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NONLINEAR FUNCTIONAL CONTRACTIONS VIA LOCALLY FINITELY ζ-TRANSITIVE BINARY RELATION AND APPLICATIONS TO ELASTIC BEAM EQUATIONS

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ABSTRACT. This paper comprises a few fixed point theorems for relational nonlinear functional contractions with a wider class of transitive relations. Our findings sharpen and subsume a number of existing results. We offer a few examples that illustrate the credibility of our findings. We adapt our findings to obtain a unique solution of a boundary value problem connected to a nonlinear cantilever beam equation that incorporates Euler-Bernoulli's hypotheses

1. Introduction

The strength of metric fixed point theory lies in its wide range of applications to different areas. For recent works related to applications of metric fixed point theory, readers are referred to [7, 8]. BCP (Banach contraction principle) being a key of metric fixed point theory continues to generalize in various directions. Alam and Imdad [2] investigated a new extension of BCP in the setup of relation metric space. Due to its ingenuity, the result of Alam and Imdad [2] has been enlarged and expanded by numerous scholars, e.g., [1, 3–6, 9–11, 14, 18–20]. The contraction employed in such results requires met for comparative elements. It is apparent that relational contractions are still deeper than corresponding usual contractions.

The significance of 4th-order two-point boundary value problems (abbreviated as "BVP") lies in analysis and engineering. These BVPs are used in material mechanics, physics, chemical sensors, micro-electromechanical systems, medical diagnostics and aircraft design to characterize the deflection of the elastic beam under equilibrium state. Several researchers are utilized monotone iterative methods to explain the presence of positive solutions for the elastic beam equations.

Let us consider the general elastic beam equation given as under:

$$\begin{cases} \mathbf{w}''''(\nu) = \hbar(\nu, \mathbf{w}(\nu), \mathbf{w}'(\nu), \mathbf{w}''(\nu), \mathbf{w}'''(\nu)), \quad 0 \le \nu \le 1, \\ \mathbf{w}(0) = \mathbf{w}'(0) = \mathbf{w}''(1) = \mathbf{w}'''(1) = 0 \end{cases} \tag{1.1}$$

where the function $\hbar : [0,1] \times [0,\infty)^3 \to [0,\infty)$ is continuous. In Eq. (1.1), $\mathbf{w}$ denotes load density stiffness, $\mathbf{w}'''(\nu)$ denotes shear force stiffness, $\mathbf{w}''(\nu)$ denotes bending moment stiffness and $\mathbf{w}'(\nu)$ remains slope of the elastic beam model.

The objective of the article is to investigate new metrical outcomes on fixed points for certain nonlinear functional contractions via a locally finitely ζ-transitive relation. A few examples are supplied to back up our findings. By applying our

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findings, we explore the guarantees of a unique solution of a BVP which remains a special case of (1.1).

2. Preliminaries

The sets of natural numbers, whole numbers, and real numbers are represented by their respective standard symbols: $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{R}$. A relation on a set $\mathcal{H}$ is defined as a subset of $\mathcal{H}^2$. In the continuation, let $\mathcal{H}$ be a set, ξ a relation on $\mathcal{H}$, d a metric on $\mathcal{H}$ and $\zeta : \mathcal{H} \to \mathcal{H}$ a mapping.

Definition 2.1 ([2])**.** We say that $p, q \in \mathcal{H}$ are ξ-comparative if $(p, q) \in \xi$, or $(q, p) \in \xi$. Such a pair is usually denoted by $[p, q] \in \xi$.

Definition 2.2 ([16])**.** $\xi^{-1} := \{(p, q) \in \mathcal{H}^2 : (q, p) \in \xi\}$ is named as inverse of ξ.

Definition 2.3 ([16])**.** $\xi^s := \xi \cup \xi^{-1}$ is called symmetric closure of ξ. Obviously, $(p, q) \in \xi^s \iff [p, q] \in \xi$.

Definition 2.4 ([2])**.** ξ is named as ζ-closed if for any $p, q \in \mathcal{H}$ with $(p, q) \in \xi$, we have

$$(\zeta p, \zeta q) \in \xi.$$

Definition 2.5 ([2])**.** $\{p_k\} \subset \mathcal{H}$ is termed as a ξ-preserving sequence if $(p_k, p_{k+1}) \in \xi$, $\forall\ k \in \mathbb{N}$.

Definition 2.6 ([3])**.** $(\mathcal{H}, d)$ is termed as ξ-complete metric space if any ξ-preserving Cauchy sequence in $\mathcal{H}$ converge.

Definition 2.7 ([3])**.** ζ is termed as ξ-continuous if for each $p \in \mathcal{H}$ and for any ξ-preserving sequence $\{p_k\} \subset \mathcal{H}$ with $p_k \xrightarrow{d} p$,

$$\zeta(p_k) \xrightarrow{d} \zeta(p).$$

Definition 2.8 ([2])**.** ξ is termed as d-self-closed if each ξ-preserving convergent sequence $\{p_k\} \subset \mathcal{H}$ with $p_k \xrightarrow{d} p \in \Upsilon$ admits a subsequence $\{p_{k_i}\}$ with $[p_{k_i}, p] \in \xi$.

