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SOLUTION OF FREDHOLM INTEGRAL EQUATION VIA WEAK CONTRACTION

MUNIRAH ALJUAID, SAMAD MUJAHID*, AND SOKHOBIDDIN AKHATKULOV

Abstract. In this article, we establish fixed point theorems for Geraghty-weak contraction in M_{rb}-metric space. Our results extend and improve many existing results in literature. Additionally, we provide an example that demonstrates the utility of our results. Lastly, we apply our main result to investigate the existence and uniqueness of a fixed point by solving an integral equation.

1. Introduction and preliminaries

If u remains invariant under $\mathfrak{H}$ (i.e., $\mathfrak{H}u = u$), then $\mathfrak{H}$ has a fixed point u. The first basic fixed point theorem was established by Banach in his dissertation in 1922 and is known as the Banach contraction principle. There are number of generalizations (see [3, 6, 9–11, 17, 18]). Indeed, the rectangular metric space is one of the most interesting generalization purposed by Branciari [5]. As an extension of rectangular metric spaces, George et al. [7] presented rectangular b-metric spaces in 2008 and demonstrated fixed point results for the same. By presenting the idea of extended rectangular b-metric spaces in the recent past, Asim et al. [4] broadened the notion of rectangular b-metric spaces and used it to prove a fixed point theorem with an application. Partial rectangular metric spaces were first developed in 2014 by Shukla et al. [19] as a generalization of rectangular metric. Parvaneh et al. [14] established the concept of partial rectangular b metric space in 2017.

On the other hand Asadi et al. introduced the idea of M-metric space as a generalization of metric space and partial metric space and proved some fixed point results with its topological properties.

Notation 1 ([1])**.** The following notations will be utilized in the definition of M-metric space.

(1) $m_{u,v} = \min\{m(u,u), m(v,v)\}$,
(2) $M_{u,v} = \max\{m(u,u), m(v,v)\}$,

In 2014, Asadi et al. [1] introduced the following definition of M-metric space.

Definition 1.1 ([1])**.** *Let $\xi \neq \emptyset$. For all $u, v, w \in \xi$, a mapping $m : \xi \times \xi \to \mathbb{R}_+$ is M-metric, if it meets the following axioms:*

(1) $m(u,u) = m(u,v) = m(v,v)$ *if and only if* $u = v$,
(2) $m_{u,v} \leq m(u,v)$,
(3) $m(u,v) = m(v,u)$,
(4) $(m(u,v) - m_{u,v}) \leq (m(u,w) - m_{u,w}) + (m(w,v) - m_{w,v})$.

Then, the pair (ξ, m) is said to be a M-metric space.

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Notation 2 ([12]). The following notations will be utilized in the definition of M_b-metric space.

(1) $m_{b_{u,v}} = \min\{m_b(u,u), m_b(v,v)\}$,
(2) $M_{b_{u,v}} = \max\{m_b(u,u), m_b(v,v)\}$,

After two years, Mlaiki et al. [12] introduced the following definition of M_b-metric space.

Definition 1.2 ([12]). *Let $\xi \neq \emptyset$. For all $u, v, w \in \xi$, a mapping $m_b : \xi \times \xi \to \mathbb{R}_+$ is M_b-metric with coefficient $s \geq 1$, if it meets the following axioms:*

(1) $m_b(u,u) = m_b(u,v) = m_b(v,v)$ *if and only if* $u = v$,
(2) $m_{b_{u,v}} \leq m_b(u,v)$,
(3) $m_b(u,v) = m_b(v,u)$,
(4) $(m_b(u,v) - m_{b_{u,v}}) \leq s[(m_b(u,w) - m_{b_{u,w}}) + (m_b(w,v) - m_{b_{w,v}})] - m_b(w,w)$.

Then, the pair (ξ, m_b) is said to be a M_b-metric space.

Notation 3 ([13]). The following notations will be utilized in the definition of rectangular M-metric space.

