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PROBABILISTIC DEGENERATE JINDALRAE AND JINDALRAE-STIRLING POLYNOMIALS OF THE SECOND KIND

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Abstract. In recent studies, probabilistic forms of some special polynomials and numbers such as probabilistic Bernoulli, probabilistic Euler, probabilistic Bell, probabilistic Fubini, and probabilistic Stirling numbers and polynomials associated with random variables have been introduced and studied in detail. In this work, we consider a probabilistic form of the degenerate Jindalrae-Stirling polynomials of the second kind and a probabilistic form of the degenerate Jindalrae polynomials. Then, we derive some of their properties and formulas, including explicit expressions, symmetric identity, recurrence relations, and summation formulas. Moreover, we investigate diverse correlations with the probabilistic degenerate Stirling numbers of the second kind associated with T, the Stirling numbers of the first kind, the partial Bell polynomials, the derangement polynomials, the degenerate Bernoulli polynomials of the second kind, and the degenerate Euler polynomials. Finally, we consider probabilistic forms of the higher-order degenerate Bernoulli and Euler polynomials and then provide some properties and relations.

1. Introduction

Special functions have various importance in a good many areas of engineering, physics, mathematics, and other linked disciplines involving issues such as quantum mechanics, mathematical physics, functional analysis, numerical analysis, differential equations, and so on (cf. [1–22, 24–29]). In the family of special functions, the family of special polynomials also possesses intensive study fields. Recently, some probabilistic special polynomials (with their corresponding numbers) including probabilistic Bell, probabilistic Fubini, probabilistic Bernoulli, probabilistic Euler, and probabilistic Stirling polynomials (and numbers), are among the most studied families of special polynomials (cf. [1–3, 6–8, 15–17, 19, 21, 23, 26–29]). For example, probabilistic Stirling numbers of the second kind in [1–3], probabilistic degenerate Stirling polynomials of the second kind in [19], probabilistic Bell polynomials in [27], probabilistic degenerate Bell polynomials associated with random variables in [15], probabilistic degenerate central Bell polynomials in [7], probabilistic derangement polynomials in [17], probabilistic Bernoulli and Euler polynomials in [6, 16], probabilistic degenerate Bernoulli and degenerate Euler polynomials in [23], probabilistic type 2 poly-Bernoulli polynomials in [21], probabilistic Fubini polynomials in [26], probabilistic degenerate Fubini polynomials in [29] and probabilistic Bernstein polynomials in [8] have been considered and many of their relations and properties have

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been examined and investigated. Motivated by these types of studies, here we consider a probabilistic degenerate Jindalrae-Stirling polynomials of the second kind and a probabilistic degenerate Jindalrae polynomials, and then we research their relations and properties. To do this, we first review some needed definitions and notations.

The Stirling numbers of the second kind ('second kind' is abbreviated by '*s.k.*') are defined, for $u \geq 0$ (see [1–3, 5, 12, 18, 19]) as

$$\sum_{\rho=u}^{\infty} u! S_2(\rho, u) \frac{z^\rho}{\rho!} = (e^z - 1)^u. \tag{1.1}$$

The Bell polynomials are provided by (see [11, 13, 15, 22, 24, 27])

$$\sum_{\rho=0}^{\infty} \phi_\rho(y) \frac{z^\rho}{\rho!} = e^{y(e^z-1)}. \tag{1.2}$$

The corresponding numbers of $\phi_\rho(y)$ are derived by taking $y = 1$, namely $\phi_\rho := \phi_\rho(1)$.

For $\lambda \in \mathbb{R}$, the degenerate model of the exponential function is given by (cf. [5, 7, 9–15, 19, 20, 22, 23, 29])

$$e_\lambda^y(z) = \sum_{\rho=0}^{\infty} (y)_{\rho,\lambda} \frac{z^\rho}{\rho!} = (1 + \lambda z)^{\frac{y}{\lambda}}, \tag{1.3}$$

where $(y)_{\rho,\lambda} = y(y - \lambda)(y - 2\lambda) \ldots (y - (\rho - 1)\lambda)$ for $\rho \geq 1$, with $(y)_{0,\lambda} = 1$. Note that $e_\lambda^1(z) := e_\lambda(z) = (1 + \lambda z)^{\frac{1}{\lambda}} = \sum_{\rho=0}^{\infty} (1)_{\rho,\lambda} \frac{z^\rho}{\rho!}$. The function $\log_\lambda(z)$ is defined to be the compositional inverse of $e_\lambda(z)$, which is termed the degenerate logarithm function, fulfilling $e_\lambda(\log_\lambda z) = \log_\lambda(e_\lambda(z)) = z$, provided as follows (see [6, 9, 21])

$$\log_\lambda(1 + z) := \sum_{\rho=1}^{\infty} \lambda^{\rho-1} (1)_{\rho,\frac{1}{\lambda}} \frac{z^\rho}{\rho!} = \frac{1}{\lambda}((1 + z)^\lambda - 1). \tag{1.4}$$

It is readily seen that $\lim_{\lambda \to 0} \log_\lambda z = \log z$.

The degenerate models of the Bernoulli and the Euler polynomials, defined by Leonard Carlitz [5], are provided as follows

$$\sum_{\rho=0}^{\infty} \beta_{\rho,\lambda}(y) \frac{z^\rho}{\rho!} = \frac{z e_\lambda^y(z)}{e_\lambda(z) - 1} \text{ and } \sum_{\rho=0}^{\infty} E_{\rho,\lambda}(y) \frac{z^\rho}{\rho!} = \frac{2 e_\lambda^y(z)}{e_\lambda(z) + 1}. \tag{1.5}$$

The corresponding numbers of $\beta_{\rho,\lambda}(y)$ and $E_{\rho,\lambda}(y)$ are derived by taking $y = 0$, namely $\beta_{\rho,\lambda} := \beta_{\rho,\lambda}(0)$ and $E_{\rho,\lambda} := E_{\rho,\lambda}(0)$, respectively. Also, the classical Bernoulli and Euler polynomials are obtained when λ approaches to 0 as $\lim_{\lambda \to 0} \beta_{\rho,\lambda}(y) = B_\rho(y)$ and $\lim_{\lambda \to 0} E_{\rho,\lambda}(y) = E_\rho(y)$.

Also, the degenerate models of higher-order Bernoulli and the Euler polynomials are provided as follows (see [5,6,9,10,16,23,24])

(1.6)
$$\sum_{\rho=0}^{\infty}\beta_{\rho,\lambda}^{(\alpha)}(y)\frac{z^{\rho}}{\rho!}=e_{\lambda}^{y}(z)\left(\frac{z}{e_{\lambda}(z)-1}\right)^{\alpha}\text{ and }\sum_{\rho=0}^{\infty}E_{\rho,\lambda}^{(\alpha)}(y)\frac{z^{\rho}}{\rho!}=e_{\lambda}^{y}(z)\left(\frac{2}{e_{\lambda}(z)+1}\right)^{\alpha}.$$

The corresponding numbers of $\beta_{\rho,\lambda}^{(\alpha)}(y)$ and $E_{\rho,\lambda}^{(\alpha)}(y)$ are derived by taking $y=0$, namely $\beta_{\rho,\lambda}^{(\alpha)}:=\beta_{\rho,\lambda}^{(\alpha)}(0)$ and $E_{\rho,\lambda}^{(\alpha)}:=E_{\rho,\lambda}^{(\alpha)}(0)$, respectively.

For $\alpha\in\mathbb{N}$, the degenerate higher-order model of Bernoulli polynomials of the *s.k.* is defined by (cf. [10])

$$\sum_{\rho=0}^{\infty}b_{\rho,\lambda}^{(\alpha)}(y)\frac{z^{\rho}}{\rho!}=\left(\frac{z}{\log_{\lambda}(1+z)}\right)^{\alpha}(1+z)^{y}. \tag{1.7}$$

The corresponding numbers of $b_{\rho,\lambda}^{(\alpha)}(y)$ are derived by taking $y=0$, namely $b_{\rho,\lambda}^{(\alpha)}:=b_{\rho,\lambda}^{(\alpha)}(0)$.

The degenerate model of Bernoulli polynomials of the *s.k.* is provided as (see [10])

$$\sum_{\rho=0}^{\infty}b_{\rho,\lambda}(y)\frac{z^{\rho}}{\rho!}=\frac{z(1+z)^{y}}{\log_{\lambda}(1+z)}. \tag{1.8}$$

The corresponding numbers of $b_{\rho,\lambda}(y)$ are derived by taking $y=0$, namely $b_{\rho,\lambda}:=b_{\rho,\lambda}(0)$.

The degenerate Stirling numbers of the *s.k.* are defined by (see [5,12,18])

$$\sum_{j=u}^{\infty}S_{2,\lambda}(j,u)\frac{z^{j}}{j!}=\frac{(e_{\lambda}(z)-1)^{u}}{u!}\ (u\geq 0). \tag{1.9}$$

The fully degenerate model of Bell polynomials of the *s.k.* is defined by (see [11,13,22])

$$\sum_{\rho=0}^{\infty}B_{\rho,\lambda}^{*}(y)\frac{z^{\rho}}{\rho!}=e_{\lambda}^{y}(e_{\lambda}(z)-1), \tag{1.10}$$

satisfying the following relation

$$B_{\rho,\lambda}^{*}(y)=\sum_{u=0}^{\rho}(y)_{u,\lambda}S_{2,\lambda}(\rho,u)\ (\rho\geq 0).$$

The corresponding numbers of $B_{\rho,\lambda}^{*}(y)$ are derived by taking $y=1$, namely $B_{\rho,\lambda}^{*}=B_{\rho,\lambda}^{*}(1)$.