Definition 2.9 ([15])**.** Given $\mathcal{K} \subseteq \mathcal{H}$, the set $\xi|_{\mathcal{K}} := \xi \cap \mathcal{K}^2$ being a relation on $\mathcal{K}$ is termed as restriction of ξ on $\mathcal{K}$.

Definition 2.10 ([4])**.** ξ is termed as locally ζ-transitive if for every ξ-preserving sequence $\{q_k\} \subset \zeta(\mathcal{H})$ with range $\mathcal{K} = \{q_k : k \in \mathbb{N}\}$, $\xi|_{\mathcal{K}}$ is transitive.

Definition 2.11 ([12])**.** Given $\aleph \in \mathbb{N} - \{1\}$, ξ is named $\aleph$-transitive if for each $p_0, p_1, \ldots, p_\aleph \in \Upsilon$,

$$(p_{i-1}, p_i) \in \xi \text{ for each } i\ (1 \leq i \leq \aleph) \Rightarrow (p_0, p_\aleph) \in \xi.$$

Thus, the ideas of usual transitivity and 2-transitivity are equivalent.

Definition 2.12 ([21])**.** ξ is termed as finitely transitive if $\exists\ \aleph \in \mathbb{N} - \{1\}$ for which ξ is $\aleph$-transitive.

Definition 2.13 ([5])**.** ξ is termed as locally finitely ζ-transitive if for every ξ-preserving sequence $\{q_k\} \subset \zeta(\mathcal{H})$ with range $\mathcal{K} = \{q_k : k \in \mathbb{N}\}$, $\xi|_{\mathcal{K}}$ is finitely transitive.

Remark 2.14. finitely transitivity implies locally finitely ζ-transitivity. Also, locally ζ-transitivity implies locally finitely ζ-transitivity.

Definition 2.15 ([17]). $\mathcal{K} \subseteq \mathcal{H}$ is termed as ξ-directed if for any $p, q \in \mathcal{K}$, $\exists\ w \in \mathcal{H}$ with $(p, w) \in \xi$ and $(q, w) \in \xi$.

Proposition 2.16 ([4]). *If ξ is ζ-closed, then ξ is ζ^k-closed, $\forall\ k \in \mathbb{N}_0$.*

Lemma 2.17 ([21]). *If $\mathcal{H}$ is a set endued with a relation ξ, $\{p_k\} \subset \Upsilon$ is ξ-preserving sequence and ξ is an $\aleph$-transitive on $\mathcal{K} = \{p_k : k \in \mathbb{N}_0\}$, then*

$$(p_k, p_{k+1+\upsilon(\aleph-1)}) \in \xi,\ \forall\ k, \upsilon \in \mathbb{N}_0.$$

Lemma 2.18 ([12]). *If a sequence $\{p_k\}$ is not Cauchy in a metric space $(\mathcal{H}, d)$, then $\exists\ \varepsilon_0 > 0$ and $\exists$ subsequences $\{p_{k_i}\}$ and $\{p_{l_i}\}$ of $\{p_k\}$ enjoying the properties:*

(i) $i \leq l_i < k_i,\ \ \forall\ i \in \mathbb{N}$,
(ii) $d(p_{l_i}, p_{k_i}) \geq \varepsilon_0,\ \ \forall\ k \in \mathbb{N}$,
(iii) $d(p_{l_i}, p_{\nu_k}) < \varepsilon_0,\ \forall\ \nu_k \in \{l_i + 1, l_i + 2, \ldots, k_i - 2, k_i - 1\}$.

Moreover, if $\lim_{k\to\infty} d(p_k, p_{k+1}) = 0$, then

$$\lim_{k\to\infty} d(p_{l_i}, p_{k_i+\upsilon}) = \varepsilon_0,\ \forall\ \upsilon \in \mathbb{N}_0.$$

The following families of functions were proposed, respectively, by Alam et al. [5] and Jleli et al. [13]:

$$\Omega = \{\sigma : [0, \infty) \to [0, \infty) : t \in (0, \infty) \Rightarrow \sigma(t) < t \text{ and } \limsup_{r\to t} \sigma(r) < t\}$$

and

$$\begin{aligned}\Psi \ = \ & \{\psi : [0, \infty)^4 \to [0, \infty) : \psi \text{ is continuous and} \\ & \psi(t_1, t_2, t_3, t_4) = 0 \Leftrightarrow t_i = 0, \text{ for some } i = 1, 2, 3, 4\}.\end{aligned}$$

Proposition 2.19. *Given $\sigma \in \Omega$ and $\psi \in \Psi$, (A) and (B) are equivalent:*

(A) $d(\zeta p, \zeta q) \leq \sigma(d(p, q)) + \psi(d(p, \zeta p), d(q, \zeta q), d(p, \zeta q), d(q, \zeta p))$,
$\forall\ p, q \in \mathcal{H}$ *with* $(p, q) \in \xi$.
(B) $d(\zeta p, \zeta q) \leq \sigma(d(p, q)) + \psi(d(p, \zeta p), d(q, \zeta q), d(p, \zeta q), d(q, \zeta p))$,
$\forall\ p, q \in \mathcal{H}$ *with* $[p, q] \in \xi$.

