mathfrak{G})]}{1 + \eth(\mathfrak{I}^{n-1}\mathfrak{G}, \mathfrak{I}^{n-1}\check{\mathfrak{S}})}, \\ &\quad \frac{\eth(\mathfrak{I}^{n-1}\mathfrak{G}, \mathfrak{I}\mathfrak{I}^{n-1}\mathfrak{G})[1 + \eth(\mathfrak{I}^{n-1}\mathfrak{G}, \mathfrak{I}\mathfrak{I}^{n-1}\mathfrak{G})]}{1 + \eth(\mathfrak{I}^{n-1}\mathfrak{G}, \mathfrak{I}^{n-1}\check{\mathfrak{S}})}\Big\} \\ &= \max\{\eth(\mathfrak{I}^{n-1}\mathfrak{G}, \check{\mathfrak{S}}), \eth(\mathfrak{I}^{n-1}\mathfrak{G}, \mathfrak{I}^n\mathfrak{G}), \eth(\check{\mathfrak{S}}, \check{\mathfrak{S}}) \\ &\quad \frac{\eth(\mathfrak{I}^{n-1}\mathfrak{G}, \check{\mathfrak{S}}) + \eth(\check{\mathfrak{S}}, \mathfrak{I}^n\mathfrak{G})}{2}, \frac{\eth(\check{\mathfrak{S}}, \check{\mathfrak{S}})[1 + \eth(\mathfrak{I}^{n-1}\mathfrak{G}, \mathfrak{I}\mathfrak{I}^{n-1}\mathfrak{G})]}{1 + \eth(\mathfrak{I}^{n-1}\mathfrak{G}, \check{\mathfrak{S}})}, \end{aligned}$$

$$\left.\frac{\eth(\Im^{n-1}\mathfrak{G},\Im^n\mathfrak{G})[1+\eth(\Im^{n-1}\mathfrak{G},\Im\Im^n\mathfrak{G})]}{1+\eth(\Im^{n-1}\mathfrak{G},\check{\mathfrak{S}})}\right\}$$
$$=\eth(\Im^{n-1}\mathfrak{G},\check{\mathfrak{S}}).$$

Similarly, for $N(\Im^{n-1}\mathfrak{G},\Im^{n-1}\check{\mathfrak{S}})=\eth(\Im^{n-1}\mathfrak{G},\check{\mathfrak{S}})$. Then (3.22) reduces to

$$\phi(\eth(\Im^{n-1}\mathfrak{G},\check{\mathfrak{S}}))\leq\phi(\eth(\Im^{n-1}\mathfrak{G},\check{\mathfrak{S}}))-\psi(\eth(\Im^{n-1}\mathfrak{G},\check{\mathfrak{S}})).\tag{3.23}$$

Using Proposition 2.3 we have

$$\eth(\Im^n\mathfrak{G},\check{\mathfrak{S}})<\eth(\Im^{n-1}\mathfrak{G},\check{\mathfrak{S}}).$$

Which amount to say that $\{\eth(\Im^n\mathfrak{G},\check{\mathfrak{S}})\}$ is a decreasing of positive real numbers, which is bounded below by 0, $\exists\ \varsigma\geq 0$ such that

$$\eth(\Im^n\mathfrak{G},\check{\mathfrak{S}})=\varsigma.$$

Our claim is that $\varsigma=0$, for that let on contrary that $\varsigma>0$ then taking upper limit to the equation (3.23) we get

$$\begin{aligned}\lim_{n\to 0}\sup\phi(\eth(\Im^n\mathfrak{G},\check{\mathfrak{S}})) &\leq \lim_{n\to 0}\sup\phi(\eth(\Im^{n-1}\mathfrak{G},\check{\mathfrak{S}}))+\lim_{n\to 0}\sup[-\psi(\eth(\Im^{n-1}\mathfrak{G},\check{\mathfrak{S}}))]\\ &\leq \lim_{n\to 0}\sup\phi(\eth(\Im^{n-1}\mathfrak{G},\check{\mathfrak{S}}))-\lim_{n\to 0}\inf\psi(\eth(\Im^{n-1}\mathfrak{G},\check{\mathfrak{S}}))\end{aligned}$$