The degenerate model of derangement polynomials $d_{\rho,\lambda}^{(\alpha)}(y)$ of order $\alpha\in\mathbb{N}$ is defined by (see [20])

$$\sum_{\rho=0}^{\infty}d_{\rho,\lambda}^{(\alpha)}(y)\frac{z^{\rho}}{\rho!}=\frac{e_{\lambda}^{y-1}(z)}{(1-z)^{\alpha}}. \tag{1.11}$$

The corresponding numbers of $d_{\rho,\lambda}^{(\alpha)}(y)$ are derived by taking $y=1$, namely $d_{\rho,\lambda}^{(\alpha)} = d_{\rho,\lambda}^{(\alpha)}(1)$.

The degenerate form of the derangement polynomials $d_{\rho,\lambda}(y)$ is defined by (see [20])

$$\sum_{\rho=0}^{\infty} d_{\rho,\lambda}(y)\frac{z^\rho}{\rho!} = \frac{e_\lambda^{y-1}(z)}{1-z}. \tag{1.12}$$

For $u \geq 0$, the partial Bell polynomials are provided by (see [11])

$$\sum_{\rho=u}^{\infty} B_{\rho,u}(y_1, y_2, \ldots, y_{\rho-u+1})\frac{z^\rho}{\rho!} = \frac{1}{u!}\left(\sum_{i=1}^{\infty} y_i \frac{z^i}{i!}\right)^u, \tag{1.13}$$

which yields

$$B_{\rho,u}(y_1, y_2, \ldots, y_{\rho-u+1}) = \sum_{\substack{l_1+l_2+\cdots+l_{\rho-u+1}=u \\ l_1+2l_2+\cdots+(\rho-u+1)l_{\rho-u+1}=\rho}} \frac{\rho!}{l_1!l_2!\ldots l_{\rho-u+1}!} \times (y_1)^{l_1}\left(\frac{y_2}{2}\right)^{l_2}\cdots\left(\frac{y_{\rho-u+1}}{(\rho-u+1)!}\right)^{l_{\rho-u+1}}. \tag{1.14}$$

For $u \geq 0$, the degenerate model of Jindalrae-Stirling numbers of the *s.k.* is given by (cf. [4])

$$\sum_{\rho=u}^{\infty} u! S_{J,\lambda}^{(2)}(\rho, u)\frac{z^\rho}{\rho!} = \left(e_\lambda(e_\lambda(z)-1)-1\right)^u. \tag{1.15}$$

The degenerate model of Jindalrae polynomials is provided (see [4]) as follows

$$\sum_{\rho=0}^{\infty} J_{\rho,\lambda}(y)\frac{z^\rho}{\rho!} = e_\lambda^y(e_\lambda(e_\lambda(z)-1)-1)). \tag{1.16}$$

When $y=1$, $J_{\rho,\lambda} := J_{\rho,\lambda}(1)$ are called the Jindalrae numbers.

Let T be a random variable ('random variable' is abbreviated by '*r.v*'). The moment generating function (abbreviated by '*m.g.f.*') of T (see [1–3, 6–8, 15–17, 19, 21, 23, 25–29]) is given by

$$E[e^{zT}] = \sum_{\rho=0}^{\infty} E[T^\rho]\frac{z^\rho}{\rho!},\ (0 < |z| < \alpha). \tag{1.17}$$

Assume that $(T_j)_{j\geq 1}$ be a sequence of mutually independent copies of the *r.v.* T, and $S_u = T_1 + T_2 + \cdots + T_u$ for $u \in \mathbb{N}$ with $S_0 = 0$. The probabilistic degenerate Stirling numbers (denoted by $S_{2,\lambda}^T(\rho, u)$) of the *s.k.* related to *r.v.* T are provided as follows (see [1–3])

$$S_{2,\lambda}^T(\rho, u) = \frac{1}{u!}\sum_{j=0}^{u}\binom{u}{j}(-1)^{u-j}E[(S_j)_{\rho,\lambda}],\ (\rho \geq u \geq 0), \tag{1.18}$$

which is equivalent to (see [1–3])

$$E[(S_u)_{\rho,\lambda}] = \sum_{j=0}^{u} \binom{u}{j} j! S_{2,\lambda}^{T}(\rho, u). \tag{1.19}$$

The generating function of $S_{2,\lambda}^{T}(\rho, u)$ is provided as follows

$$\sum_{\rho=u}^{\infty} S_{2,\lambda}^{T}(\rho, u) \frac{z^{\rho}}{\rho!} = \frac{\left(E[e_{\lambda}^{T}(z)] - 1\right)^{u}}{u!}. \tag{1.20}$$

In the case $T = 1$, the numbers $S_{2,\lambda}^{T}(\rho, u)$ become the numbers $S_{2,\lambda}(\rho, u)$ in (1.9). We abbreviate the probabilistic degenerate Stirling numbers of the *s.k.* as the *PDSNSK* numbers. When $\lambda \to 0$, the numbers $S_{2,\lambda}^{T}(\rho, u)$ become the probabilistic Stirling numbers of the *s.k.* $S_{2}^{T}(\rho, u)$ given by

$$\sum_{\rho=u}^{\infty} S_{2}^{T}(\rho, u) \frac{z^{\rho}}{\rho!} = \frac{\left(E[e^{zT}] - 1\right)^{u}}{u!}. \tag{1.21}$$

The probabilistic degenerate model of Stirling polynomials of the *s.k.* associated with the *r.v.* T is considered by (see [19])

$$\sum_{\rho=u}^{\infty} S_{2,\lambda}^{T}(\rho, u|y) \frac{z^{\rho}}{\rho!} = \frac{e_{\lambda}^{y}(z) \left(E[e_{\lambda}^{T}(z)] - 1\right)^{u}}{u!}. \tag{1.22}$$

When $y = 0$, the polynomials $S_{2,\lambda}^{T}(\rho, u|y)$ reduces to the polynomials $S_{2,\lambda}^{T}(\rho, u)$ in (1.20).

The probabilistic degenerate model of Bell polynomials $\phi_{\rho,\lambda}^{T}(y)$ associated with *r.v.* T is defined as follows (see [15])

$$\sum_{\rho=0}^{\infty} \phi_{\rho,\lambda}^{T}(y) \frac{z^{\rho}}{\rho!} = e^{y(E[e_{\lambda}^{T}(z)]-1)}. \tag{1.23}$$

In the case $\lambda \to 0$, the polynomials $\phi_{\rho,\lambda}^{T}(y)$ become the probabilistic Bell polynomials $\phi_{\rho}^{T}(y)$ associated with *r.v.* T given by (see [15])

$$\sum_{\rho=0}^{\infty} \phi_{\rho}^{T}(y) \frac{z^{\rho}}{\rho!} = e^{y(E[e^{zT}]-1)}. \tag{1.24}$$

The corresponding numbers of $\phi_{\rho}^{T}(y)$ are derived by taking $y = 1$, namely $\phi_{\rho}^{T} := \phi_{\rho}^{T}(1)$. When $T = 1$, the polynomials $\phi_{\rho,\lambda}^{T}(y)$ becomes the degenerate Bell polynomials $\phi_{\rho,\lambda}(y)$ provided by

$$\sum_{\rho=0}^{\infty} \phi_{\rho,\lambda}(y) \frac{z^{\rho}}{\rho!} = e^{y(e_{\lambda}(z)-1)}.$$

We see from (1.20) and (1.23) that

$$\phi_{\rho,\lambda}^{T}(y) = \sum_{u=0}^{\rho} S_{2,\lambda}^{T}(\rho, u) y^{u}, \ (\rho \geq 0).$$

The probabilistic fully degenerate model of Bell polynomials of the *s.k.* associated with *r.v.* T which are given by

$$\sum_{\rho=0}^{\infty}\mathbb{B}_{\rho,\lambda}^{(*,T)}(y)\frac{z^{\rho}}{\rho!}=e_{\lambda}^{y}(E[e_{\lambda}^{T}(z)]-1). \tag{1.25}$$

For $T=1$, we have $\mathbb{B}_{\rho,\lambda}^{(*,T)}(y)=\mathbb{B}_{\rho,\lambda}^{*}(y)$ in (1.10). The corresponding numbers of $\mathbb{B}_{\rho,\lambda}^{(*,T)}(y)$ are derived by taking $y=1$, namely $\mathbb{B}_{\rho,\lambda}^{(*,T)}:=\mathbb{B}_{\rho,\lambda}^{(*,T)}(1)$.

We observe that

$$\mathbb{B}_{\rho,\lambda}^{(*,T)}(y)=\sum_{u=0}^{\rho}(y)_{u,\lambda}S_{2,\lambda}^{T}(\rho,u).$$

The probabilistic degenerate model of Fubini polynomials associated with *r.v.* T are defined by (cf. [29])

$$\sum_{\rho=0}^{\infty}F_{\rho,\lambda}^{T}(y)\frac{z^{\rho}}{\rho!}=\frac{1}{1-y\left(E[e_{\lambda}^{T}(z)]-1\right)}. \tag{1.26}$$

We abbreviate the probabilistic degenerate Fubini polynomials as the *PDF* polynomials. The corresponding numbers of $F_{\rho,\lambda}^{T}$ are derived by taking $y=1$, namely $F_{\rho,\lambda}^{T}=F_{\rho,\lambda}^{T}(1)$. From (1.20) and (1.26), we provide that

$$F_{\rho,\lambda}^{T}(y)=\sum_{u=0}^{\rho}y^{u}u!S_{2,\lambda}^{T}(\rho,u),$$

for $\rho\geq u\geq 0$. Some other detailed properties and relations of the *PDF* polynomials were derived in [29].