Proof. The conclusion arrives instantly utilising the symmetric character of metric d. □

3. Main results

This section resolves the fixed point results to meet a relational nonlinear functional contraction.

Theorem 3.1. *Let $(\mathcal{H}, d)$ be a metric space, $\zeta : \mathcal{H} \to \mathcal{H}$ a map, and ξ a relation on $\mathcal{H}$. Also,*

(i) $\exists\ p_0 \in \mathcal{H}$ *satisfying* $(p_0, \zeta p_0) \in \xi$,
(ii) ξ *remains ζ-closed and locally finitely ζ-transitive,*
(iii) $(\mathcal{H}, d)$ *continues to be ξ-complete,*
(iv) $\mathcal{H}$ *remains ξ-continuous, or ξ continues to be d-self-closed,*

(v) $\exists\ \sigma \in \Omega$ *and* $\psi \in \Psi$ *verifying*

$$d(\zeta p, \zeta q) \leq \sigma(d(p,q)) + \psi(d(p,\zeta p), d(q,\zeta q), d(p,\zeta q), d(q,\zeta p)),$$
$$\forall\ p, q \in \mathcal{H} \text{ with } (p,q) \in \xi.$$

Then, ζ admits a fixed point.

Proof. Beginning with $p_0 \in \mathcal{H}$, we construct a sequence $\{p_k\} \subset \mathcal{H}$ that fulfills

$$p_k := \zeta^k(p_0) = \zeta(p_{k-1}), \quad \forall\ k \in \mathbb{N}. \tag{3.1}$$

Using (i), (ii) and Proposition 2.16, we get

$$(\zeta^k p_0, \zeta^{k+1} p_0) \in \xi,$$

which owing to (3.1) becomes

$$(p_k, p_{k+1}) \in \xi, \quad \forall\ k \in \mathbb{N}_0. \tag{3.2}$$

This yields that $\{p_k\}$ is a ξ-preserving sequence.

Define

$$d_k := d(p_k, p_{k+1}).$$

If $d_{k_0} = 0$ for some $k_0 \in \mathbb{N}_0$, then we have $\zeta(p_{k_0}) = p_{k_0}$ and hence p_{k_0} is a fixed point of ζ. Thus the proof is concluded.

In either case, we have $d_k > 0,\ \forall\ k \in \mathbb{N}_0$. Using (3.1), (3.2) and condition (v), we get

$$\begin{aligned} d_k &= d(p_k, p_{k+1}) = d(\zeta p_{k-1}, \zeta p_k) \\ &\leq \sigma(d(p_{k-1}, p_k)) + \psi(d(p_{k-1}, p_k), d(p_k, p_{k+1}), d(p_{k-1}, p_{k+1}), 0), \end{aligned}$$

which using the property of Ψ reduces to

$$d_k \leq \sigma(d_{k-1}), \quad \forall\ k \in \mathbb{N}_0. \tag{3.3}$$

Utilizing the property of Ω in (3.3), we get

$$d_k < d_{k-1}, \quad \forall\ k \in \mathbb{N}.$$

Thus $\exists\ d^* \geq 0$ verifying

$$\lim_{k\to\infty} d_k = d^*. \tag{3.4}$$

We now claim that $d^* = 0$. In either case $d^* > 0$, letting the upper limit in (3.3) and employing (3.4) and the definition of Ω, we get

$$d^* = \limsup_{k\to\infty} d_k \leq \limsup_{k\to\infty} \sigma(d_{k-1}) = \limsup_{d_k \to l^+} \sigma(d_{k-1}) < d^*,$$

which is not possible. Hence

$$\lim_{k\to\infty} d_k = 0. \tag{3.5}$$

If possible, assume that $\{p_k\}$ is not Cauchy. According to Lemma 2.18, $\exists\ \varepsilon_0 > 0$ and $\exists$ subsequences $\{p_{k_i}\}$ and $\{p_{l_i}\}$ of $\{p_k\}$ satisfying

$$k \leq l_i < k_i,\ d(p_{l_i}, p_{k_i}) \geq \varepsilon_0 > d(p_{l_i}, p_{\nu_k}),\ \forall\ k \in \mathbb{N},\ \nu_k \in \{l_i+1, l_i+2, \ldots, k_i-2, k_i-1\}.$$

By (3.4) and Lemma 2.18, we find

$$\lim_{k\to\infty} d(p_{l_i}, p_{k_i+\upsilon}) = \varepsilon_0,\ \forall\ \upsilon \in \mathbb{N}_0. \tag{3.6}$$

Due to (3.1), we have $\mathcal{K} := \{p_k : k \in \mathbb{N}_0\} \subset \zeta(\mathcal{H})$. Making use of locally finitely ζ-transitivity of ξ, $\exists\ \aleph \geq 2$, such that $\xi|_{\mathcal{K}}$ is $\aleph$-transitive.

