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A NEW PROOF TO THE FIXED POINT THEORY FOR PARTIAL METRIC SPACES BY (L, c)-EXPANSION MAPPINGS

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Abstract. In this paper, by using the Kummer's test, we present a new relation between $p(Tx, Ty)$ and $p(x, y)$ as a new fixed point result in partial metric space.

1. Introduction

The study of fixed point theory in partial metric spaces and their results by Kummer's test, is our goal in the new point of view. Our results improve very recent topics which is verified by F. Khojasteh el. al [3].

Theorem 1.1 (Kummer's Test, [4]). *Let $\sum_{n=1}^{\infty} u_n$ be a positive series.*

(1) *$\sum_{n=1}^{\infty} u_n$ is convergent if and only if there is a positive series $\sum_{n=1}^{\infty} k_n$ and a constant $c > 0$, such that $k_n(\frac{u_n}{u_{n+1}}) - k_{n+1} \geq c$.*

(2) *$\sum_{n=1}^{\infty} u_n$ is divergent if and only if there is a positive series $\sum_{n=1}^{\infty} k_n$ such that $\sum_{n=1}^{\infty} \frac{1}{k_n}$ diverges and $k_n(\frac{u_n}{u_{n+1}}) - k_{n+1} \leq 0$.*

In the simple case when $k_n := k$ be constant. By Kummer's test converts to form:

Let $\sum_{n=1}^{\infty} u_n$ be a positive series. Then, we hve

(1)

$$\sum_{n=1}^{\infty} u_n < \infty \iff \exists k, c > 0 \quad \text{such that} \quad k\left(\frac{u_n}{u_{n+1}}\right) - k \geq c.$$

(2)

$$\sum_{n=1}^{\infty} u_n > \infty \iff \frac{u_{n+1}}{u_n} \geq 1.$$

$$k\left(\frac{u_n}{u_{n+1}}\right) - k \geq c \iff \frac{u_{n+1}}{u_n} \leq \frac{k}{k+c} \leq 1 \iff \limsup_{n\to\infty} \frac{u_{n+1}}{u_n} \leq 1.$$

Definition 1.2 ([1, 2, 5]). Let X be a nonempty set and $p : X \times X \to \mathbb{R}^+$ be a self mapping of X such that for all $x, y, z \in X$ the followings are satisfied:

p1 $x = y \iff p(x, x) = p(x, y) = p(y, y)$,
p2 $p(x, x) \leq p(x, y)$,
p3 $p(x, y) = p(y, x)$,
p4 $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

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Then p is called partial metric on X and the pair (X, p) is called partial metric space (in short PMS).

We note that in a partial metric space (X, p):

(1) $p(x, y) = 0 \Rightarrow x = y$;
(2) $\lim_{n\to\infty} p(x_n, x) = p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$;
(3) by the references [1, 2], which is proved recently, the condition $p2$ is redundant;

for every $x, y, x_n \in X$.

2. Main Results

In this section, by using the Kummer's test, we present a new relation between $p(Tx, Ty)$ and $p(x, y)$ as a new fixed point result in partial metric space.

Definition 2.1. Let (X, p) be a PMS. The function $L : X \times X \to \mathbb{R}^+$ is called G-function if it satisfies the following items:

(i) $L(x, y) = L(y, x)$, for all $x, y \in X$,
(ii) for each sequence $\{x_n\} \subset X$ and each $y \in X$ and $c > 0$

$$x_n \to x \quad \textit{implies} \quad \frac{L(x_n, x)}{c + L(x_{n+1}, y)} \quad \textit{is bounded}$$

for sufficiently large $n \in \mathbb{N}$.

The set of all G-functions is denoted by $\mathfrak{S}_G(X)$.

Example 2.2. Let $X = \mathbb{R}$ endowed

$$p(x, y) = \max\{x, y\}, L(x, y) = xy \quad \text{or} \quad L(x, y) = |x + y| + 1$$

and let $c > 0$ be arbitrary. Then L is a $G-$function.

Example 2.3. Let (X, p) be partial metric space, $L(x, y) = p(x, y)$ and let $c > 0$ be arbitrary. Then L is a $G-$function.

Definition 2.4. Let (X, p) be a partial metric space. We say that $T : X \to X$ is (L, c)-expansion mapping, if there exist $L \in \mathfrak{S}_G(X)$ and $c > 0$ such that

$$p(Tx, Ty) \leq \frac{L(x, y)}{c + L(Tx, Ty)} p(x, y), \quad \forall x, y \in X. \tag{2.1}$$

As you know in finding a fixed point, we always put $a_n = d(x_n, x_{n-1})$, where $x_n = T^n(x_0)$ as an iterative sequence induced by the mapping T, initiated at x_0. So the convergence of the series $\sum_{n=1}^{\infty} a_n$, leads us to the limit point of x_n as the best candidate for the fixed point.