by right continuity of ϕ we get

$$\phi(\varsigma)\leq\phi(\varsigma)-\lim_{n\to\infty}\inf\psi(\varsigma)$$

which amounts to say that $\lim_{n\to\infty}\inf\psi(\varsigma)\leq 0$, which is contradiction to the condition Ψ_2. Hence, $\lim_{n\to\infty}\{\eth(\Im^n\mathfrak{G},\check{\mathfrak{S}})\}=0$ means $\lim_{n\to\infty}\Im^n\mathfrak{G}=\check{\mathfrak{S}}$. Analogously, we can proved that $\lim_{n\to\infty}\Im^n\mathfrak{G}=\varkappa$. Then by unicity of limit we have $\check{\mathfrak{S}}=\varkappa$. Hence, $\Im$ admits unique fixed point. □

On setting $\mathrm{Y}=\check{G}$ in Theorem 3.1, we deduce the following:

Corollary 3.3. *Let $(\check{G},\eth)$ be a DMS equipped with a binary relation $\mathcal{H}$ and $\Im$ a self-mapping on $\check{G}$. Suppose that the condition (ii), (iii), (v), (vi) together with following conditions are satisfied:*

(vii) : $(\check{G},\eth)$ is $\mathcal{H}$-complete,
(viii) : either $\Im$ is $\mathcal{H}$-sequentially-continuous or $\mathcal{H}$ is $\eth$-self-closed.

Then $\Im$ admits unique fixed point.

Corollary 3.4. *If we replace the contractive condition of Theorem 3.1 and 3.2, by the following condition*

$$\phi(\eth(\Im\check{\mathfrak{S}},\Im\varkappa))\leq\phi(M(\check{\mathfrak{S}},\varkappa))-\psi(M(\check{\mathfrak{S}},\varkappa))$$

where

$$\begin{aligned}M(\check{\mathfrak{S}},\varkappa)=\max\Big\{&\eth(\check{\mathfrak{S}},\varkappa),\eth(\check{\mathfrak{S}},\Im\check{\mathfrak{S}}),\eth(\varkappa,\Im\varkappa),\frac{\eth(\check{\mathfrak{S}},\Im\varkappa)+\eth(\varkappa,\Im\check{\mathfrak{S}})}{2},\\ &\frac{\eth(\varkappa,\Im\varkappa)[1+\eth(\check{\mathfrak{S}},\Im\check{\mathfrak{S}})]}{1+\eth(\check{\mathfrak{S}},\varkappa)},\frac{\eth(\check{\mathfrak{S}},\Im\check{\mathfrak{S}})[1+\eth(\check{\mathfrak{S}},\Im\check{\mathfrak{S}})]}{1+\eth(\check{\mathfrak{S}},\varkappa)}\Big\},\end{aligned}$$

for all $\phi\in\Phi$ and $\psi\in\Psi$. Then $\Im$ has a fixed point.

Taking ϕ as identity mapping and $\psi(t) = (1-k)t$ for $k \in (0,1)$. We deduce the several versions of our newly proved results in the context of metric, universal relation and rational expression.

Remark 3.5. Under the setting of $\eth = p$ (i.e., partial metric), $\mathcal{H} = \check{G} \times \check{G}$ in Theorem 3.1 then our results deduce to Corollary 2.4 of Kumar et al. [17].

Remark 3.6. Under the setting of $\eth = p$ (i.e., partial metric), $\mathcal{H} = \check{G} \times \check{G}$ and replacing $M(\check{\mathfrak{S}}, \varkappa)$ by $N(\check{\mathfrak{S}}, \varkappa)$ in Theorem 3.1 then our results deduce to Corollary 2.5 of Kumar et al. [17].