We have $l_i < k_i$ and $\aleph - 1 > 0$, Thus by division algorithm, we have

$$k_i - l_i = (\aleph - 1)(\tau_k - 1) + (\aleph - \theta_k)$$

$$\tau_k - 1 \geq 0,\ 0 \leq \aleph - \theta_k < \aleph - 1$$

$$\iff \begin{cases} k_i + \theta_k = l_i + 1 + (\aleph - 1)\tau_k \\ \quad \tau_k \geq 1,\ 1 < \theta_k \leq \aleph. \end{cases}$$

Obviously, $\theta_k \in (1, \aleph]$. Assume that the subsequences $\{p_{k_i}\}$ and $\{p_{l_i}\}$ of $\{p_k\}$ (satisfying (3.6)) are chosen such that $\theta_k = \theta$ is a constant and so

$$l_i' = k_i + \theta = l_i + 1 + (\aleph - 1)\tau_k. \tag{3.7}$$

By (3.6) and (3.7), we get

$$\lim_{k\to\infty} d(p_{l_i}, p_{l_i'}) = \lim_{k\to\infty} d(p_{l_i}, p_{k_i+\theta}) = \epsilon. \tag{3.8}$$

Making use of triangular inequality, we get

$$d(p_{l_i+1}, p_{l_i'+1}) \quad \leq \quad d(p_{l_i+1}, p_{l_i}) + d(p_{l_i}, p_{l_i'}) + d(p_{l_i'}, p_{l_i'+1})$$

and

$$d(p_{l_i}, p_{l_i'}) \quad \leq \quad d(p_{l_i}, p_{l_i+1}) + d(p_{\mu_{k+1}}, p_{l_i'+1}) + d(p_{l_i'+1}, p_{l_i'}).$$

Therefore, we have

$$\begin{aligned} d(p_{l_i}, p_{l_i'}) - d(p_{l_i}, p_{l_i+1}) - d(p_{l_i'+1}, p_{l_i'}) &\leq d(p_{\mu_{k+1}}, p_{l_i'+1}) \\ &\leq d(p_{l_i+1}, p_{l_i}) + d(p_{l_i}, p_{l_i'}) + d(p_{l_i'}, p_{l_i'+1}) \end{aligned}$$

which on using $k \to \infty$ and by (3.6) and (3.8) reduces to

$$\lim_{k\to\infty} d(p_{l_i+1}, p_{l_i'+1}) = \epsilon. \tag{3.9}$$

Using (3.7) and Lemma 2.18, we find $(p_{l_i}, p_{l_i'}) \in \xi$. Denote $\delta_k := d(p_{l_i}, p_{\mu_k'})$. By condition (v), we get

$$\begin{aligned} d(p_{l_i+1}, p_{\mu_k'+1}) &= d(\zeta p_{l_i}, \zeta p_{\mu_k'}) \\ &\leq \sigma(d(p_{l_i}, p_{\mu_k'})) \\ &\quad + \psi\{d(p_{l_i}, \zeta p_{l_i}), d(p_{\mu_k'}, \zeta p_{\mu_k'}), d(p_{l_i}, \zeta p_{\mu_k'}), d(p_{\mu_k'}, \zeta p_{l_i})\} \end{aligned}$$

so that

$$d(p_{l_i+1}, p_{\mu_k'+1}) \leq \sigma(\delta_k) + \psi\{d_{l_i}, d_{\mu_k'}, d(p_{l_i}, p_{\mu_k'+1}), d(p_{\mu_k'}, p_{l_i+1})\}. \tag{3.10}$$

Employing limit superior in (3.10) and by Lemma (2.18) and the properties of Ω and Ψ, we obtain

$$\varepsilon_0 = \limsup_{k\to\infty} d(p_{l_i+1}, p_{\mu_k'+1}) \leq \limsup_{k\to\infty} \sigma(\delta_k) + \psi(0, 0, \varepsilon_0, \varepsilon_0) = \limsup_{t\to\varepsilon_0^+} \sigma(t) < \varepsilon_0,$$

which remains contradiction. Hence $\{p_k\}$ remains Cauchy. But $\{p_k\}$ is also ξ-preserving; so by ξ-completeness of $\mathcal{H}$, $\exists\ \bar{p} \in \mathcal{H}$ such that $p_k \xrightarrow{d} \bar{p}$.

In view of (iv), assuming firstly that ζ is ξ-continuous. Consequently, $\{p_k\}$ being ξ-preserving verifying $p_k \xrightarrow{d} \bar{p}$, provides $p_{k+1} = \zeta(p_k) \xrightarrow{d} \zeta(\bar{p})$. This implies that $\zeta(\bar{p}) = \bar{p}$.