(1) $m_{r_{u,v}} = \min\{m_r(u,u), m_r(v,v)\}$,
(2) $M_{r_{u,v}} = \max\{m_r(u,u), m_r(v,v)\}$.

In 2018, Özgür et al. [13] introduced the following definition of rectangular M-metric space.

Definition 1.3 ([13]). *Let $\xi \neq \emptyset$. For all $u, v \in \xi$ and all distinct $p, q \in \xi \setminus \{u, v\}$, a mapping $m_r : \xi \times \xi \to \mathbb{R}_+$ is rectangular M-metric, if it meets the following axioms:*

(1) $m_r(u,u) = m_r(u,v) = m_r(v,v)$ *if and only if* $u = v$,
(2) $m_{r_{u,v}} \leq m_r(u,v)$,
(3) $m_r(u,v) = m_r(v,u)$,
(4) $(m_r(u,v) - m_{r_{u,v}}) \leq (m_r(u,p) - m_{r_{u,p}}) + (m_r(p,q) - m_{r_{p,q}}) + (m_r(q,v) - m_{r_{q,v}})$.

Then the pair (ξ, m_r) is said to be a rectangular M-metric space.

Notation 4. The following notations will be used in the definition of rectangular M_b-metric space.

(1) $m_{rb_{u,v}} = \min\{m_{rb}(u,u), m_{rb}(v,v)\}$,
(2) $M_{rb_{u,v}} = \max\{m_{rb}(u,u), m_{rb}(v,v)\}$.

Rectangular M_b-metric space was first described as a generalization of both rectangular M-metric space and M_b-metric space by Asim et al. [2] in 2019.

Definition 1.4 ([2]). *Let $\xi \neq \emptyset$. $\forall\ u, v \in \xi$ and all distinct $p, q \in \xi \setminus \{u, v\}$, a mapping $m_{rb} : \xi \times \xi \to \mathbb{R}_+$ is M_b-metric with coefficient $s \geq 1$, if it meets the following axioms:*

$(1m_{rb})$ $m_{rb}(u,u) = m_{rb}(u,v) = m_{rb}(v,v)$ *if and only if* $u = v$,

$(2m_{rb})$ $m_{rb_{u,v}} \leq m_{rb}(u,v)$,

$(3m_{rb})$ $m_{rb}(u,v) = m_{rb}(v,u)$,

$(4m_{rb})$ $(m_{rb}(u,v)-m_{rb_{u,v}}) \leq s[(m_{rb}(u,p)-m_{rb_{u,p}})+(m_{rb}(p,q)-m_{rb_{p,q}})+(m_{rb}(q,v)-m_{rb_{q,v}})] - m_{rb}(p,p) - m_{rb}(q,q)$.

Then, (ξ, m_{rb}) is said to be a rectangular M_b-metric space.

Remark 1.5. *The pair (ξ, m_rb) reduces to rectangular metric space with $s = 1$.*

Now, we adopt an example of a rectangular M_b-metric space

Example 1.6. *Let $\xi = [0, \infty)$ and $l > 1$ any positive integer. Define $m_{rb} : \xi \times \xi \to \mathbb{R}_+$ by ($\forall$ $u, v \in \xi$):*

$$m_{rb}(u,v) = \max\{u,v\}^l + |u-v|^l.$$

Then (ξ, m_{rb}) is a rectangular M_b-metric space with coefficient $s = 3^{l-1}$.It is simple to verify, by using basic computation that (ξ, m_{rb}) is not a rectangular M-metric space.

Definition 1.7. *A sequence $\{u_n\}$ in (ξ, m_{rb}) is said to be convergent to $u \in \xi$ iff*

$$\lim_{n\to\infty}(m_{rb}(u_n,u) - m_{rb_{u_n,u}}) = 0.$$

Definition 1.8. *A sequence $\{u_n\}$ in (ξ, m_{rb}) is said to be Cauchy iff*

$$\lim_{n,m\to\infty}(m_{rb}(u_n,u_m) - m_{rb_{u_n,u_m}}) \text{ and } \lim_{n,m\to\infty}(M_{rb_{u_n,u_m}} - m_{rb_{u_n,u_m}})$$

exist and finite.