4. Application to a fractional differential equation

Now we are going to find the solution of a fractional differential equation boundary value problem by means of fixed point theorem. Let us consider the problem

$$D^{\alpha}_{0+}u(\check{\mathfrak{S}}) = \lambda f(\check{\mathfrak{S}}, u(\check{\mathfrak{S}})), 0 < \check{\mathfrak{S}} < 1 \tag{4.1}$$

$$u(0) = u(1) = u'(0) = u'(1) = 0, \tag{4.2}$$

where $3 < \alpha \leq 4$, $\lambda \in \mathbb{R}^{+}$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and D^{α}_{0+} denotes the standard Riemann-Liouville fractional derivative. Define a relation $\mathcal{H}$ on $\check{G}$ as

$$u\mathcal{H}v \iff u(\check{\mathfrak{S}}) \leq v(\check{\mathfrak{S}}) \ \forall\, u, v \in C[0,1] \ and\ \check{\mathfrak{S}} \in [0,1].$$

In 2009, Xu et al. [23] transform the above boundary value problem in to an integral equation, which as follows:

Lemma 4.1. *Boundary value problem* (4.1), (4.2) *is analogous to the integral equation*

$$u(\check{\mathfrak{S}}) = \lambda \int_0^1 K(\check{\mathfrak{S}}, \varkappa) f(\varkappa, u(\varkappa)) \eth\varkappa, \tag{4.3}$$

where

$$K(\check{\mathfrak{S}}, \varkappa) = \begin{cases} \frac{(\check{\mathfrak{S}}-\varkappa)^{\alpha-1}+(1-\varkappa)^{\alpha-2}\check{\mathfrak{S}}^{\alpha-2}[(\varkappa-\check{\mathfrak{S}})\alpha-2(1-\check{\mathfrak{S}})\varkappa]}{\Gamma(\alpha)}, & if\ 0 \leq \varkappa \leq \check{\mathfrak{S}} \leq 1 \\ \frac{(1-\varkappa)^{\alpha-2}\check{\mathfrak{S}}^{\alpha-2}[(\varkappa-\check{\mathfrak{S}})^{\alpha-2}(1-\check{\mathfrak{S}})\varkappa]}{\Gamma(\alpha)}, & if\ 0 \leq \check{\mathfrak{S}} \leq \varkappa \leq 1. \end{cases}$$

To reduce the problem into more simpler one, Xu et al. [23] introduced another lemma to bound the value of the kernel of the integral equation (4.3) *as follows:*

Lemma 4.2. *For all* $\check{\mathfrak{S}}, \varkappa \in (0,1)$, *we have* $A_1 \leq K(\check{\mathfrak{S}}, \varkappa) \leq A_0$, *where* $A_1 = \frac{(\alpha-2)\varkappa^2(1-\varkappa)^{\alpha-2}\check{\mathfrak{S}}^{\alpha-2}(1-\check{\mathfrak{S}})^2}{\Gamma(\alpha)}$, $A_0 = \frac{\check{\mathfrak{S}}^{\alpha-2}(1-\check{\mathfrak{S}})^{\alpha-2}}{\Gamma(\alpha)}B_0$ *and* $B_0 = max\{\alpha-1, (\alpha-2)^2\}$.

Now for existence and uniqueness of the solution of boundary value problem we have following theorem.

Theorem 4.3. *Consider the boundary value problem* (4.1), (4.2) *and assume that any of the following conditions hold:*

(*i*) $\exists$ *a real number* β *with* $0 \leq \beta \leq 1$*, such that*

$$|f(\varkappa, u(\varkappa)) - f(\varkappa, v(\varkappa))| \leq \beta |f(\varkappa, u(\varkappa))| - \beta \sup_{0 \leq \check{\mathfrak{S}} \leq 1} |u(\check{\mathfrak{S}})|$$

for all real valued continuous functions $u(\varkappa)$*,* $v(\varkappa)$ *defined on* $[0,1]$ *and* $\lambda A_1 \geq 1$*.*