2. Probabilistic Degenerate Jindalrae and Jindalrae-Stirling Polynomials of the Second Kind

In this part, we introduce a probabilistic degenerate Jindalrae-Stirling polynomials of the *s.k.* and a probabilistic degenerate Jindalrae polynomials. Then, we derive some of their properties and formulas, including explicit expressions, recurrence relations, and summation formulas. Moreover, we investigate diverse correlations with the probabilistic degenerate Stirling numbers of the *s.k.* associated with T, the Stirling numbers of the first kind, the partial Bell polynomials, the derangement polynomials, the degenerate Bernoulli polynomials of the *s.k.* and the degenerate Euler polynomials.

Along the study, we suppose that T is a *r.v.* such that the *m.g.f.* of T is given as follows

$$E[e^{zT}]=\sum_{\rho=0}^{\infty}E[T^{\rho}]\frac{z^{\rho}}{\rho!},\ (|z|<\alpha)$$

which exists for some $0<\alpha$. We assume that

$$S_0=0 \text{ and } S_u=T_1+T_2+\cdots+T_u \text{ for } u\in\mathbb{N} \tag{2.1}$$

where $(T_j)_{j\geq 1}$ is a sequence of mutually independent copies of the *r.v.* T.

Let

$$\frac{\left(e^{E[e^{zT}]-1}-1\right)^u}{u!}=\sum_{\rho=u}^{\infty}\mathfrak{T}_T(\rho,u)\frac{z^\rho}{\rho!}. \tag{2.2}$$

Theorem 2.1. *For $\rho \geq \mu \geq 0$, we possess*

$$\mathfrak{T}_T(\rho,u)=\sum_{\mu=u}^{\rho}S_2(\mu,u)S_2^T(\rho,\mu). \tag{2.3}$$

Proof. Utilizing (1.21) and substituting z by $E[e^{zT}]-1$ in (1.1), we observe that

$$\begin{aligned}\frac{1}{u!}\left(e^{E[e^{zT}]-1}-1\right)^u &= \sum_{\mu=u}^{\infty}S_2(\mu,u)\frac{1}{\mu!}(E[e^{zT}]-1)^\mu\\ &= \sum_{\mu=u}^{\infty}S_2(\mu,u)\sum_{\rho=\mu}^{\infty}S_2^T(\rho,\mu)\frac{z^\rho}{\rho!}\\ &= \sum_{\rho=u}^{\infty}\left(\sum_{\mu=u}^{\rho}S_2(\mu,u)S_2^T(\rho,\mu)\right)\frac{z^\rho}{\rho!}. \qquad (2.4)\end{aligned}$$

As a consequence, by (2.2) and (2.4), we discover the asserted result (2.3). □

Theorem 2.2. *For $\rho_1,\ldots,\rho_u \geq 0$ with $\rho_1+\rho_2+\cdots+\rho_u=\rho$, we possess*

$$\mathfrak{T}_T(\rho,u)=\frac{1}{u!}\sum_{\rho_1+\rho_2+\cdots+\rho_u=\rho}\binom{\rho}{\rho_1,\ldots,\rho_u}\phi_{\rho_1}^T\ldots\phi_{\rho_u}^T=\sum_{\mu=u}^{\rho}S_2(\mu,u)S_2^T(\rho,\mu).$$

Proof. By utilizing (1.14) and (2.2), we possess

$$\frac{1}{u!}\left(e^{E[e^{zT}]-1}-1\right)^u=\frac{1}{u!}\left(\sum_{\rho=1}^{\infty}\phi_\rho^T\frac{z^\rho}{\rho!}\right)^u$$

$$=\frac{1}{u!}\sum_{\rho=u}^{\infty}\left(\sum_{\rho_1+\rho_2+\cdots+\rho_u=\rho}\binom{\rho}{\rho_1,\ldots,\rho_u}\phi_{\rho_1}^T\ldots\phi_{\rho_u}^T\right)\frac{z^\rho}{\rho!}. \tag{2.5}$$

As a consequence, by (1.24) and (2.5), we discover the desired equation in the theorem. □

We observe from (1.21), (1.24) and (2.2) that

$$\mathfrak{T}_T(\rho,1)=\sum_{\mu=1}^{\rho}S_2^T(\rho,\mu)=\phi_\rho^T,\ (\rho\geq 1).$$

For $u \geq 0$, as an extension of the notion of the probabilistic degenerate Stirling polynomials of the *s.k.*, we define a probabilistic degenerate Jindalrae-Stirling polynomials of the *s.k.* associated with *r.v.* T by

$$\frac{\left(e_\lambda(E[e_\lambda^T(z)]-1)-1\right)^u}{u!}e_\lambda^y(z)=\sum_{\rho=u}^{\infty}S_{J,\lambda}^{(2,T)}(\rho,u:y)\frac{z^\rho}{\rho!}. \tag{2.6}$$

The corresponding numbers of $S_{J,\lambda}^{(2,T)}(\rho, u : y)$ are derived by taking $y = 0$, namely $S_{J,\lambda}^{(2,T)}(\rho, u) = S_{J,\lambda}^{(2,T)}(\rho, u : 0)$ given as

$$\frac{\left(e_\lambda(E[e_\lambda^T(z)] - 1) - 1\right)^u}{u!} = \sum_{\rho=u}^{\infty} S_{J,\lambda}^{(2,T)}(\rho, u)\frac{z^\rho}{\rho!}. \tag{2.7}$$

We abbreviate the probabilistic degenerate Jindalrae-Stirling polynomials and numbers of the *s.k.* as the *PDJSPSK* polynomials and *PDJSNSK* numbers, respectively. We note from (2.7) that when $T = 1$, the new polynomials and numbers $S_{J,\lambda}^{(2,T)}(\rho, u : y)$ reduce to the degenerate Jindalrae-Stirling polynomials and numbers of the *s.k.*, provided (cf. [18]) by

$$\frac{(e_\lambda(e_\lambda(z) - 1) - 1)^u}{u!} e_\lambda^y(z) = \sum_{\rho=u}^{\infty} S_{J,\lambda}^{(2)}(\rho, u : y)\frac{z^\rho}{\rho!},$$

and

$$\frac{(e_\lambda(e_\lambda(z) - 1) - 1)^u}{u!} = \sum_{\rho=u}^{\infty} S_{J,\lambda}^{(2)}(\rho, u)\frac{z^\rho}{\rho!}.$$

Now, we analyze some properties and relations for the *PDJSPSK* polynomials and *PDJSNSK* numbers by the consecutive theorems (with their detailed proofs) given below. We first state a relation between the *PDJSNSK* numbers and the *PDSNSK* numbers.

Theorem 2.3. *For $\rho \geq u \geq 0$, we possess*

$$S_{J,\lambda}^{(2,T)}(\rho, u) = \sum_{\mu=u}^{\rho} S_{2,\lambda}(\mu, u) S_{2,\lambda}^T(\rho, \mu). \tag{2.8}$$

Proof. In view of (1.20) and (2.7), we acquire

$$\begin{aligned} \frac{1}{u!}\left(e_\lambda(E[e_\lambda^T(z)] - 1) - 1\right)^u &= \sum_{\mu=u}^{\infty} S_{2,\lambda}(\mu, u) \sum_{\rho=\mu}^{\infty} S_{2,\lambda}^T(\rho, \mu)\frac{z^\rho}{\rho!} \\ &= \sum_{\rho=u}^{\infty}\left(\sum_{\mu=u}^{\rho} S_{2,\lambda}(\mu, u) S_{2,\lambda}^T(\rho, \mu)\right)\frac{z^\rho}{\rho!}. \end{aligned} \tag{2.9}$$

As a consequence, by (2.7) and (2.9), we discover the equality in the theorem. □

When $u = 1$ in (2.7), we possess

$$\sum_{\rho=1}^{\infty} S_{j,\lambda}^{(2,T)}(\rho, 1)\frac{z^\rho}{\rho!} = e_\lambda(E[e_\lambda^T(z)] - 1) - 1 = \sum_{\rho=0}^{\infty} \mathbb{B}_{\rho,\lambda}^{(*,T)}\frac{z^\rho}{\rho!} - 1 = \sum_{\rho=1}^{\infty} \mathbb{B}_{\rho,\lambda}^{(*,T)}\frac{z^\rho}{\rho!},$$

which gives the following relation.

Corollary 2.4. *For $\rho \geq 1$, we possess*

$$S_{j,\lambda}^{(2,T)}(\rho, 1) = \mathbb{B}_{\rho,\lambda}^{(*,T)} = \sum_{\mu=1}^{\rho} S_{2,\lambda}(\mu, 1) S_{2,\lambda}^T(\rho, \mu) = \sum_{\mu=1}^{\rho} S_{2,\lambda}^T(\rho, \mu)(1)_{\mu,\lambda}. \tag{2.10}$$

A correlation between the *PDJSPSK* polynomials and *PDJSNSK* numbers is given as follows.