Secondly, assuming that ξ is d-self-closed; therefore $\{p_{k_i}\}$ of $\{p_k\}$ with $[p_{k_i}, \bar{p}] \in \xi$, $\forall\ i \in \mathbb{N}$. Utilizing (v), Proposition 2.19 and $[p_{k_i}, \bar{p}] \in \xi$, we obtain

$$\begin{aligned} d(p_{k_i+1}, \zeta\bar{p}) &= d(\zeta p_{k_i}, \zeta\bar{p}) \\ &\leq \sigma(d(p_{k_i}, \bar{p})) + \psi(d(p_{k_i}, p_{k_i+1}), 0, d(p_{k_i}, \bar{p}), d(\bar{p}, p_{k_i+1})) \\ &= \sigma(d(p_{k_i}, \bar{p})). \end{aligned}$$

We claim that

$$d(p_{k_i+1}, \zeta\bar{p}) \leq d(p_{k_i}, \bar{p}), \quad \forall\ i \in \mathbb{N}. \tag{3.11}$$

If for some $k_0 \in \mathbb{N}$, $d(p_{k_{k_0}}, \bar{p}) = 0$, then we get $d(\zeta p_{k_{k_0}}, \zeta\bar{p}) = 0$, i.e., $d(p_{k_{k_0}+1}, \zeta\bar{p}) = 0$ and hence (3.11) holds for these $k_0 \in \mathbb{N}$. If $d(p_{k_i}, \bar{p}) > 0$, $\forall\ i \in \mathbb{N}$, then we have $d(p_{k_i+1}, \zeta\bar{p}) \leq \sigma(d(p_{k_i}, \bar{p})) < d(p_{k_i}, \bar{p})$, $\forall\ i \in \mathbb{N}$. Thus, the inequality (3.11) is verified. Letting limit of (3.11) and $p_{k_i} \xrightarrow{d} \bar{p}$, we conclude $p_{k_i+1} \xrightarrow{d} \zeta(\bar{p})$ and so $\zeta(\bar{p}) = \bar{p}$. Therefore, $\bar{p}$ serves as a fixed point of ζ. □

Theorem 3.2. *In the collaboration to assertions of Theorem 3.1, if $\zeta(\mathcal{H})$ is ξ^s-directed, then ζ owns a unique fixed point.*

Proof. If possible, $\exists\ \bar{p}, \bar{q} \in \mathcal{H}$ such that

$$\zeta(\bar{p}) = \bar{p} \text{ and } \zeta(\bar{q}) = \bar{q}. \tag{3.12}$$

Owing to $\bar{p}, \bar{q} \in \zeta(\mathcal{H})$, $\exists\ w \in \mathcal{H}$ such that

$$[\bar{p}, w] \in \xi \quad \text{and} \quad [\bar{q}, w] \in \xi. \tag{3.13}$$

Set $\rho_k := d(\bar{p}, \zeta^k w)$. By (3.12), (3.13) and (v), we conclude

$$\begin{aligned} \rho_k = d(\bar{p}, \zeta^k w) &= d(\zeta\bar{p}, \zeta(\zeta^{k-1} w)) \\ &\leq \sigma(d(\bar{p}, \zeta^{k-1} w)) + \psi(0, d(\zeta^{k-1} w, \zeta^k k), d(\bar{p}, \zeta^k w), d(\zeta^{k-1} w, \bar{p})) \\ &= \sigma(\rho_{k-1}) \end{aligned}$$

so that

$$\rho_k \leq \sigma(\rho_{k-1}). \tag{3.14}$$

Firstly, assume that $\exists\ k_0 \in \mathbb{N}$ such that $\rho_{k_0} = 0$. Then $\rho_{k_0} \leq \rho_{k_0-1}$. Secondly, we have $\rho_k > 0$, $\forall\ k \in \mathbb{N}$. In this case, (3.14) becomes $\rho_k < \rho_{k-1}$. Thus in each of the cases, we conclude

$$\rho_k \leq \rho_{k-1}.$$

Similar to Theorem 3.1, above inequality concludes

$$\lim_{k\to\infty} \rho_k = \lim_{k\to\infty} d(\bar{p}, \zeta^k w) = 0. \tag{3.15}$$

In the similar manner, we find

$$\lim_{k\to\infty} d(\bar{q}, \zeta^k w) = 0. \tag{3.16}$$

Combining (3.15) and (3.16), we get

$$d(\bar{p}, \bar{q}) = d(\bar{p}, \zeta^k w) + d(\zeta^k w, \bar{q}) \to 0 \quad \text{as} \quad k \to \infty$$

i.e., $\bar{p} = \bar{q}$. This verifies uniqueness of fixed point. □

Remark 3.3. Under the restriction $\psi(t_1, t_2, t_3, t_4) = 0$, Theorems 3.1 and 3.2 deduce corresponding theorems of Alam et al. [5].