(*ii*) $\exists$ *a real number* β *with* $0 \leq \beta < 1$*, such that*

$$|f(\varkappa, u(\varkappa)) - f(\varkappa, v(\varkappa))| \leq \beta |u(\varkappa) - v(\varkappa)|$$

for all real valued continuous functions $u(\varkappa), v(\varkappa)$ *defined on* $[0,1]$ *and* $\lambda A_0 \leq 1$*.*

Then, the problem has a unique solution in $C[0,1]$*.*

Proof. We know that $C[0,1]$ with $\eth$ the supremum metric is complete DMS. Define a selfmap $\mathfrak{I}$ on $C[0,1]$ by

$$(\mathfrak{I}u)(\check{\mathfrak{S}}) = \lambda \int_0^1 K(\check{\mathfrak{S}}, \varkappa) f(\varkappa, u(\varkappa)) \eth\varkappa,$$

for $u \in C[0,1]$ and $\check{\mathfrak{S}} \in [0,1]$. Then the fixed point of $\mathfrak{I}$ is the solution of the boundary value problem (4.1) and (4.2).
Now we will show that all the hypothesis of Theorem 3.1 and contractivity condition of Corollary 3.4 are satisfied .
To prove that the relation $\mathcal{H}$ is $\mathfrak{I}$-closed, take $u, v \in \check{G}$ such that $u\mathcal{H}v$, i.e., By the assumption, we have $u_0 = 0 \in C[0,1]$ such that

$$\begin{aligned} 0 = u(\check{\mathfrak{S}}) &= \lambda \int_0^1 K(\check{\mathfrak{S}}, \varkappa) f(\varkappa, u(\varkappa)) \eth\varkappa, \\ &\leq \mathfrak{I}u_0 = 0, \end{aligned}$$

this implies that $u_0 \mathcal{H} \mathfrak{I} u_0$, implying thereby $\check{G}(\mathfrak{I}, \mathcal{H})$ is non-empty.
Now we prove that the relation $\mathcal{H}$ to be $\mathfrak{I}$-closed, choose $u, v \in C[0,1]$ such that $u\mathcal{H}v$, then for $v(t) \geq u(t)$

$$\begin{aligned} K(\check{\mathfrak{S}}, \varkappa) f(\varkappa, u(\varkappa)) &\leq K(\check{\mathfrak{S}}, \varkappa) f(\varkappa, v(\varkappa)) \\ \int_0^1 K(\check{\mathfrak{S}}, \varkappa) f(\varkappa, u(\varkappa)) \eth\varkappa &\leq \int_0^1 K(\check{\mathfrak{S}}, \varkappa) f(\varkappa, v(\varkappa)) \eth\varkappa \\ \mathfrak{I}u(\check{\mathfrak{S}}) &\leq \mathfrak{I}v(\check{\mathfrak{S}}), \end{aligned}$$

implies that $(\mathfrak{I}u, \mathfrak{I}v) \in \mathcal{H}$. Hence, $\mathcal{H}$ is $\mathfrak{I}$-closed. Firstly, assume that conditions (*i*) holds. Then for $v(t) \geq u(t)$ we have

$$\begin{aligned} &\lambda K(\check{\mathfrak{S}}, \varkappa) |f(\varkappa, u(\varkappa)) - f(\varkappa, v(\varkappa))| \\ &\qquad \leq \beta \lambda K(\check{\mathfrak{S}}, \varkappa) |f(\varkappa, u(\varkappa))| - \beta \lambda K(\check{\mathfrak{S}}, \varkappa) \sup_{0 \leq \check{\mathfrak{S}} \leq 1} |u(\check{\mathfrak{S}})|. \end{aligned}$$