Theorem 2.5. *Let (X, p) be a complete PMS and let T from $X \to X$ be a (L, c)-expansion mapping. Then T has a unique fixed point.*

Proof. Let $x_0 \in X$ and let $x_1 = Tx_0$. If $x_0 = x_1$, then x_0 is the fixed point and the proof is completed. If x_n be selected, then we can define $x_{n+1} = Tx_n$, inductively.

Without loss of generality, we can suppose that $x_{n+1} \neq x_n$. Considering (2.1), we have

$$p(x_{n+1}, x_n) \leq \frac{L(x_n, x_{n-1})}{c + L(x_{n+1}, x_n)} p(x_n, x_{n-1}).$$

Letting $u_n = p(x_n, x_{n-1})$ and $k_n = L(x_n, x_{n-1})$, we yields

$$k_n \left(\frac{u_n}{u_{n+1}} \right) - k_{n+1} \geq c.$$

Thus, by the Kummer's test, the series $\sum_{n=1}^{\infty} u_n$ is convergent.
Now we have to show that: $\{x_n\}$ is a Cauchy sequence and so is convergent to a some unique $x \in X$.

Let $m, n \in \mathbb{N}$ and $m > n$. Then

$$p(x_n, x_m) \leq \sum_{k=n}^{m-1} p(x_i, x_{i+1}) = \sum_{k=n}^{m-1} u_{i+1} \to 0 \quad (m, n \to \infty).$$

Therefore, $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m \geq n\} = 0$. So, the sequence $\{x_n\}$ is Cauchy and since X is complete, there exists $x \in X$ such that $x_n \to x$, as $n \to \infty$.

$$\lim_{n\to\infty} p(x_n, x_m) = 0 = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

$$p(x_{n+1}, Tx) \leq \frac{L(x_n, x)}{c + L(x_{n+1}, Tx)} p(x_n, x). \tag{2.2}$$

According to (ii) of Definition 2.1, we have $\frac{L(x_n,x)}{c+L(x_{n+1},Tx)}$ is bounded. So by letting n tends to ∞, the right hand both side of (2.2), tends to zero i.e. $p(x_{n+1}, Tx) \to 0$ and it deduces that by uniqueness $Tx = x$.

To prove uniqueness, assume $x, y \in X$ are two fixed points. We have

$$p(x, y) = p(Tx, Ty) \leq \frac{L(x, y)}{c + L(Tx, Ty)} p(x, y) = \frac{L(x, y)}{c + L(x, y)} p(x, y).$$

$$p(x, y) \left(1 - \frac{L(x, y)}{c + L(x, y)} \right) \leq 0.$$

$$0 \leq p(x, y) \frac{c}{c + L(x, y)} \leq 0,$$

so we have $x = y$ and hence the fixed point is unique. □

Theorem 2.6. *Let (X, p) be a complete PMS and let T from $X \to X$ be a Banach contraction mapping. Then T has a unique fixed point in X.*

Proof. We just show that constant of A, here is $A \neq 1$.

$$p(Tx, Ty) \leq Ap(x, y)$$

for any $A \neq 1$. Put $k_n := \frac{cA}{|1-A|} \geq 0$ with $c > 0$. By Kummar's test we have

$$\frac{cA}{|1 - A|} \frac{p(x, y)}{p(Tx, Ty)} - \frac{cA}{|1 - A|} \geq c \iff p(Tx, Ty) \leq Ap(x, y).$$

If $A<1$ it's clear that

$$\frac{cA}{1-A}\frac{p(x,y)}{p(Tx,Ty)}-\frac{cA}{1-A}\geq c \iff p(Tx,Ty)\leq Ap(x,y).$$

By $A>1$

$$\frac{cA}{A-1}\frac{p(x,y)}{p(Tx,Ty)}-\frac{cA}{A-1}\geq c \iff \frac{A}{A-1}\left(\frac{p(x,y)}{p(Tx,Ty)}-1\right)\geq 1$$

$$\iff \frac{p(x,y)}{p(Tx,Ty)}\geq\frac{2A-1}{A}\geq\frac{1}{A} \iff p(Tx,Ty)\leq Ap(x,y).$$