Definition 1.9. *If every Cauchy in ξ is convergent to some point in ξ, then the rectangular M_b-metric space (ξ, m_{rb}) is complete.*

2. Main results

Lemma 2.1. *Assume a sequence $\{u_n\}$ in M_{rb}-metric space (ξ, m_{rb}) such that $\lim_{n\to\infty} m_{rb}(u_n,u_n) = 0$. Then,*

1. *The sequence $\{u_n\}$ is Cauchy in (ξ, m_{rb}) iff the sequence $\{u_n\}$ is Cauchy in (ξ, r_b).*
2. *The (ξ, m_{rb}) is complete iff (ξ, r_b) is complete.*

Geraghty [8] established a fixed point result and broadened the Banach contraction principle in 1973. Later, using Geraghty-weak contractions, Roshan et al. [15] demonstrated fixed point results in b-metric spaces.
Following [8], let $\theta : [0,\infty) \to [0, \frac{1}{s})$ (with $s \geq 1$) for any sequence $t_n \in [0,\infty)$ satisfies the following:

$$\limsup_{n\to\infty} \theta(t_n) = \frac{1}{s} \;\Rightarrow\; \lim_{n\to\infty} t_n = 0.$$

Let Θ denotes for the set of all θ.

Definition 2.2. *Let (ξ, m_{rb}) be a M_{rb}-metric space. A mapping $\mathfrak{H} : \xi \to \xi$ is said to be Geraghty-weak contraction if there exists $\theta \in \Theta$ such that*

$$m_{rb}(\mathfrak{H}u, \mathfrak{H}v) \leq \theta(m_{rb}(u,v))N(m_{rb}(u,v)), \tag{2.1}$$

where

$$N(m_{rb}(u,v)) \;=\; \max\left\{m_{rb}(u,v), \frac{m_{rb}(u,\mathfrak{H}u)m_{rb}(v,\mathfrak{H}v)}{1+m_{rb}(\mathfrak{H}u,\mathfrak{H}v)}\right\}.$$

Theorem 2.3. *Let (ξ, m_{rb}) be an M_{rb}-metric space with coefficient $s \geq 1$ and $\mathfrak{H} : \xi \to \xi$ a Geraghty-weak contraction. Then, $\mathfrak{H}$ has a unique fixed point u such that $m_{rb}(u, u) = 0$.*

Proof. Assume that $u_0 \in \xi$ and the iterative sequence $\{u_n\}$ can be constructed by:

$$u_1 = \mathfrak{H}u_0, \ u_2 = \mathfrak{H}^2 u_0, \ u_3 = \mathfrak{H}^3 u_0, ..., u_n = \mathfrak{H}^n u_0, \cdots .$$

If $m_{rb}(u_n, u_{n+1}) = 0$ for some $n \in \mathbb{N}_0$, then u_n is a fixed point of $\mathfrak{H}$ and we are done. Henceforth, we assume that $m_{rb}(u_n, u_{n+1}) > 0$ for all $n \in \mathbb{N}_0$. We assert that $\lim_{n\to\infty} m_{rb}(u_n, u_{n+1}) = 0$. On setting $u = u_{n-1}$ and $v = u_n$ in (2.1), we get

$$\begin{aligned} m_{rb}(u_n, u_{n+1}) &= m_{rb}(\mathfrak{H}u_{n-1}, \mathfrak{H}u_n) \leq \theta(m_{rb}(u_{n-1}, u_n))N(m_{rb}(u_{n-1}, u_n)) \\ &< \frac{1}{s}N(m_{rb}(u_{n-1}, u_n)) \leq N(m_{rb}(u_{n-1}, u_n)). \end{aligned} \tag{2.2}$$