So

$$\int_0^1 \lambda K(\check{\mathfrak{S}},\varkappa)|f(\varkappa,u(\varkappa))-f(\varkappa,v(\varkappa))| \le \int_0^1 \beta\lambda K(\check{\mathfrak{S}},\varkappa)|f(\varkappa,u(\varkappa))| - \int_0^1 \beta\lambda K(\check{\mathfrak{S}},\varkappa)\sup_{0\le\check{\mathfrak{S}}\le 1}|u(\check{\mathfrak{S}})|\eth\varkappa.$$

$$\begin{aligned}\Longrightarrow \int_0^1 &|\lambda K(\check{\mathfrak{S}},\varkappa)f(\varkappa,u(\varkappa))-\lambda K(\check{\mathfrak{S}},\varkappa)f(\varkappa,v(\varkappa))|\\ &\le \int_0^1 \beta\lambda K(\check{\mathfrak{S}},\varkappa)|f(\varkappa,u(\varkappa))| - \int_0^1 \beta\lambda K A_1 \sup_{0\le\check{\mathfrak{S}}\le 1}|u(\check{\mathfrak{S}})|\eth\varkappa.\\ &\le \int_0^1 \beta\lambda K(\check{\mathfrak{S}},\varkappa)|f(\varkappa,u(\varkappa))|\eth\varkappa - \beta\lambda K A_1|u(\check{\mathfrak{S}})|\\ &\le \int_0^1 \beta\lambda K(\check{\mathfrak{S}},\varkappa)|f(\varkappa,u(\varkappa))|\eth\varkappa - \beta|u(\check{\mathfrak{S}})|\\ &\le \beta\Big|\int_0^1 \lambda K(\check{\mathfrak{S}},\varkappa)f(\varkappa,u(\varkappa))\eth\varkappa - u(\check{\mathfrak{S}})\Big|.\end{aligned}$$

Therefore, for any $u,v\in C[0,1]$ and $\check{\mathfrak{S}}\in[0,1]$, we have

$$\begin{aligned}\big(|(\Im u)(\check{\mathfrak{S}})-(\Im v)(\varkappa)|\big) &= \lambda\Big|\int_0^1 K(\check{\mathfrak{S}},\varkappa)f(\varkappa,u(\varkappa))\eth\varkappa - \int_0^1 K(\check{\mathfrak{S}},\varkappa)f(\varkappa,v(\varkappa))\eth\varkappa\Big|\\ &\le \lambda\int_0^1 K(\check{\mathfrak{S}},\varkappa)|f(\varkappa,u(\varkappa))-f(\varkappa,v(\varkappa))|\eth\varkappa\\ &\le \beta\Big|\int_0^1 \beta\lambda K(\check{\mathfrak{S}},\varkappa)f(\varkappa,u(\varkappa))\eth\varkappa - u(\check{\mathfrak{S}})\Big|\\ &= \beta|(Tu)(\check{\mathfrak{S}})-u(\check{\mathfrak{S}})|.\end{aligned}$$

So,

$$\eth(\Im u,\Im v)\le\beta\eth(\Im u,u)\le M(u,v).$$

Similarly,

$$\eth(\Im u,\Im v)\le\beta\eth(\Im u,u)\le N(u,v).$$

Now, if we choose the condition (ii), then

$$\begin{aligned}\big(|(\Im u)(\check{\mathfrak{S}})-(\Im v)(\check{\mathfrak{S}})|\big) &= \lambda\Big|\int_0^1 K(\check{\mathfrak{S}},\varkappa)f(\varkappa,u(\varkappa))\eth\varkappa - \int_0^1 K(\check{\mathfrak{S}},\varkappa)f(\varkappa,v(\varkappa))\eth\varkappa\Big|\\ &\le \lambda\int_0^1 K(\check{\mathfrak{S}},\varkappa)|f(\varkappa,u(\varkappa))\eth\varkappa - f(\varkappa,v(\varkappa))|\eth\varkappa\\ &\le \lambda\beta A_0\int_0^1 |u(\varkappa)-u(\check{\mathfrak{S}})|\\ &= \lambda\beta A_0\eth(u,v)\\ &\le \beta\eth(u,v).\end{aligned}$$