Theorem 2.5. *For $\rho \geq \mu \geq 0$, we acquire*

$$S_{J,\lambda}^{(2,T)}(\rho, u : y) = \sum_{\mu=0}^{\rho} \binom{\rho}{\mu} S_{J,\lambda}^{(2,T)}(\rho, \mu)(y)_{\rho-\mu,\lambda}.$$

Proof. By (2.6) and (2.7), we have

$$\begin{aligned} \sum_{\rho=u}^{\infty} S_{J,\lambda}^{(2,T)}(\rho, u : y) \frac{z^\rho}{\rho!} &= \frac{\left(e_\lambda(E[e_\lambda^T(z)] - 1) - 1\right)^u}{u!} e_\lambda^y(z) \\ &= \sum_{\rho=u}^{\infty} S_{J,\lambda}^{(2,T)}(\rho, u) \frac{z^\rho}{\rho!} \sum_{\rho=0}^{\infty} (y)_{\rho,\lambda} \frac{z^\rho}{\rho!} \\ &= \sum_{\rho=0}^{\infty} \left(\sum_{\mu=0}^{\rho} \binom{\rho}{\mu} S_{J,\lambda}^{(2,T)}(\rho, \mu)(y)_{\rho-\mu,\lambda} \right) \frac{z^\rho}{\rho!}, \end{aligned}$$

which yields the claimed equality in the theorem. □

Theorem 2.6. *For $\rho_1, \ldots, \rho_u \geq 0$ with $\rho_1 + \rho_2 + \cdots + \rho_u = \rho$, we have*

(2.11)

$$S_{J,\lambda}^{(2,T)}(\rho, u) = \frac{1}{u!} \sum_{\rho_1+\rho_2+\cdots+\rho_u=\rho} \binom{\rho}{\rho_1, \ldots, \rho_u} \mathbb{B}_{\rho_1,\lambda}^{(*,T)} \ldots \mathbb{B}_{\rho_u,\lambda}^{(*,T)} = \sum_{\mu=u}^{\rho} S_2(\mu, u) S_{2,\lambda}^T(\rho, \mu).$$

Proof. By utilizing (1.25) and (2.7), we have

$$\frac{1}{u!}\left(e_\lambda(E[e_\lambda^T(z)] - 1) - 1\right)^u = \frac{1}{u!} \left(\sum_{\rho=1}^{\infty} \mathbb{B}_{\rho_1,\lambda}^{(*,T)} \frac{z^\rho}{\rho!} \right)^u$$

$$= \frac{1}{u!} \sum_{\rho=u}^{\infty} \left(\sum_{\rho_1+\rho_2+\cdots+\rho_u=\rho} \binom{\rho}{\rho_1, \ldots, \rho_u} \mathbb{B}_{\rho_1,\lambda}^{(*,T)} \ldots \mathbb{B}_{\rho_u,\lambda}^{(*,T)} \right) \frac{z^\rho}{\rho!}. \tag{2.12}$$

As a consequence, by (1.14) and (2.12), we establish the equality (2.11). □

Note that $S_{J,\lambda}^{(2,T)}(\rho, 1) = \sum_{\mu=1}^{\rho} S_2(\mu, 1) S_{2,\lambda}^T(\rho, \mu)$ for $\rho \geq 1$.

Theorem 2.7. *For $\rho \geq u \geq 0$, we have*

$$S_{J,\lambda}^{(2,T)}(\rho, u) = \frac{1}{\mu!} \sum_{\mu=u}^{\rho} S_{2,\lambda}(\mu, u) \sum_{j=0}^{\mu} \binom{\mu}{j} (-1)^{\mu-j} E[(S_j)_{\rho,\lambda}]. \tag{2.13}$$

Proof. Substituting z by $E[e_\lambda^T(z)] - 1$ in (1.9), we have

$$\frac{1}{u!}\left(e_\lambda(E[e_\lambda^T(z)] - 1) - 1\right)^u = \sum_{\mu=u}^{\infty} S_{2,\lambda}(\mu, u) \frac{[E[e_\lambda^T(z)] - 1]^\mu}{\mu!}$$

$$= \sum_{\mu=u}^{\infty} S_{2,\lambda}(\mu,u) \frac{1}{\mu!} \sum_{j=0}^{\mu} \binom{\mu}{j} (-1)^{\mu-j} (E[e_{\lambda}^{T}(z))^{j}$$

$$= \frac{1}{\mu!} \sum_{\mu=u}^{\rho} S_{2,\lambda}(\mu,u) \sum_{j=0}^{\mu} \binom{\mu}{j} (-1)^{\mu-j} \underbrace{E[e_{\lambda}^{T}(z)] \ldots E[e_{\lambda}^{T}(z)]}_{j-\text{times}}$$

$$= \frac{1}{\mu!} \sum_{\mu=u}^{\rho} S_{2,\lambda}(\mu,u) \sum_{j=0}^{\mu} \binom{\mu}{j} (-1)^{\mu-j} E[e_{\lambda}^{T_1+\cdots+T_j}(z)]$$

$$= \sum_{\rho=0}^{\infty} \left(\frac{1}{\mu!} \sum_{\mu=u}^{\rho} S_{2,\lambda}(\mu,u) \sum_{j=0}^{\mu} \binom{\mu}{j} (-1)^{\mu-j} E[(T_1+T_2+\cdots+T_j)_{\rho,\lambda}] \right) \frac{z^{\rho}}{\rho!}$$

$$= \sum_{\rho=0}^{\infty} \left(\frac{1}{\mu!} \sum_{\mu=u}^{\rho} S_{2,\lambda}(\mu,u) \sum_{j=0}^{\mu} \binom{\mu}{j} (-1)^{\mu-j} E[(S_j)_{\rho,\lambda}] \right) \frac{z^{\rho}}{\rho!} \tag{2.14}$$

and

$$\frac{1}{u!} \left(e_{\lambda}(E[e_{\lambda}^{T}(z)]-1)-1 \right)^{u} = \sum_{\rho=u}^{\infty} S_{J,\lambda}^{(2,T)}(\rho,u) \frac{z^{\rho}}{\rho!}. \tag{2.15}$$

By (2.14) and (2.15), we discover the equality (2.13). □

Now, we define the probabilistic degenerate Jindalrae polynomials associated with the *r.v.* T by

$$\sum_{\rho=0}^{\infty} J_{\rho,\lambda}^{T}(y) \frac{z^{\rho}}{\rho!} = e_{\lambda}^{y} \left(e_{\lambda}(E[e_{\lambda}^{T}(z)]-1)-1 \right). \tag{2.16}$$

Note that $T=1$, then $J_{\rho,\lambda}^{T}(y) = J_{\rho,\lambda}(y)$ in (1.12). In the case $y=1$, $J_{\rho,\lambda}^{T} = J_{\rho,\lambda}^{T}(1)$ are called the probabilistic degenerate Jindalrae numbers associated with the *r.v.* T.

Theorem 2.8. *For $\rho \geq u \geq 0$, we have*

$$J_{\rho,\lambda}^{T}(y) = \sum_{u=0}^{\rho} (y)_{u,\lambda} S_{J,\lambda}^{(2,T)}(\rho,u). \tag{2.17}$$

Proof. We observe from (1.10) and (2.16) that

$$e_{\lambda}^{y} \left(e_{\lambda}(E[e_{\lambda}^{T}(z)]-1)-1 \right) = \sum_{u=0}^{\infty} (y)_{u,\lambda} \frac{1}{u!} \left(e_{\lambda}(E[e_{\lambda}^{T}(z)]-1)-1 \right)^{u}$$

$$= \sum_{u=0}^{\infty} (y)_{u,\lambda} \sum_{\rho=u}^{\infty} S_{J,\lambda}^{(2,T)}(\rho,u) \frac{z^{\rho}}{\rho!}$$

$$= \sum_{\rho=0}^{\infty} \left(\sum_{u=0}^{\rho} (y)_{u,\lambda} S_{J,\lambda}^{(2,T)}(\rho,u) \right) \frac{z^{\rho}}{\rho!}. \tag{2.18}$$

From (2.18), we discover the equality (2.17). □

In particular, for $y = 1$, we have

$$J_{\rho,\lambda}^{T} = \sum_{u=0}^{\rho} (1)_{u,\lambda} S_{J,\lambda}^{(2,T)}(\rho, u). \tag{2.19}$$

Theorem 2.9. *For $\rho \geq \mu \geq 0$, we possess*

$$J_{\rho,\lambda}^{T}(y) = \sum_{\mu=0}^{\rho} B_{\mu,\lambda}^{*}(y) S_{2,\lambda}^{T}(\rho, \mu). \tag{2.20}$$

Proof. We observe from (2.16) that

$$\begin{aligned} e_{\lambda}^{y}\left(e_{\lambda}(E[e_{\lambda}^{T}(z)] - 1) - 1\right) &= \sum_{\mu=0}^{\infty} B_{\mu,\lambda}^{*}(y) \frac{1}{\mu!} (E[e_{\lambda}^{T}(z)] - 1)^{\mu} \\ &= \sum_{\mu=0}^{\infty} B_{\mu,\lambda}^{*}(y) \sum_{\rho=\mu}^{\infty} S_{2,\lambda}^{T}(\rho, \mu) \frac{z^{\rho}}{\rho!} \\ &= \sum_{\rho=0}^{\infty} \left(\sum_{\mu=0}^{\rho} B_{\mu,\lambda}^{*}(y) S_{2,\lambda}^{T}(\rho, \mu) \right) \frac{z^{\rho}}{\rho!}. \end{aligned} \tag{2.21}$$