Remark 3.4. In particular for $\psi(t_1, t_2, t_3, t_4) = r \cdot \min\{t_1, t_2, t_3, t_4\}$ (where $r \geq 0$), Theorems 3.1 and 3.2 deduce the corresponding theorems of Khan et al. [14].

4. Examples

In demonstrating Theorems 3.1 and 3.2, we go to the following examples.

Example 4.1. Let $\mathcal{H} = [0, \infty)$ with usual metric d. On $\mathcal{H}$, define a relation $\xi := \{(p, q) \in \mathcal{H}^2 : p - q > 0\}$. Define the map $\zeta : \mathcal{H} \to \mathcal{H}$ by $\zeta(p) = \frac{p}{p+1}$. Clearly, ξ is locally finitely ζ-transitive and ζ-closed, $(\mathcal{H}, d)$ is ξ-complete and ζ is ξ-continuous.

Define the test functions $\sigma(t) = \frac{t}{t+1}$ and $\psi(t_1, t_2, t_3, t_4) = t_1 t_2 t_3 t_4$. Then $\sigma \in \Omega$ and $\psi \in \Psi$. Now, for any $(p, q) \in \xi$, we conclude

$$\begin{aligned} d(\zeta p, \zeta q) &= \left| \frac{p - q}{1 + p + q + pq} \right| \leq \frac{d(p, q)}{1 + d(p, q)} \\ &\leq \sigma(d(p, q)) + \psi(d(p, \zeta p), d(q, \zeta q), d(p, \zeta q), d(q, \zeta p)). \end{aligned}$$

Thus, assertion (v) of Theorem 3.1 is verified. Remaining presumptions of Theorem 3.1 and Theorem 3.2 also hold; so ζ owns a unique fixed point, $\bar{p} = 0$.

Example 4.2. Let $\mathcal{H} = [0, 1]$ with usual metric d. On $\mathcal{H}$, define a relation $\xi := \leq$. Consider the map $\zeta : \mathcal{H} \to \mathcal{H}$ defined by

$$\zeta(p) = \begin{cases} p^2, & \text{if } p \in [0, 1/4) \\ 0, & \text{if } p \in [1/4, 1]. \end{cases}$$

Here, ξ is locally finitely ζ-transitive, ϕ-closed and d-self-closed. Also, $(\mathcal{H}, d)$ is ξ-complete. Define the test functions $\sigma(t) = t/2$ and $\psi(t_1, t_2, t_3, t_4) = \min\{t_3, t_4\}$. Then $\sigma \in \Omega$ and $\psi \in \Psi$. Condition (v) of Theorem 3.1 can easily be verified. Remaining presumptions of Theorem 3.1 and Theorem 3.2 also hold; so ζ owns a unique fixed point, $\bar{p} = 0$.

5. Solutions of nonlinear elastic beam equations

As an specific case of (1.1), consider the following differential equation:

$$\begin{cases} \mathbf{w}''''(\nu) = \hbar(\nu, \mathbf{w}(\nu)), \quad 0 \leq \nu \leq 1, \\ \mathbf{w}(0) = \mathbf{w}'(0) = \mathbf{w}''(1) = \mathbf{w}'''(1) = 0, \end{cases} \tag{5.1}$$

where $\hbar : [0,1] \times [0,\infty) \to [0,\infty)$ remains a continuous function. Eq. (5.1) is called cantilever beam in material mechanics. This equation describes the bending equilibria of elastic beam model having a length 1 with rigidly fastened ends.

Now, we present the existence and uniqueness theorem to compute a solution for the BVP (5.1).

Theorem 5.1. *In a furtherance of the problem* (5.1), *let us assume that* $\exists$ *a monotonic increasing function* $\sigma \in \Omega$ *that fulfills*

$$0 \leq \hbar(\nu, x) - \hbar(\nu, y) \leq \sigma(x - y), \tag{5.2}$$

$\forall\ \nu \in [0,1]$ *and* $\forall\ x, y \in \mathbb{R}$ *with* $x \geq y$. *If* $\exists\ \gamma \in C[0,1]$ *such that*

$$\gamma(\nu) \geq \int_0^1 \digamma(\nu, s)\hbar(s, \gamma(s))ds,\ \forall\ \nu \in [0,1], \tag{5.3}$$

where Green function $\digamma(\nu, s)$ *is defined as*

$$\digamma(\nu, s) = \frac{1}{6}\begin{cases} s^2(3\nu - s), & 0 \leq s \leq \nu \leq 1, \\ \nu^2(3s - \nu), & 0 \leq \nu \leq s \leq 1, \end{cases}$$

then BVP (5.1) *owns a unique solution.*

Proof. BVP (5.1) is equivalent to the integral equation:

$$\mathbf{w}(\nu) = \int_0^1 \digamma(\nu, s)\hbar(s, \mathbf{w}(s))ds,\ \forall\ \nu \in [0,1]. \tag{5.4}$$

It can be easily verified that

$$0 \leq \digamma(\nu, s) \leq \frac{1}{2}\nu^2 s,\ \forall\ \nu, s \in [0,1]. \tag{5.5}$$