Therefore,

$$\eth(\Im u,\Im v)\le\beta\eth(u,u)\le M(u,v).$$

Similarly,

$$\eth(\Im u, \Im v) \leq \beta \eth(u, u) \leq N(u, v).$$

Then for $u, v \in C[0,1]$ and the assumption of (i) and (ii) we have

$$\phi(\eth(\Im u, \Im v)) \leq \phi(M(\eth(u, v))) - \psi(M(\eth(u, v))). \tag{4.4}$$

Using Remark 2.4, we have

$$\phi(\eth(\Im u, \Im v)) \leq \phi(M(\eth(u, v))) - \psi(N(\eth(u, v))). \tag{4.5}$$

For $\phi(t) = t$ and $\psi(t) = (1 - \beta)t$, equation (4.3) holds true. Then we can say that $\Im$ satisfy the contractive condition of the Theorem 3.1.

For $\mathcal{H}$ to be $\eth$-self closed, consider $\{u_n\}$ a $\mathcal{H}$-preserving Cauchy sequence converging to $u \in C[0,1]$. As $\{u_n\}$ is $\mathcal{H}$- preserving, we have

$$u_0(\check{\mathfrak{S}}) \leq u_1(\check{\mathfrak{S}}) \leq u_2(\check{\mathfrak{S}}) \leq \cdots \leq u_n(\check{\mathfrak{S}}) \leq \check{\mathfrak{S}}_{n+1}(\check{\mathfrak{S}}) \leq \cdots \leq u(\check{\mathfrak{S}}) \qquad \check{\mathfrak{S}} \in [0,1],$$

then we have $u_n \mathcal{H} u$ $\forall n \in \mathbb{N}$. Therefore, $\mathcal{H}$ is $\eth$- self closed.

Hence, we verify that all conditions of Theorem 3.1 are satisfied. Hence $\Im$ has a fixed point, which amounts to say that the integral equation (4.3) has a solution. Finally we can say that the fractional differential equations (4.1) & (4.2) have a solution in $C[0,1]$. □

Now we can give an example in support of Theorem 3.1.

Example 4.4. Suppose $\check{G} = [0,1]$ and define $\theta : \check{G} \times \check{G} \to \mathbb{R}^+$ by

$$\theta(\check{\mathfrak{S}}, \varkappa) = \begin{cases} 6\check{\mathfrak{S}} & if\ \check{\mathfrak{S}} = \varkappa \\ \max\{\check{\mathfrak{S}}, \varkappa\}, & elsewhere. \end{cases}$$

and define binary relation $\mathcal{H}$ by $\mathcal{H} = \{(\check{\mathfrak{S}}, \varkappa) \in \check{G}^2 : \varkappa \geq \check{\mathfrak{S}}\ and\ \varkappa < 1\}$, Then $(\check{G}, \theta)$ is a metric-like space which is neither a partial metric space nor a metric space. Also $(\check{G}, \theta)$ is a $\mathcal{H}$-complete DMS. Take $\Im : \check{G} \to \check{G}$ defined by

$$\Im(\check{\mathfrak{S}}) = \begin{cases} \frac{\check{\mathfrak{S}}}{6}\ if\ \check{\mathfrak{S}} \in [0,1) \\ 0,\ if\ \check{\mathfrak{S}} = 1. \end{cases}$$

Then $\mathcal{H}$ is $\Im$-closed. Let $\{\check{\mathfrak{S}}_n\}$ be an arbitrary sequence such that $\check{\mathfrak{S}}_n \xrightarrow{\tau_\theta} \check{\mathfrak{S}}$ (for some $\check{\mathfrak{S}} \in \check{G}$), i.e., $\{\check{\mathfrak{S}}_n\}$ is a sequence in $[0,1)$ such that $\check{\mathfrak{S}}_n \leq \check{\mathfrak{S}}_{n+1}$ $\forall$ n with $\lim_{n\to\infty} \theta(\check{\mathfrak{S}}_n, \check{\mathfrak{S}}) = \theta(\check{\mathfrak{S}}, \check{\mathfrak{S}})$. Then for $\check{\mathfrak{S}} \in [0,1)$ and