As a consequence, by (2.21), we discover the equality (2.20). ☐

In particular, from (2.20), we have

$$J_{\rho,\lambda}^{T} = \sum_{\mu=0}^{\rho} B_{\mu,\lambda}^{*} S_{2,\lambda}^{T}(\rho, \mu). \tag{2.22}$$

Theorem 2.10. *For $\rho \geq \mu \geq 0$, we have*

$$J_{\rho,\lambda}^{T}(y) = \frac{1}{\mu!} \sum_{\mu=0}^{\infty} B_{\mu,\lambda}^{*}(y) \sum_{j=0}^{\mu} \binom{\mu}{j} (-1)^{\mu-j} E[(S_j)_{\rho,\lambda}]. \tag{2.23}$$

Proof. We observe from (2.16) that

$$e_{\lambda}^{y}\left(e_{\lambda}(E[e_{\lambda}^{T}(z)] - 1) - 1\right) = \sum_{\mu=0}^{\infty} B_{\mu,\lambda}^{*}(y) \frac{1}{\mu!} (E[e_{\lambda}^{T}(z)] - 1)^{\mu}$$

$$\begin{aligned} &= \sum_{\mu=0}^{\infty} B_{\mu,\lambda}^{*}(y) \frac{1}{\mu!} \sum_{j=0}^{\mu} \binom{\mu}{j} (-1)^{\mu-j} (E[e_{\lambda}^{T}(z))^{j} \\ &= \frac{1}{\mu!} \sum_{\mu=0}^{\infty} B_{\mu,\lambda}^{*}(y) \sum_{j=0}^{\mu} \binom{\mu}{j} (-1)^{\mu-j} \underbrace{E[e_{\lambda}^{T}(z)] \dots E[e_{\lambda}^{T}(z)]}_{j-\text{times}} \\ &= \frac{1}{\mu!} \sum_{\mu=0}^{\infty} B_{\mu,\lambda}^{*}(y) \sum_{j=0}^{\mu} \binom{\mu}{j} (-1)^{\mu-j} E[e_{\lambda}^{T_1+\cdots+T_j}(z)] \\ &= \sum_{\rho=0}^{\infty} \left(\frac{1}{\mu!} \sum_{\mu=0}^{\infty} B_{\mu,\lambda}^{*}(y) \sum_{j=0}^{\mu} \binom{\mu}{j} (-1)^{\mu-j} E[(T_1 + T_2 + \cdots + T_j)_{\rho,\lambda}] \right) \frac{z^{\rho}}{\rho!} \end{aligned}$$

$$=\sum_{\rho=0}^{\infty}\left(\frac{1}{\mu!}\sum_{\mu=0}^{\infty}B^{*}_{\mu,\lambda}(y)\sum_{j=0}^{\mu}\binom{\mu}{j}(-1)^{\mu-j}E[(S_j)_{\rho,\lambda}]\right)\frac{z^{\rho}}{\rho!}. \tag{2.24}$$

As a consequence, by (2.24), we discover the equality (2.23). □

Theorem 2.11. *For $\rho_1,\ldots,\rho_u\geq 0$ with $\rho_1+\rho_2+\cdots+\rho_u=\rho$, we have*

$$J^{T}_{\rho,\lambda}(y)=\sum_{u=0}^{\rho}(y)_{u,\lambda}S^{(2,T)}_{J,\lambda}(\rho,u)$$

$$=\sum_{u=0}^{\rho}(y)_{u,\lambda}\frac{1}{u!}\sum_{\rho_1+\rho_2+\cdots+\rho_u=\rho}\binom{\rho}{\rho_1,\ldots,\rho_u}\mathbb{B}^{(*,T)}_{\rho_1,\lambda}\ldots\mathbb{B}^{(*,T)}_{\rho_u,\lambda}. \tag{2.25}$$

Proof. By utilizing (1.14) and (1.25), we have

$$e^{y}_{\lambda}\left(e_{\lambda}(E[e^{T}_{\lambda}(z)]-1)-1\right)=\sum_{u=0}^{\infty}(y)_{u,\lambda}\frac{1}{u!}(e_{\lambda}(E[e^{T}_{\lambda}(z)]-1)-1)^{u}$$

$$=\sum_{u=0}^{\infty}(y)_{u,\lambda}\frac{1}{u!}\left(\sum_{\rho=1}^{\infty}\mathbb{B}^{(*,T)}_{\rho,\lambda}\frac{z^{\rho}}{\rho!}\right)^{u}$$

$$=\sum_{u=0}^{\infty}(y)_{u,\lambda}\frac{1}{u!}\sum_{\rho=u}^{\infty}\left(\sum_{\rho_1+\rho_2+\cdots+\rho_u=\rho}\binom{\rho}{\rho_1,\ldots,\rho_u}\mathbb{B}^{(*,T)}_{\rho_1,\lambda}\ldots\mathbb{B}^{(*,T)}_{\rho_u,\lambda}\right)\frac{z^{\rho}}{\rho!}. \tag{2.26}$$

As a consequence, by (1.14), (2.16), and (2.26), we find the equality (2.25). □

Theorem 2.12. *For $\rho\geq u\geq 0$, we have*

$$J^{T}_{\rho,\lambda}(y_1+y_2)=\sum_{u=0}^{\rho}\binom{\rho}{u}J^{T}_{u,\lambda}(y_1)J^{T}_{\rho-u,\lambda}(y_2)=\sum_{u=0}^{\rho}\binom{\rho}{u}J^{T}_{u,\lambda}(y_2)J^{T}_{\rho-u,\lambda}(y_1).$$

Proof. Now, we observe that

$$\begin{aligned}\sum_{\rho=0}^{\infty}J^{T}_{\rho,\lambda}(y_1+y_2)\frac{z^{\rho}}{\rho!} &= e^{(y_1+y_2)}_{\lambda}\left(e_{\lambda}(E[e^{T}_{\lambda}(z)]-1)-1\right)\\ &= e^{y_1}_{\lambda}\left(e_{\lambda}(E[e^{T}_{\lambda}(z)]-1)-1\right)e^{y_2}_{\lambda}\left(e_{\lambda}(E[e^{T}_{\lambda}(z)]-1)-1\right)\\ &= \sum_{u=0}^{\infty}J^{T}_{u,\lambda}(y_1)\frac{z^{u}}{u!}\sum_{\rho=0}^{\infty}J^{T}_{\rho,\lambda}(y_2)\frac{z^{\rho}}{\rho!}\\ &= \sum_{\rho=0}^{\infty}\left(\sum_{u=0}^{\rho}\binom{\rho}{u}J^{T}_{u,\lambda}(y_1)J^{T}_{\rho-u,\lambda}(y_2)\right)\frac{z^{\rho}}{\rho!}. \end{aligned}\tag{2.27}$$

As a consequence, by (2.27), we discover the asserted equality in the theorem. □

Theorem 2.13. *For $\rho\geq u\geq 0$, we have*

$$\sum_{j=0}^{\rho-u}\binom{\rho}{u}u^{j}(y)_{j,\lambda}S^{(2,T)}_{J,\lambda}(\rho-u,j) \tag{2.28}$$

$$= B_{\rho,u}\left(J_{0,\lambda}^{T}(y), 2J_{1,\lambda}^{T}(y), 3J_{2,\lambda}^{T}(y), \dots, (\rho-u+1)J_{\rho-u,\lambda}^{T}(y)\right).$$

Proof. We consider that

$$ze_{\lambda}^{y}\left(e_{\lambda}(E[e_{\lambda}^{T}(z)]-1)-1\right) = z\sum_{\rho=0}^{\infty} J_{\rho,\lambda}^{T}(y)\frac{z^{\rho}}{\rho!} = \sum_{\rho=1}^{\infty}\rho J_{\rho-1,\lambda}^{T}(y)\frac{z^{\rho}}{\rho!}. \tag{2.29}$$

By (2.16), we acquire

$$\begin{aligned}
\left(\sum_{\rho=1}^{\infty}\rho J_{\rho-1,\lambda}^{T}(y)\frac{z^{\rho}}{\rho!}\right)^{u} &= \left(ze_{\lambda}^{y}\left(e_{\lambda}(E[e_{\lambda}^{T}(z)]-1)-1\right)\right)^{u} \\
&= z^{u}\sum_{j=0}^{\infty}u^{j}(y)_{j,\lambda}\frac{1}{j!}(e_{\lambda}(E[e_{\lambda}^{T}(z)]-1)-1)^{j} \\
&= z^{u}\sum_{j=0}^{\infty}u^{j}(y)_{j,\lambda}\sum_{\rho=j}^{\infty}S_{J,\lambda}^{(2,T)}(\rho,j)\frac{z^{\rho}}{\rho!} \\
&= \sum_{\rho=0}^{\infty}\left(\sum_{j=0}^{\rho}u^{j}(y)_{j,\lambda}S_{J,\lambda}^{(2,T)}(\rho,j)\right)\frac{z^{\rho+u}}{\rho!} \\
&= \sum_{\rho=u}^{\infty}\left(\sum_{j=0}^{\rho-u}u^{j}(y)_{j,\lambda}S_{J,\lambda}^{(2,T)}(\rho-u,j)\right)\frac{z^{\rho}}{(\rho-u)!} \\
&= \sum_{\rho=u}^{\infty}\left(\sum_{j=0}^{\rho-u}\binom{\rho}{u}u!u^{j}(y)_{j,\lambda}S_{J,\lambda}^{(2,T)}(\rho-u,j)\right)\frac{z^{\rho}}{\rho!}. \qquad (2.30)
\end{aligned}$$