where,

$$\begin{aligned} N(m_{rb}(u_{n-1}, u_n)) &= \max\left\{m_{rb}(u_{n-1}, u_n), \frac{m_{rb}(u_{n-1}, \mathfrak{H}u_{n-1})m_{rb}(u_n, \mathfrak{H}u_n)}{1 + m_{rb}(\mathfrak{H}u_{n-1}, \mathfrak{H}u_n)}\right\} \\ &= \max\left\{m_{rb}(u_{n-1}, u_n), \frac{m_{rb}(u_{n-1}, u_n)m_{rb}(u_n, u_{n+1})}{1 + m_{rb}(u_n, u_{n+1})}\right\} \\ &\leq \{m_{rb}(u_{n-1}, u_n), m_{rb}(u_{n-1}, u_n)\} \\ &= m_{rb}(u_{n-1}, u_n). \end{aligned}$$

Therefore, (2.2) gives rise

$$m_{rb}(u_n, u_{n+1}) < m_{rb}(u_{n-1}, u_n). \tag{2.3}$$

Hence, $\{m_{rb}(u_n, u_{n+1})\}$ is a decreasing sequence of positive real numbers. So that, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} m_{rb}(u_n, u_{n+1}) = r.$$

Assume that $r > 0$. Then from (2.2), we have

$$\lim_{n\to\infty} m_{rb}(u_n, u_{n+1}) \leq \lim_{n\to\infty} \left[\theta(m_{rb}(u_{n-1}, u_n))N(m_{rb}(u_{n-1}, u_n))\right].$$

Using the definition of θ we obtain $r < \frac{1}{s}r$, a contradiction. Thus,

$$\lim_{n\to\infty} m_{rb}(u_n, u_{n+1}) = 0. \tag{2.4}$$

Now, by taking $u = u_{n-1}$ and $q = u_{n+1}$ in (2.1), we get

$$\begin{aligned} m_{rb}(u_n, u_{n+2}) = m_{rb}(\mathfrak{H}u_{n-1}, \mathfrak{H}u_{n+1}) &\leq \theta(m_{rb}(u_{n-1}, u_{n+1}))N(m_{rb}(u_{n-1}, u_{n+1})) \\ &< \frac{1}{s}N(m_{rb}(u_{n-1}, u_{n+1})) \leq N(m_{rb}(u_{n-1}, u_{n+1})), \end{aligned} \tag{2.5}$$

where,

$$\begin{aligned} N(m_{rb}(u_{n-1}, u_{n+1})) &= \max\left\{m_{rb}(u_{n-1}, u_{n+1}), \frac{m_{rb}(u_{n-1}, \mathfrak{H}u_{n-1})m_{rb}(u_{n+1}, \mathfrak{H}u_{n+1})}{1 + m_{rb}(\mathfrak{H}u_{n-1}, \mathfrak{H}u_{n+1})}\right\} \\ &= \max\left\{m_{rb}(u_{n-1}, u_{n+1}), \frac{m_{rb}(u_{n-1}, u_n)m_{rb}(u_{n+1}, u_{n+2})}{1 + m_{rb}(u_n, u_{n+2})}\right\} \\ &\leq \max\left\{m_{rb}(u_{n-1}, u_{n+1}), [m_{rb}(u_{n-1}, u_n)m_{rb}(u_{n+1}, u_{n+2})]\right\}. \end{aligned}$$

Thanks to (2.3), we have some obliteration, that is, we have

$$N(m_{rb}(u_{n-1},u_{n+1})) \leq \max\left\{m_{rb}(u_{n-1},u_{n+1}),[m_{rb}(u_{n-1},u_n)]^2\right\}.$$

Here, we assume that

$$\max\left\{m_{rb}(u_{n-1},u_{n+1}),[m_{rb}(u_{n-1},u_n)]^2\right\} = m_{rb}(u_{n-1},u_n) \text{ or } [m_{rb}(u_{n-1},u_n)]^2.$$