$$\begin{aligned} \theta(\Im\check{\mathfrak{S}}, \Im\check{\mathfrak{S}}) = \theta(\frac{\check{\mathfrak{S}}}{6}, \frac{\check{\mathfrak{S}}}{6}) = \check{\mathfrak{S}} &= \frac{1}{6}(\theta(\check{\mathfrak{S}}, \check{\mathfrak{S}})) \\ &= \frac{1}{6}(\lim_{n\to\infty} \theta(\check{\mathfrak{S}}_n, \check{\mathfrak{S}})) \\ &= \frac{1}{6}\left(\lim_{n\to\infty} \left(\begin{cases} 6\check{\mathfrak{S}}_n & if\ \check{\mathfrak{S}}_n = \varkappa \\ \max\{\check{\mathfrak{S}}_n, \varkappa\}, & elsewhere. \end{cases} \right) \right) \\ &= \lim_{n\to\infty} \begin{cases} 6\frac{\check{\mathfrak{S}}_n}{6} & if\ \check{\mathfrak{S}}_n = \varkappa \\ \max\{\frac{\check{\mathfrak{S}}_n}{6}, \frac{\varkappa}{6}\}, & elsewhere. \end{cases} \end{aligned}$$

$$= \lim_{n\to\infty} \theta(\Im\check{\mathfrak{S}}_n, \Im\check{\mathfrak{S}}).$$

This shows that $\Im\check{\mathfrak{S}}_n \xrightarrow{\tau_\theta} \Im\check{\mathfrak{S}}$ and hence $\Im$ is an $\mathcal{H}$-sequentially-continuous. Now, for any $(\check{\mathfrak{S}}, \varkappa) \in \check{\boldsymbol{G}}\times\check{\boldsymbol{G}}$ with $(\check{\mathfrak{S}}, \varkappa) \in \mathcal{H}$, one can easily verify that $\frac{1}{6}\theta(\check{\mathfrak{S}}, \varkappa) \leq M(\check{\mathfrak{S}}, \varkappa)$ and $\frac{1}{6}\theta(\check{\mathfrak{S}}, \varkappa) \leq N(\check{\mathfrak{S}}, \varkappa)$, then we have

$$\begin{aligned}
\theta(\Im\check{\mathfrak{S}}, \Im\varkappa) &= \theta\Big(\frac{\check{\mathfrak{S}}}{6}, \frac{\varkappa}{6}\Big) \\
&= \begin{cases} \check{\mathfrak{S}} & if\ \check{\mathfrak{S}} = \varkappa \\ \max\{\frac{\check{\mathfrak{S}}}{6}, \frac{\varkappa}{6}\}, & elsewhere. \end{cases} \\
&= \frac{1}{6}\begin{cases} 6\check{\mathfrak{S}} & if\ \check{\mathfrak{S}} = \varkappa \\ \max\{\check{\mathfrak{S}}, \varkappa\}, & elsewhere. \end{cases} \\
&= \frac{1}{6}\theta(\check{\mathfrak{S}}, \varkappa) \\
&= \frac{1}{6}\theta(\check{\mathfrak{S}}, \varkappa) - \frac{[\frac{1}{6}]\theta(\check{\mathfrak{S}}, \varkappa)}{10} \\
&\leq M(\check{\mathfrak{S}}, \varkappa) - \frac{[N(\check{\mathfrak{S}}, \varkappa)]}{10} \\
\phi(\theta(T\check{\mathfrak{S}}, T\varkappa)) &\leq \phi(M(\check{\mathfrak{S}}, \varkappa)) - \psi(N(\check{\mathfrak{S}}, \varkappa))
\end{aligned}$$

which implies that $\Im$ satisfies the assumption (v) of Theorem 3.1 for $\phi(t) = t$ and $\psi(t) = \frac{[t]}{10}$. Consequently all the conditions of Theorem 3.1 are satisfied and hence $\Im$ has fixed point namely $\check{\mathfrak{S}} = 0$. Further the fixed point is also unique.

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