From (1.14) and (2.30), we note that

$$\begin{aligned}
&\sum_{\rho=u}^{\infty}\left(\sum_{j=0}^{\rho-u}\binom{\rho}{u}u^{j}(y)_{j,\lambda}S_{J,\lambda}^{(2,T)}(\rho-u,j)\right)\frac{z^{\rho}}{\rho!} = \frac{1}{u!}\left(\sum_{j=1}^{\infty}jJ_{j-1,\lambda}^{T}(y)\frac{z^{j}}{j!}\right)^{u} \\
&= \sum_{\rho=u}^{\infty}B_{\rho,u}\left(J_{j-1,\lambda}^{T}(y), 2J_{1,\lambda}^{T}(y), 3J_{2,\lambda}^{T}(y), \dots, (\rho-u+1)J_{\rho-u,\lambda}^{T}(y)\right)\frac{z^{\rho}}{\rho!}. \qquad (2.31)
\end{aligned}$$

As a consequence, by (2.31), we discover the equality (2.28). ∎

Theorem 2.14. *For $\rho \geq u \geq 0$, we have*

$$B_{\rho,u}\left(J_{0,\lambda}^{T}(y), J_{1,\lambda}^{T}(y), J_{2,\lambda}^{T}(y), \dots, J_{\rho-u,\lambda}^{T}(y)\right) = \sum_{j=u}^{\rho}S_{2,\lambda}(j,u)(y)_{j,\lambda}S_{J,\lambda}^{(2,T)}(\rho,\mu).$$

Proof. We observe from (1.14) and (2.31) that

$$\sum_{\rho=u}^{\infty}B_{\rho,u}\left(J_{0,\lambda}^{T}(y), J_{2,\lambda}^{T}(y), J_{3,\lambda}^{T}(y), \dots, J_{\rho-u,\lambda}^{T}(y)\right)\frac{z^{\rho}}{\rho!}$$

$$= \frac{1}{u!}\left(\sum_{j=1}^{\infty} J_{j,\lambda}^{T}(y)\frac{z^j}{j!}\right)^u$$
$$= \frac{1}{u!}\left(e_\lambda^y\left(e_\lambda(E[e_\lambda^T(z)]-1)-1\right)\right)^u$$
$$= \sum_{j=u}^{\infty} S_{2,\lambda}(j,u)(y)_{j,\lambda}\frac{1}{j!}\left(e_\lambda(E[e_\lambda^T(z)]-1)-1\right)^j$$
$$= \sum_{j=u}^{\infty} S_{2,\lambda}(j,u)(y)_{j,\lambda}\sum_{\rho=j}^{\infty} S_{J,\lambda}^{(2,T)}(\rho,\mu)\frac{z^\rho}{\rho!}$$
$$= \sum_{\rho=u}^{\infty}\left(\sum_{j=u}^{\rho} S_{2,\lambda}(j,u)(y)_{j,\lambda}S_{J,\lambda}^{(2,T)}(\rho,\mu)\right)\frac{z^\rho}{\rho!}. \tag{2.32}$$

By comparing the coefficients of z on both sides of (2.32), we discover the claimed equality in the theorem. □

Theorem 2.15. *For $\rho \geq u \geq 0$, we have*

$$\sum_{u=\alpha}^{\rho}\binom{\rho}{u}S_{2,\lambda}^{T}(u,\alpha)\mathbb{B}_{\rho-u,\lambda}^{(*,T)}(y) = \sum_{u=\alpha}^{\rho}\sum_{\mu=0}^{\rho-u}\binom{\rho}{u}S_{J,\lambda}^{(2,T)}(u,\alpha)\beta_{\mu,\lambda}^{(\alpha)}(y)\, S_{2,\lambda}^{T}(\rho-u,\mu).$$

Proof. Substituting z by $E[e_\lambda^T(z)]-1$ in (1.6), utilizing (1.20), (1.25), and (2.6), we acquire

$$\frac{(E[e_\lambda^T(z)]-1)^\alpha}{(e_\lambda(E[e_\lambda^T(z)]-1)-1)^\alpha}e_\lambda^y\left(E[e_\lambda^T](z)-1\right) = \sum_{\mu=0}^{\infty}\beta_{\mu,\lambda}^{(\alpha)}(y)\frac{1}{\mu!}(E[e_\lambda^T(z)]-1)^\mu$$
$$\Longleftrightarrow$$

$$\frac{(E[e_\lambda^T(z)]-1)^\alpha}{\alpha!}e_\lambda^y\left(E[e_\lambda^T](z)-1\right)$$
$$= \frac{(e_\lambda(E[e_\lambda^T(z)]-1)-1)^\alpha}{\alpha!}\sum_{\mu=0}^{\infty}\beta_{\mu,\lambda}^{(\alpha)}(y)\sum_{\rho=\mu}^{\infty}S_{2,\lambda}^{T}(\rho,\mu)\frac{z^\rho}{\rho!}$$
$$= \sum_{u=\alpha}^{\infty}S_{J,\lambda}^{(2,T)}(u,\alpha)\frac{z^u}{u!}\sum_{\rho=0}^{\infty}\sum_{\mu=0}^{\rho}\beta_{\mu,\lambda}^{(\alpha)}(y)S_{2,\lambda}^{T}(\rho,\mu)\frac{z^\rho}{\rho!}$$
$$= \sum_{\rho=u}^{\infty}\left(\sum_{u=\alpha}^{\rho}\sum_{\mu=0}^{\rho-u}\binom{\rho}{u}S_{J,\lambda}^{(2,T)}(u,\alpha)\beta_{\mu,\lambda}^{(\alpha)}(y)S_{2,\lambda}^{T}(\rho-u,\mu)\right)\frac{z^\rho}{\rho!} \tag{2.33}$$

and

$$\frac{(E[e_\lambda^T(z)]-1)^\alpha}{\alpha!}e_\lambda^y\left(E[e_\lambda^T](z)-1\right) = \sum_{u=\alpha}^{\infty}S_{2,\lambda}^{T}(u,\alpha)\frac{z^u}{u!}\sum_{\rho=0}^{\infty}\mathbb{B}_{\rho,\lambda}^{(*,T)}(y)\frac{z^\rho}{\rho!}$$

$$= \sum_{\rho=u}^{\infty}\left(\sum_{u=\alpha}^{\rho}\binom{\rho}{u}S_{2,\lambda}^{T}(u,\alpha)\mathbb{B}_{\rho-u,\lambda}^{(*,T)}(y)\right)\frac{z^{\rho}}{\rho!}. \tag{2.34}$$

From (2.33) and (2.34), we discover the claimed equality in the theorem. □

Theorem 2.16. *For $\rho \geq u \geq 0$, we have*

$$\sum_{u=\alpha}^{\rho}\binom{\rho}{u}S_{J,\lambda}^{T}(u,\alpha)\mathbb{B}_{\rho-u,\lambda}^{(*,T)}(y) = \sum_{u=\alpha}^{\rho}\sum_{\mu=0}^{\rho-u}\binom{\rho}{u}b_{\mu,\lambda}^{(\alpha)}(y)S_{J,\lambda}^{T}(\rho-u,\mu)S_{2,\lambda}^{T}(u,\alpha).$$

Proof. Substituting z by $e_\lambda(E[e_\lambda^T(z)]-1)-1$ in (1.7), utilizing (1.20) and (1.25), we acquire

$$\left(\frac{e_\lambda(E[e_\lambda^T(z)]-1)-1}{E[e_\lambda^T(z)]-1}\right)^{\alpha}e_\lambda^{y}\left(E[e_\lambda^T](z)-1\right) = \sum_{\mu=0}^{\infty}b_{\mu,\lambda}^{(\alpha)}(y)\frac{1}{\mu!}(e_\lambda(E[e_\lambda^T(z)]-1)-1)^{\mu}$$

$$\Longleftrightarrow$$

$$\frac{1}{\alpha!}\left(E[e_\lambda^T(z)]-1\right)^{\alpha}\sum_{\mu=0}^{\infty}b_{\mu,\lambda}^{(\alpha)}(y)\sum_{\rho=\mu}^{\infty}S_{J,\lambda}^{T}(\rho,\mu)\frac{z^{\rho}}{\rho!}$$

$$= \sum_{u=\alpha}^{\infty}S_{2,\lambda}^{T}(u,\alpha)\frac{z^{u}}{u!}\sum_{\rho=0}^{\infty}\sum_{\mu=0}^{\rho}b_{\mu,\lambda}^{(\alpha)}(y)S_{J,\lambda}^{T}(\rho,\mu)\frac{z^{\rho}}{\rho!}$$

$$= \sum_{\rho=u}^{\infty}\left(\sum_{u=\alpha}^{\rho}\sum_{\mu=0}^{\rho-u}\binom{\rho}{u}b_{\mu,\lambda}^{(\alpha)}(y)S_{J,\lambda}^{T}(\rho-u,\mu)S_{2,\lambda}^{T}(u,\alpha)\right)\frac{z^{\rho}}{\rho!} \tag{2.35}$$

and

$$\frac{1}{\alpha!}\left(e_\lambda(E[e_\lambda^T(z)]-1)-1\right)^{\alpha}e_\lambda^{y}\left(E[e_\lambda^T](z)-1\right) = \sum_{u=\alpha}^{\infty}S_{J,\lambda}^{T}(u,\alpha)\frac{z^{u}}{u!}\sum_{\rho=0}^{\infty}\mathbb{B}_{\rho,\lambda}^{(*,T)}(y)\frac{z^{\rho}}{\rho!}$$

$$= \sum_{\rho=u}^{\infty}\left(\sum_{u=\alpha}^{\rho}\binom{\rho}{u}S_{J,\lambda}^{T}(u,\alpha)\mathbb{B}_{\rho-u,\lambda}^{(*,T)}(y)\right)\frac{z^{\rho}}{\rho!}. \tag{2.36}$$