On $\mathcal{H}$:=C[0,1], define a metric d by

$$d(\mathbf{w}, \mathbf{v}) = \max_{\nu \in [0,1]} |\mathbf{w}(\nu) - \mathbf{v}(\nu)|,\ \forall\ \mathbf{w}, \mathbf{v} \in \mathcal{H}.$$

Also, take a relation ξ on $\mathcal{H}$ as

$$(\mathbf{w}, \mathbf{v}) \in \xi \Leftrightarrow \mathbf{w}(\nu) \geq \mathbf{v}(\nu),\ \forall\ \mathbf{w}, \mathbf{v} \in \mathcal{H},\ \forall\ \nu \in [0,1].$$

Define a function $\zeta : \mathcal{H} \to \mathcal{H}$ by

$$\zeta(\mathbf{w})(\nu) = \int_0^1 \digamma(\nu, s)\hbar(s, \mathbf{w}(s))ds,\ \forall\ \nu \in [0,1],\ \forall\ \mathbf{w} \in \mathcal{H}.$$

We now check each of the assumptions of Theorems 3.1 and 3.2.

(i) By (5.3), we have $\gamma(\nu) \geq \zeta(\gamma)(\nu)$ so that $(\gamma, \zeta\gamma) \in \xi$.

(ii) ξ being transitive relation is locally finitely ζ-transitive. Now, take $\mathbf{w}, \mathbf{v} \in \mathcal{H}$ verifying $(\mathbf{w}, \mathbf{v}) \in \xi$. Using (5.2), (5.4) and the fact that $\digamma(\nu, s) > 0$, $\forall\ \nu, s \in [0,1]$, we get

$$\begin{aligned} \zeta(\mathbf{w})(\nu) &= \int_0^1 \digamma(\nu, s)\hbar(s, \mathbf{w}(s))ds \\ &\geq \int_0^1 \digamma(\nu, s)\hbar(s, \mathbf{v}(s))ds \\ &= \zeta(\mathbf{v})(\nu), \quad \forall\ \nu \in [0,1], \end{aligned}$$

so that $(\zeta\mathbf{w}, \zeta\mathbf{v}) \in \xi$ and hence ξ is ζ-closed.

(iii) Clearly, $(\mathcal{H}, d)$ is ξ-complete.

(iv) If $\{\mathbf{w}_k\} \subset \mathcal{H}$ is a ξ-preserving sequence converging to $\mathbf{w} \in \mathcal{H}$, then convergence theory in $\mathbb{R}$ yields that $\mathbf{w}_k(\nu) \geq \mathbf{w}(\nu)$, $\forall\ k \in \mathbb{N}$ and $\forall\ \nu \in [0,1]$. It turns out that $(\mathbf{w}_k, \mathbf{w}) \in \xi$, $\forall\ k \in \mathbb{N}$ so that ξ is d-self-closed.

(v) By (5.2), for all $\nu \in [0,1]$ and for all $\mathbf{w}, \mathbf{v} \in \mathcal{H}$ with $(\mathbf{w}, \mathbf{v}) \in \xi$, we have

$$\begin{aligned}
|\zeta(\mathbf{w})(\nu) - \zeta(\mathbf{v})(\nu)| &= \int_0^1 F(\nu, s)(\hbar(s, \mathbf{w}(s)) - \hbar(s, \mathbf{v}(s)))ds \\
&\leq \int_0^1 F(\nu, s)\sigma(\mathbf{w}(s) - \mathbf{v}(s))ds \\
&\leq \left(\int_0^1 F(\nu, s)ds\right)(\sigma(d(\mathbf{w}, \mathbf{v})) \qquad \text{(as } \sigma \text{ is increasing)} \\
&\leq \frac{\sigma(d(\mathbf{w}, \mathbf{v}))}{4} \qquad \text{(using (5.5))} \\
&\leq \sigma(d(\mathbf{w}, \mathbf{v})) \\
&\leq \sigma(d(\mathbf{w}, \mathbf{v})) + \psi(d(\mathbf{w}, \zeta\mathbf{w}), d(\mathbf{v}, \zeta\mathbf{v}), d(\mathbf{w}, \zeta\mathbf{v}), d(\mathbf{v}, \zeta\mathbf{w})), \\
&\quad \text{(where } \psi \in \Psi \text{ is arbitrary)}
\end{aligned}$$

so that

$$d(\zeta\mathbf{w}, \zeta\mathbf{v}) \leq \sigma(d(\mathbf{w}, \mathbf{v})) + \psi(d(\mathbf{w}, \zeta\mathbf{w}), d(\mathbf{v}, \zeta\mathbf{v}), d(\mathbf{w}, \zeta\mathbf{v}), d(\mathbf{v}, \zeta\mathbf{w})).$$