Since $\lim_{n\to\infty} m_{rb}(u_{n-1},u_n) = 0$, then from (2.5), we have

$$\lim_{n\to\infty} m_{rb}(u_n,u_{n+2}) = 0.$$

On similar way, one can easily obtain

$$\lim_{n\to\infty} m_{rb}(u_n,u_n) = 0. \tag{2.6}$$

By the definition of r_b and conditions (2.4) and (2.6), we have

$$\lim_{n\to\infty} r_b(u_n,u_{n+1}) = \lim_{n\to\infty} r_b(u_n,u_{n+2}) = \lim_{n\to\infty} m_{rb}(u_n,u_n) = 0. \tag{2.7}$$

Firstly, we show that $u_n \neq u_m$ for any $n \neq m$. Let on contrary that, $u_n = u_m$ for some $n > m$, then we have $u_{n+1} = \mathfrak{H}u_n = \mathfrak{H}u_m = u_{m+1}$. Then, from (2.2) we have

$$\begin{aligned} m_{rb}(u_m,u_{m+1}) &= m_{rb}(u_n,u_{n+1}) = m_{rb}(\mathfrak{H}u_{n-1},\mathfrak{H}u_n) \\ &\leq \theta(m_{rb}(u_{n-1},u_n))N(m_{rb}(u_{n-1},u_n)) \\ &< \frac{1}{s}N(m_{rb}(u_{n-1},u_n)) \\ &\leq N(m_{rb}(u_{n-1},u_n)), \\ &\leq m_{rb}(u_{n-1},u_n). \end{aligned}$$

Then we have

$$\begin{aligned} m_{rb}(u_m,u_{m+1}) &= m_{rb}(u_n,u_{n+1}) \\ &< m_{rb}(u_{n-1},u_n) \\ &< m_{rb}(u_{n-2},u_{n-1}) \\ &< \vdots \\ &< m_{rb}(u_m,u_{m+1}) \end{aligned}$$

a contradiction. Thus, in what follows, we can assume that $u_n \neq u_m$ for all $n \neq m$. Now, we have to show that $\{u_n\}$ is Cauchy sequence in (ξ, m_{rb}). For this we have to show that the sequence $\{u_n\}$ is Cauchy in (ξ, r_b) (see Lemma 2.1). On the contrary suppose that, $\{u_n\}$ is not Cauchy sequence. Then, there exists $\epsilon > 0$ for which we can find two subsequences $\{n_k\}$ and $\{m_k\}$ such that n_k is the smallest index for which

$$n_k > m_k > k \text{ and } r_b(u_{m_k},u_{n_k}) \geq \epsilon. \tag{2.8}$$

This means that

$$r_b(u_{m_k},u_{n_k-1}) < \epsilon. \tag{2.9}$$

By using (2.8) and rectangular inequality of r_b, we obtain

$$\epsilon \leq r_b(u_{m_k},u_{n_k}) \leq sr_b(u_{m_k},u_{n_k-1}) + sr_b(u_{n_k-1},u_{n_k+1}) + sr_b(u_{n_k+1},u_{n_k}). \tag{2.10}$$

Taking, upper limit as $k \to \infty$ and using (2.7) and (2.9), we have

$$\epsilon \leq \limsup_{k\to\infty} r_b(u_{m_k}, u_{n_k}) \leq s\epsilon. \tag{2.11}$$

On the other hand, by the definition of r_b and using (2.6), we have

$$\lim_{k\to\infty} r_b(u_{m_k}, u_{n_k}) = 2 \lim_{k\to\infty} m_{rb}(u_{m_k}, u_{n_k}).$$

This shows that

$$\frac{\epsilon}{2} \leq \limsup_{k\to\infty} m_{rb}(u_{m_k}, u_{n_k}) \leq \frac{s\epsilon}{2}. \tag{2.12}$$