As a consequence, by (2.35) and (2.36), we discover the claimed equality in the theorem. □

Theorem 2.17. *For $\rho \geq \mu \geq 0$, we have*

$$\sum_{\mu=0}^{\rho}(y-1)_{\mu,\lambda}(-1)^{\mu}S_{2,-\lambda}^{T}(\rho,\mu) = \sum_{u=0}^{\rho}\sum_{\mu=0}^{\rho-u}\binom{\rho}{u}E[(S_\alpha)_{u,-\lambda}]d_{\mu,\lambda}^{(\alpha)}(y)(-1)^{\mu}S_{2,-\lambda}^{T}(\rho-u,\mu).$$

Proof. Substituting z by $1-E[e_{-\lambda}^T(z)]$ in (1.11), utilizing (1.20) and (1.25), we acquire

$$\frac{1}{[E[e_{-\lambda}^T(z)]]^{\alpha}}e_\lambda^{y-1}\left(1-E[e_{-\lambda}^T(z)]\right) = \sum_{\mu=0}^{\infty}d_{\mu,\lambda}^{(\alpha)}(y)\frac{(-1)^{\mu}}{\mu!}(E[e_{-\lambda}^T(z)]-1)^{\mu}$$

$$\Longleftrightarrow$$

$$[E[e^T_{-\lambda}(z)]]^\alpha \sum_{\mu=0}^{\infty} d^{(\alpha)}_{\mu,\lambda}(y)\frac{(-1)^\mu}{\mu!}(E[e^T_{-\lambda}(z)]-1)^\mu$$
$$=\sum_{u=0}^{\infty} E[(T_1+T_2+\cdots+T_\alpha)_{u,-\lambda}]\frac{z^u}{u!}\sum_{\mu=0}^{\infty} d^{(\alpha)}_{\mu,\lambda}(y)(-1)^\mu \sum_{\rho=\mu}^{\infty} S^T_{2,-\lambda}(\rho,\mu)\frac{z^\rho}{\rho!}$$
$$=\sum_{u=0}^{\infty} E[(T_1+T_2+\cdots+T_\alpha)_{u,-\lambda}]\frac{z^u}{u!}\sum_{\rho=0}^{\infty}\sum_{\mu=0}^{\rho} d^{(\alpha)}_{\mu,\lambda}(y)(-1)^\mu S^T_{2,-\lambda}(\rho,\mu)\frac{z^\rho}{\rho!}$$
$$=\sum_{\rho=0}^{\infty}\left(\sum_{u=0}^{\rho}\sum_{\mu=0}^{\rho-u}\binom{\rho}{u}E[(S_\alpha)_{u,-\lambda}]d^{(\alpha)}_{\mu,\lambda}(y)(-1)^\mu S^T_{2,-\lambda}(\rho-u,\mu)\right)\frac{z^\rho}{\rho!} \tag{2.37}$$

and

$$e^{y-1}_\lambda\left(1-E[e^T_{-\lambda}(z)]\right) = \sum_{\mu=0}^{\infty}(y-1)_{\mu,\lambda}\frac{1}{\mu!}(1-E[e^T_{-\lambda}(z)])^\mu$$
$$= \sum_{\mu=0}^{\infty}(y-1)_{\mu,\lambda}(-1)^\mu\frac{1}{\mu!}(E[e^T_{-\lambda}(z)]-1)^\mu$$
$$= \sum_{\mu=0}^{\infty}(y-1)_{\mu,\lambda}(-1)^\mu\sum_{\rho=\mu}^{\infty}S^T_{2,-\lambda}(\rho,\mu)\frac{z^\rho}{\rho!}$$
$$= \sum_{\rho=0}^{\infty}\left(\sum_{\mu=0}^{\rho}(y-1)_{\mu,\lambda}(-1)^\mu S^T_{2,-\lambda}(\rho,\mu)\right)\frac{z^\rho}{\rho!}. \tag{2.38}$$

From (2.37) and (2.38), we discover the claimed equality in the theorem. □

Theorem 2.18. *For $\rho \geq u \geq 0$, we have*

$$\sum_{u=0}^{\rho}\sum_{\mu=0}^{u}\binom{\rho}{u}E^{(\alpha)}_{\mu,\lambda}S^T_{2,\lambda}(u,\mu)\mathbb{B}^{(*,T)}_{\rho-u,\lambda}(y)=\sum_{\mu=0}^{\rho}E^{(\alpha)}_{\mu,\lambda}(y)S^T_{2,\lambda}(\rho,\mu).$$

Proof. Substituting z by $E[e^T_\lambda(z)]-1$ in (1.6), then we note from (1.21), (1.25), and (3.2) that

$$\left(\frac{2}{e_\lambda(E[e^T_\lambda(z)]-1)+1}\right)^\alpha e^y_\lambda\left(E[e^T_\lambda(z)]-1\right)=\sum_{\mu=0}^{\infty}E^{(\alpha)}_{\mu,\lambda}(y)\frac{1}{\mu!}(E[e^T_\lambda(z)]-1)^\mu$$
$$=\sum_{\mu=0}^{\infty}E^{(\alpha)}_{\mu,\lambda}(y)\sum_{\rho=\mu}^{\infty}S^T_{2,\lambda}(\rho,\mu)\frac{z^\rho}{\rho!}$$
$$=\sum_{\rho=0}^{\infty}\left(\sum_{\mu=0}^{\rho}E^{(\alpha)}_{\mu,\lambda}(y)S^T_{2,\lambda}(\rho,\mu)\right)\frac{z^\rho}{\rho!} \tag{2.39}$$

and

$$\left(\frac{2}{e_\lambda(E[e^T_\lambda(z)]-1)+1}\right)^\alpha e^y_\lambda\left(E[e^T_\lambda(z)]-1\right)$$

$$= \sum_{\mu=0}^{\infty} E_{\mu,\lambda}^{(\alpha)} \frac{1}{\mu!} (E[e_\lambda^T(z)] - 1)^\mu \sum_{\rho=0}^{\infty} \mathbb{B}_{\rho,\lambda}^{(*,T)}(y) \frac{z^\rho}{\rho!}$$

$$= \sum_{\rho=0}^{\infty} \left(\sum_{\mu=0}^{\rho} E_{\mu,\lambda}^{(\alpha)} S_{2,\lambda}^T(\rho,\mu) \right) \frac{z^\rho}{\rho!} \left(\sum_{\rho=0}^{\infty} \mathbb{B}_{\rho,\lambda}^{(*,T)}(y) \frac{z^\rho}{\rho!} \right)$$

$$= \sum_{\rho=0}^{\infty} \left(\sum_{u=0}^{\rho} \sum_{\mu=0}^{u} \binom{\rho}{u} E_{\mu,\lambda}^{(\alpha)} S_{2,\lambda}^T(u,\mu) \mathbb{B}_{\rho-u,\lambda}^{(*,T)}(y) \right) \frac{z^\rho}{\rho!}. \tag{2.40}$$

As a consequence, by (2.39) and (2.40), we discover the claimed equality in the theorem. □

3. Further remarks

In this part, the probabilistic higher-order degenerate Bernoulli and Euler polynomials are provided and some properties and relations are then provided.