Moreover, $\zeta(\mathcal{H})$ is ξ^s-directed. All the hypotheses of Theorems 3.1 and 3.2 are thus met; so $\exists$ a unique $\overline{\mathbf{w}} \in C([0,1])$ such that $\zeta(\overline{\mathbf{w}}) = \overline{\mathbf{w}}$. Thus, $\overline{\mathbf{w}}$ is a (unique) solution of (5.1). $\square$

6. Conclusions

In near past, certain outcomes on fixed points through an amorphous relation on functional contraction via (c)-comparison function are investigated by Ansari et al. [10]. Our work utilizes a functional contraction involving yet another control function. The involved relation in our outcomes being locally finitely ζ-transitive is restrictive a relation is required; but the class of functional contraction is weaken. By means of illustration of the results, we constructed several examples. Our findings served to find a unique positive solution for certain elastic beam equation, which shown the efficiency of our findings. In the immediate future, learners might employ our results for a couple of mappings.

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References

[1] M. Abbas, H. Iqbal and A. Petruşel, *Fixed Points for multivalued Suzuki type* $(\theta, \mathcal{R})$*-contraction mapping with applications*, J. Func. Spaces **2019** (2019): 9565804.

[2] A. Alam and M. Imdad, *Relation-theoretic contraction principle*, J. Fixed Point Theory Appl. **17** (2015), 693–702.

[3] A. Alam and M. Imdad, *Relation-theoretic metrical coincidence theorems*, Filomat **31** (2017), 4421–4439.

[4] A. Alam and M. Imdad, *Nonlinear contractions in metric spaces under locally* T*-transitive binary relations*, Fixed Point Theory **19** (2018), 13–24.

[5] A. Alam, M. Arif and M. Imdad, *Metrical fixed point theorems via locally finitely* T*-transitive binary relations under certain control functions*, Miskolc Math. Notes **20** (2019) 59–73.

[6] E. A. Algehyne, N. H. Altaweel, M. Areshi and F. A. Khan, *Relation-theoretic almost* ϕ*-contractions with an application to elastic beam equations*, AIMS Mathematics **8** (2023), 18919–18929.

[7] J. Ali, M. Jubair and F. Ali, *Stability and convergence of* F *iterative scheme with an application to the fractional differential equation*, Engineering with Computers **38** (2022), 693–702.

[8] J. Ali and M. Jubair, *Existence and estimation of the fixed points of enriched Berinde nonexpansive mappings*, Miskolc Math. Notes **24** (2023), 541–552.

[9] B. Almarri, S. Mujahid and I. Uddin, *New fixed point results for Geraghty contractions and their applications*, J. Appl. Anal. Comp. **13** (2023), 2788–2798.

[10] K. J. Ansari, S. Sessa and A. Alam, *A class of relational functional contractions with applications to nonlinear integral equations*, Mathematics **11** (2023): 3408.

[11] M. Arif, M. Imdad and A. Alam, *Fixed point theorems under locally* T*-transitive binary relations employing Matkowski contractions*, Miskolc Math. Notes **23** (2022), 71–83.

[12] M. Berzig and E. Karapinar, *Fixed Point results for* $(\alpha\psi, \beta\phi)$*-contractive mappings for a generalized altering distance*, Fixed Point Theory Appl. **2013** (2013): 205.

[13] M. Jleli, V. C. Rajic, B. Samet and C. Vetro, *Fixed point theorems on ordered metric spaces and applications to nonlinear elastic beam equations*, J. Fixed Point Theory Appl. **12** (2012), 175–192.

[14] F. A. Khan, M. S. Aldhabani, A. Alamer, E. Alshaban, F. M. Alamrani and H. I. A. Mohammed, *Almost nonlinear contractions under locally finitely transitive relations with applications to integral equations*, Mathematics **11** (2023): 4749 .

[15] B. Kolman, R. C. Busby and S. Ross, *Discrete Mathematical Structures*, 6th Ed, New Jersey: Pearson/Prentice Hall, 2009.

[16] S. Lipschutz, *Schaum's Outlines of Theory and Problems of Set Theory and Related Topics*, New York: McGraw-Hill, 1964.

[17] B. Samet and M. Turinici, *Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications*, Commun. Math. Anal. **13** (2012), 82–97.

[18] K. Sawangsup, W. Sintunavarat and A. F. Roldán-López-de-Hierro, *Fixed point theorems for* $F_{\mathcal{R}}$*-contractions with applications to solution of nonlinear matrix equations*, J. Fixed Point Theory Appl. **19** (2017), 1711–1725.

[19] N. Shahzad, A. F. Roldán-López-de-Hierro and F. Khojasteh, *Some new fixed point theorems under* $(\mathcal{A}, \mathcal{S})$*-contractivity conditions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. (RACSAM) **111** (2017), 307–324.

[20] F. Sk, F. A. Khan, Q. H. Khan and A. Alam, *Relation-preserving generalized nonlinear contractions and related fixed point theorems*, AIMS Mathematics **7** (2021), 6634–6649.

[21] M. Turinici, *Contractive maps in locally transitive relational metric spaces*, Sci. World J. **2014** (2014): 169358.

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