Using (2.1) and the definition of r_b, we have

$$\begin{aligned} \lim_{k\to\infty} m_{rb}(u_{m_k}, u_{n_k}) &\leq \lim_{k\to\infty} sm_{rb}(u_{m_k+1}, u_{m_k}) + \lim_{k\to\infty} sm_{rb}(u_{m_k+1}, u_{n_k+1}) \\ &\quad + \lim_{k\to\infty} sm_{rb}(u_{n_k+1}, u_{n_k}), \\ &\leq s \lim_{k\to\infty} \theta(m_{rb}(u_{m_k}, u_{n_k}))N(m_{rb}(u_{m_k}, u_{n_k})), \end{aligned} \tag{2.13}$$

where,

$$\begin{aligned} (m_{rb}(u_{m_k}, u_{n_k})) &= \max\left\{ m_{rb}(u_{m_k}, u_{n_k}), \frac{m_{rb}(u_{m_k}, \mathfrak{H}u_{m_k})m_{rb}(u_{n_k}, \mathfrak{H}u_{n_k})}{1 + m_{rb}(\mathfrak{H}u_{m_k}, \mathfrak{H}u_{n_k})} \right\} \\ &= \max\left\{ m_{rb}(u_{n_k}, u_{m_k}), \frac{m_{rb}(u_{m_k}, u_{m_k+1})m_{rb}(u_{n_k}, u_{n_k+1})}{1 + m_{rb}(u_{m_k+1}, u_{n_k+1})} \right\}. \end{aligned} \tag{2.14}$$

On making the upper limit as $k \to \infty$ and using (2.6) in (2.14), we have

$$\limsup_{n\to\infty} N(m_{rb}(u_{m_k}, u_{n_k})) = \limsup_{n,m\to\infty} m_{rb}(u_{m_k}, u_{n_k}).$$

Hence, from (2.13), we obtain

$$\limsup_{n\to\infty} m_{rb}(u_{m_k}, u_{n_k}) \leq s \limsup_{n,m\to\infty} \theta(m_{rb}(u_{m_k}, u_{n_k})) \limsup_{n,m\to\infty} m_{rb}(u_{m_k}, u_{n_k}).$$

In view of our assumption, $\limsup_{k\to\infty} m_{rb}(u_{m_k}, u_{n_k}) \neq 0$, so from above inequality, we get

$$\frac{1}{s} \leq \limsup_{k\to\infty} \theta(m_{rb}(u_{m_k}, u_{n_k})).$$

Since $\theta \in \Theta$, we deduce that $\lim_{n,m\to\infty} m_{rb}(u_{n_k}, u_{m_k}) = 0 \Rightarrow \lim_{n,m\to\infty} r_b(u_{n_k}, u_{m_k}) = 0$, which is a contradiction. Therefore, $\{u_n\}$ is Cauchy sequence in (ξ, r_b). Consequently, $\{u_n\}$ is Cauchy sequence in (ξ, m_{rb}), that is

$$\lim_{n,m\to\infty} (m_{rb}(u_n, u_m) - m_{rb_{u_n,u_m}}) = 0.$$

and

$$\lim_{n,m\to\infty} \left(M_{rb_{u_n,u_m}} - m_{rb_{u_n,u_m}}\right) = 0.$$

Since (ξ, m_{rb}) is complete then there exists $u \in \xi$ such that

$$\lim_{n\to\infty} (m_{rb}(u_n, u) - m_{rb_{u_n,u}}) = 0.$$

To show that u is a fixed point of $\mathfrak{H}$, by the continuity of $\mathfrak{H}$, we have

$$u = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} fu_n = f(\lim_{n\to\infty} u_n) = \mathfrak{H}u.$$

Hence, u is a fixed point of $\mathfrak{H}$.