The probabilistic forms of the higher-order degenerate Bernoulli and the Euler polynomials are introduced as follows:

$$\sum_{\rho=0}^{\infty} \beta_{\rho,\lambda}^{(\alpha,T)}(y) \frac{z^\rho}{\rho!} = \left(\frac{z}{E[e_\lambda^T(z)] - 1} \right)^\alpha e_\lambda^y(z) \tag{3.1}$$

$$\text{and } \sum_{\rho=0}^{\infty} E_{\rho,\lambda}^{(\alpha,T)}(y) \frac{z^\rho}{\rho!} = \left(\frac{2}{E[e_\lambda^T(z)] + 1} \right)^\alpha e_\lambda^y(z).$$

The corresponding numbers of $\beta_{\rho,\lambda}^{(\alpha,T)}(y)$ and $E_{\rho,\lambda}^{(\alpha,T)}(y)$ are derived by taking $y = 0$, namely $\beta_{\rho,\lambda}^{(\alpha,T)} := \beta_{\rho,\lambda}^{(\alpha,T)}(0)$ and $E_{\rho,\lambda}^{(\alpha,T)} := E_{\rho,\lambda}^{(\alpha,T)}(0)$, respectively. Also, the probabilistic Bernoulli and Euler polynomials are obtained when $\alpha = 1$ and λ approaches to 0 as $\lim_{\lambda\to 0} \beta_{\rho,\lambda}^{(\alpha,T)}(y) = \beta_\rho^T(y)$ and $\lim_{\lambda\to 0} E_{\rho,\lambda}^{(\alpha,T)}(y) = E_\rho^T(y)$, cf. [16, 23]. Moreover, when $\alpha = 1$ in (3.1), we acquire the probabilistic forms of the degenerate Bernoulli and the Euler polynomials as follows

$$\sum_{\rho=0}^{\infty} \beta_{\rho,\lambda}^{T}(y) \frac{z^\rho}{\rho!} = \frac{z}{E[e_\lambda^T(z)] - 1} e_\lambda^y(z) \tag{3.2}$$

$$\text{and } \sum_{\rho=0}^{\infty} E_{\rho,\lambda}^{(\alpha,T)}(y) \frac{z^\rho}{\rho!} = \frac{2}{E[e_\lambda^T(z)] + 1} e_\lambda^y(z).$$

The corresponding numbers of $\beta_{\rho,\lambda}^T(y)$ and $E_{\rho,\lambda}^T(y)$ are derived by taking $y = 0$, namely $\beta_{\rho,\lambda}^T := \beta_{\rho,\lambda}^T(0)$ and $E_{\rho,\lambda}^T := E_{\rho,\lambda}^T(0)$, respectively.

We observe from (3.1) that

$$\sum_{\rho=0}^{\infty} \beta_{\rho,\lambda}^{(\alpha,T)}(y) \frac{z^\rho}{\rho!} = \left(\frac{z}{E[e_\lambda^T(z)] - 1} \right)^\alpha e_\lambda^y(z)$$

$$= \sum_{\rho=u}^{\infty} \beta_{\rho,\lambda}^{(\alpha,T)} \frac{z^{\rho}}{\rho!} \sum_{\rho=0}^{\infty} (y)_{\rho,\lambda} \frac{z^{\rho}}{\rho!}$$

$$= \sum_{\rho=0}^{\infty} \left(\sum_{\mu=0}^{\rho} \binom{\rho}{\mu} \beta_{\mu,\lambda}^{(\alpha,T)} (y)_{\rho-\mu,\lambda} \right) \frac{z^{\rho}}{\rho!},$$

and

$$\sum_{\rho=0}^{\infty} E_{\rho,\lambda}^{(\alpha,T)}(y) \frac{z^{\rho}}{\rho!} = \left(\frac{2}{E[e_{\lambda}^{T}(z)]+1} \right)^{\alpha} e_{\lambda}^{y}(z)$$

$$= \sum_{\rho=u}^{\infty} E_{\rho,\lambda}^{(\alpha,T)} \frac{z^{\rho}}{\rho!} \sum_{\rho=0}^{\infty} (y)_{\rho,\lambda} \frac{z^{\rho}}{\rho!}$$

$$= \sum_{\rho=0}^{\infty} \left(\sum_{\mu=0}^{\rho} \binom{\rho}{\mu} E_{\mu,\lambda}^{(\alpha,T)} (y)_{\rho-\mu,\lambda} \right) \frac{z^{\rho}}{\rho!},$$

which yield

$$\beta_{\rho,\lambda}^{(\alpha,T)}(y) = \sum_{\mu=0}^{\rho} \binom{\rho}{\mu} \beta_{\mu,\lambda}^{(\alpha,T)} (y)_{\rho-\mu,\lambda}$$

and

$$E_{\rho,\lambda}^{(\alpha,T)}(y) = \sum_{\mu=0}^{\rho} \binom{\rho}{\mu} E_{\mu,\lambda}^{(\alpha,T)} (y)_{\rho-\mu,\lambda}.$$

We observe from (1.26) and (3.2) that

$$\sum_{\rho=0}^{\infty} F_{\rho,\lambda}^{T}(-\frac{1}{2}) \frac{z^{\rho}}{\rho!} = \frac{1}{1+\frac{1}{2}\left(E[e_{\lambda}^{T}(z)]-1\right)} = \sum_{\rho=0}^{\infty} E_{\rho,\lambda}^{T} \frac{z^{\rho}}{\rho!},$$

which gives

$$F_{\rho,\lambda}^{T}(-\frac{1}{2}) = E_{\rho,\lambda}^{T}.$$

More properties of $\beta_{\rho,\lambda}^{(\alpha,T)}(y)$ and $E_{\rho,\lambda}^{(\alpha,T)}(y)$ can be examined similar to those of completed for $S_{J,\lambda}^{(2,T)}(\rho, u : y)$ in (2.6) and $J_{\rho,\lambda}^{T}(y)$ in (2.16). We here provide a correlation as follows.

Theorem 3.1. *We possess, for $\rho \geq \mu \geq 0$, that*

(3.3)
$$\sum_{\mu=0}^{\rho} b_{\mu,\lambda}(y) S_{J,\lambda}^{T}(\rho,\mu) = (\rho+1) \sum_{\mu=0}^{\rho+1} \binom{\rho+1}{\mu} \frac{\mathbb{B}_{\mu,\lambda}^{(*,T)}(y+1) - \mathbb{B}_{\mu,\lambda}^{(*,T)}(y)}{\mu+1} \beta_{\rho+1-\mu,\lambda}^{T}.$$

Proof. Substituting z by $e_{\lambda}(E[e_{\lambda}^{T}](z)-1)-1$ in (1.8), then we note from (1.22), (1.25), and (3.2) that

$$\frac{e_{\lambda}(E[e_{\lambda}^{T}(z)]-1)-1}{E[e_{\lambda}^{T}(z)]-1} e_{\lambda}^{y}\left(E[e_{\lambda}^{T}(z)]-1\right) = \sum_{\mu=0}^{\infty} b_{\mu,\lambda}(y) \frac{1}{\mu!} (e_{\lambda}(E[e_{\lambda}^{T}(z)]-1)-1)^{\mu}$$

$$
\begin{aligned}
&= \sum_{\mu=0}^{\infty} b_{\mu,\lambda}(y) \sum_{\rho=\mu}^{\infty} S_{J,\lambda}^{T}(\rho,\mu) \frac{z^{\rho}}{\rho!} \\
&= \sum_{\rho=0}^{\infty} \left(\sum_{\mu=0}^{\rho} b_{\mu,\lambda}(y) S_{J,\lambda}^{T}(\rho,\mu) \right) \frac{z^{\rho}}{\rho!} \qquad (3.4)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{e_{\lambda}(E[e_{\lambda}^{T}(z)]-1)-1}{E[e_{\lambda}^{T}(z)]-1} e_{\lambda}^{y}\left(E[e_{\lambda}^{T}(z)]-1\right) \\
&= \left(\frac{z}{E[e_{\lambda}^{T}(z)]-1} \right) \frac{1}{z} \frac{e_{\lambda}^{y+1}(E[e_{\lambda}^{T}(z)]-1) - e_{\lambda}^{y}(E[e_{\lambda}^{T}(z)]-1)}{E[e_{\lambda}^{T}(z)]-1} \\
&= \left(\sum_{\rho=0}^{\infty} \beta_{\rho,\lambda}^{T} \frac{z^{\rho}}{\rho!} \right) \frac{1}{z} \sum_{\mu=0}^{\infty} \left(\mathbb{B}_{\mu,\lambda}^{(*,T)}(y+1) - B_{\mu,\lambda}^{(*,T)}(y) \right) \frac{z^{\mu}}{\mu!} \\
&= \left(\sum_{\rho=0}^{\infty} \beta_{\rho,\lambda}^{T} \frac{z^{\rho}}{\rho!} \right) \left(\sum_{\mu=0}^{\infty} \frac{\mathbb{B}_{\mu,\lambda}^{(*,T)}(y+1) - \mathbb{B}_{\mu,\lambda}^{(*,T)}(y)}{\mu+1} \frac{z^{\mu-1}}{\mu!} \right) \\
&= \sum_{\rho=0}^{\infty} \left(\sum_{\mu=0}^{\rho} \binom{\rho}{\mu} \frac{\mathbb{B}_{\mu,\lambda}^{(*,T)}(y+1) - \mathbb{B}_{\mu,\lambda}^{(*,T)}(y)}{\mu+1} \beta_{\rho-\mu,\lambda}^{T} \right) \frac{z^{\rho-1}}{\rho!}. \qquad (3.5)
\end{aligned}
$$

From (3.4) and (3.5), we discover the claimed equality (3.3). □

4. Conclusions

In recent years, probabilistic special polynomials and numbers such as probabilistic Bell, probabilistic Fubini, and probabilistic Stirling numbers and polynomials associated with random variables have been defined and studied in detail. In this work, we have considered a probabilistic degenerate Jindalrae-Stirling polynomials of the second kind and a probabilistic degenerate Jindalrae polynomials. Then, we have derived some of their properties and formulas, including explicit expressions, symmetric identity, recurrence relations, and summation formulas. Moreover, we have investigated diverse correlations with the probabilistic degenerate Stirling numbers of the second kind associated with T, the Stirling numbers of the first kind, the partial Bell polynomials, the derangement polynomials, the degenerate Bernoulli polynomials of the second kind and the degenerate Euler polynomials. Finally, we have defined probabilistic higher-order degenerate Bernoulli and Euler polynomials and then provided some properties and relations.

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