Let's assume for the uniqueness component that u, v exist in ξ such that $\mathfrak{H}u = u$ and $\mathfrak{H}v = v$. Then from (2.1), we obtain

$$\begin{aligned} m_{rb}(u,v) &= m_{rb}(\mathfrak{H}u,\mathfrak{H}v) \leq \theta(m_{rb}(u,v))N(m_{rb}(u,v)) \\ &< \frac{1}{s}N(m_{rb}(u,v)) \leq N(m_{rb}(u,v)) \\ &= \max\left\{m_{rb}(u,v), \frac{m_{rb}(u,\mathfrak{H}u)m_{rb}(v,\mathfrak{H}v)}{1+m_{rb}(\mathfrak{H}u,\mathfrak{H}v)}\right\} \\ &= m_{rb}(u,v) \end{aligned}$$

a contradiction, unless $m_{rb}(u,v) = 0 \implies u = v$. Hence $\mathfrak{H}$ has unique fixed point in ξ.

Finally, we demonstrate that $m_{rb}(u,u) = 0$ if u is a fixed point. Let u be a fixed point of $\mathfrak{H}$ to achieve this

$$\begin{aligned} m_{rb}(u,u) &= m_{rb}(\mathfrak{H}u,\mathfrak{H}u) \\ &\leq \theta(m_{rb}(u,u))N(m_{rb}(u,u)) \\ &< \frac{1}{s}N(m_{rb}(u,u)) \leq N(m_{rb}(u,u)) \\ &= \max\left\{m_{rb}(u,u), \frac{m_{rb}(u,\mathfrak{H}u)m_{rb}(u,\mathfrak{H}u)}{1+m_{rb}(\mathfrak{H}u,\mathfrak{H}u)}\right\} \\ &= \max\left\{m_{rb}(u,u), \frac{m_{rb}(u,u)m_{rb}(u,u)}{1+m_{rb}(u,u)}\right\} \\ &< m_{rb}(u,u), \end{aligned}$$

yielding thereby $m_{rb}(u,u) = 0$. This concludes the proof. □

We now present the following example which shows the utility of our result.

Example 2.4. *Let $\xi = [0,1]$ and let $m_{rb} : \xi \times \xi \to \mathbb{R}^+$ is defined by:*

$$m_{rb}(u,v) = \left(\frac{u+v}{2}\right)^2$$

Then it is clear that (ξ, m_{rb}) is an complete m_{rb}-metric space with $s = 3$. Consider a mapping $\mathfrak{H} : \xi \to \xi$ defined by:

$$\mathfrak{H} = \frac{u}{3}, \quad \forall\ u \in \xi.$$

It is easy to check that all the conditions of Theorem 2.3 are fulfilled with $\theta(u) = \frac{2}{7}$ for each $u > 0$ and $\theta(u) \in [0, 1/3)$. Then the contractive condition (2.1) is trivially holds. Now, by taking $u = 4$ and $v \in \{1, 2, 3, 5\}$, such that $u \preceq v$, we obtain $\mathfrak{H}u = 1$ and $\mathfrak{H}v = 3$. Then by contractive condition (2.1), we have

$$\begin{aligned} m_{rb}(\mathfrak{H}u,\mathfrak{H}v) &= \left(\frac{\frac{u}{3}+\frac{v}{3}}{2}\right)^2 = \frac{1}{9}\left(\frac{u+v}{2}\right)^2 \\ &\leq \frac{2}{7}\left(\frac{u+v}{2}\right)^2 \\ &= \theta(u)d(u,v) \end{aligned}$$

$$\leq \theta\big(m_{rb}(u,v)\big)N(u,v).$$

It follows that $\mathfrak{H}$ has a unique fixed point (which is $u=0$).

The following corollary is proved by Asim et al. [2].

Corollary 2.5. *Let (ξ, m_{rb}) be a complete rectangular M_b-metric space with coefficient $s \geq 1$. Suppose $\mathfrak{H} : \xi \to \xi$ satisfies the following conditions:*

$$m_{rb}(\mathfrak{H}u, \mathfrak{H}v) \leq \lambda m_{rb}(u,v) \ \forall \ u, v \in \xi$$

where $\lambda \in [0, \frac{1}{s})$. Then, f has a unique fixed point u such that $m_{rb}(u,u) = 0$.

3. Application

In this section, we use Theorem 2.3 to examine the existence and uniqueness of solution of the nonlinear Fredholm integral equation. Consider the following Fredholm type integral equation:

$$u(t) = \int_a^b \mathfrak{G}(t, s, u(t))ds, \ \text{for } t, s \in [a, b] \tag{3.1}$$

where $\mathfrak{G}, h \in \xi = \mathbb{C}([a,b], \mathbb{R})$, the set of continuous real valued functions defined on $[a, b]$. Define $m_{rb} : \xi \times \xi \to \mathbb{R}^+$ and $\theta : [0, \infty) \to [0, \frac{1}{s})$ by $\theta(u) = \frac{2}{7}$.

$$m_{rb}(u(t), v(t)) = \sup_{t\in[a,b]} \left(\frac{|u(t)| + |v(t)|}{2}\right)^2 \ \text{for all } u, v \in \xi.$$

Then, (ξ, m_{rb}) is an complete m_{rb}-metric space.

Now, we are equipped to assert and demonstrate our result as follows:

Theorem 3.1. *Assume that (for all $u, v \in \mathbb{C}([a,b], \mathbb{R})$)*

$$|\mathfrak{G}(t, s, u(t)) + \mathfrak{G}(t, s, v(t))| \leq \frac{1}{3(b-a)}|u(t) + v(t)|, \ \textit{for all } t, s \in [a, b]. \tag{3.2}$$

Then the integral equation (3.1) *has a unique solution.*

Proof. Define $\mathfrak{H} : \xi \to \xi$ by

$$\mathfrak{H}u(t) = \int_a^b \mathfrak{G}(t, s, u(t))ds, \ \text{for all } t, s \in [a, b].$$

Observe that, existence of a fixed point of the operator f is equivalent to the existence of a solution of the integral equation (3.1). Now, for all $u, v \in \xi$, we have

$$\begin{aligned}
m_{rb}(\mathfrak{H}u, \mathfrak{H}v) = \left|\frac{\mathfrak{H}u(t) + \mathfrak{H}v(t)}{2}\right|^2 &= \left|\int_a^b \left(\frac{\mathfrak{G}(t, s, u(t)) + \mathfrak{G}(t, s, v(t))}{2}\right)ds\right|^2 \\
&\leq \left(\int_a^b \left|\frac{\mathfrak{G}(t, s, u(t)) + \mathfrak{G}(t, s, v(t))}{2}\right|ds\right)^2 \\
&\leq \left(\int_a^b \frac{1}{3(b-a)}\left|\frac{u(t) + v(t)}{2}\right|ds\right)^2 \\
&\leq \frac{1}{9(b-a)^2} \sup_{t\in[a,b]} \left(\frac{|u(t)| + |v(t)|}{2}\right)^3 \left(\int_a^b ds\right)^2
\end{aligned}$$

$$\leq \frac{1}{9} m_{rb}(u,v) \leq \theta(u) d(u,v) \leq \theta\big(m_{rb}(u,v)\big) N(u,v).$$

As a result, Theorem's 2.3 requirements are all met. Hence, the Fredholm integral Equation (3.1) has a unique solution since the operator $\mathfrak{H}$ has a fixed point, which is unique. The evidence is now complete. □

4. Conclusions

Due to the fact that m_{rb}-metric space is a really sharpened form of both m_r-metric space and m_b-metric space. In m_{rb}-metric space, we demonstrated a fixed point result for Geraghty-weak contraction. Additionally, a model is created to illustrate the usefulness of our findings.

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M. Aljuaid
Department of Mathematics, Faculty of Science, Northern Border University, Arar 73222, Saudi Arabia
E-mail address: `moneera.mutlak@nbu.edu.sa`

S. Mujahid
Department of Mathematics, Jamia Millia Islamia, New Delhi-110025, India
E-mail address: `mujahidsamad721@gmail.com`

S. Akhatkulov
Samarkand State University, Samarkand, Uzbekistan
E-mail address: `a-soxobiddin@samdu.uz`