□

Example 2.7. Let $X=\{0,1,\frac{1}{2}\}$ and let $p(x,y)=\max\{x,y\}$, for all $x,y\in X$. Now define

$$T0=0,\ T1=\frac{1}{2},\quad \text{and}\quad T\frac{1}{2}=0.$$

Also, define $L(x,y)=x+y+1$ and put $c=\frac{1}{2}$, we obtain

$$p(T0,T1)=p\left(0,\frac{1}{2}\right)=\frac{1}{2}\leq\frac{L(0,1)}{c+L(0,\frac{1}{2})}p(0,1)=\frac{4}{4}=1$$

$$p\left(T0,T\frac{1}{2}\right)=p(0,0)=0\leq\frac{L(0,\frac{1}{2})}{c+L(0,0)}p\left(0,\frac{1}{2}\right)=\frac{\frac{3}{2}}{\frac{3}{2}}\frac{1}{2}=\frac{1}{2}$$

$$p\left(T1,T\frac{1}{2}\right)=p\left(\frac{1}{2},0\right)=\frac{1}{2}\leq\frac{L(1,\frac{1}{2})}{c+L(\frac{1}{2},0)}p\left(1,\frac{1}{2}\right)=\frac{\frac{5}{2}}{\frac{4}{2}}=\frac{5}{4}$$

Thus, T satisfies in Theorem (2.5) and so is $(L,\frac{1}{2})$-expansion mapping. Note that the coefficient

$$\frac{L(1,\frac{1}{2})}{c+L(\frac{1}{2},0)}>1\quad\text{and}\quad\frac{L(0,1)}{c+L(0,\frac{1}{2})}=\frac{L(0,\frac{1}{2})}{c+L(0,0)}=1$$

which show that it is not needed to be less than one.

Our Theorems improve the following results in [3].

Corollary 2.8 ([3])**.** *Let (X,d) be a complete metric space and let T from $X\to X$ be a (L,c)-expansion mapping. Then T has a unique fixed point in X.*

Corollary 2.9 ([3])**.** *Let (X,d) be a complete metric space and let T from X into itself is a contraction mapping. Then T has a unique fixed point in X.*

Corollary 2.10 ([3])**.** *Let (X,d) be a complete metric space and let T from X itself be a mapping. Suppose that there exists $c>0$ such that*

$$d^2(Tx,Ty)+cd(Tx,Ty)\leq d^2(x,y), \tag{2.3}$$

for all $x,y\in X$. Then T has a unique fixed point in X.

Corollary 2.11 ([3])**.** *Let (X,d) be a complete metric space and let T from X itself be a mapping. Suppose that there exists $c>0$ and $\varphi:[0,+\infty)\to[0,+\infty)$ such that for all sequence $\{t_n\}\subset[0,+\infty)$ implies $\limsup_{n\to\infty}\varphi(t_n)$ exists. Also,*

$$d(Tx,Ty)(c+\varphi(d(Tx,Ty)))\leq\varphi(d(x,y))d(x,y), \tag{2.4}$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Corollary 2.12 (Caristi type, [3])**.** *Let (X, d) be a complete metric space and let T from X itself be a mapping. Suppose that there exists $c > 0$ and $\varphi : X \to [0, +\infty)$ such that for all sequence $\{t_n\} \subset [0, +\infty)$ implies $\limsup_{n\to\infty} \varphi(t_n)$ exists. Also,*

$$cd(Tx, Ty) \leq \varphi(x)\varphi(y) - \varphi(Tx)\varphi(Ty), \tag{2.5}$$

for all $x, y \in X$. Then T has a unique fixed point.

Definition 2.13 ([3])**.** Let $F^+ : \mathbb{R}^+ \to \mathbb{R}^+$, be an strictly increasing mapping A mapping $T : X \to X$ is said to be F^+-contraction if there exists $\tau > 0$ such that for all $x, y \in X$ such that

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \tag{2.6}$$

Corollary 2.14 ([3])**.** *Let (X, d) be a complete metric space and let T from X into itself be a F^+-contraction mapping. Then T has a unique fixed point in X.*

The following lemma plays a crucial rule in the next theorem.

Lemma 2.15 ([3])**.** *Let $\{a_n\} \subset (0, +\infty)$ and let $\sum_{n=1}^{\infty} a_n < \infty$. Then there exists a monotonic sequence $\{\gamma_n\} \subset (0, +\infty)$ such that $\lim_{n\to\infty} \gamma_n = \infty$ and $\sum_{n=1}^{\infty} a_n\gamma_n < \infty$.*

Proof. Put $\rho_n = \sum_{k=n+1}^{\infty} a_k \geq 0$. Therefore, the series $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{\rho_{k-1}}} < \infty$. Pick $\gamma_n = \frac{1}{\rho_{n-1}}$, to get desired result. □

Theorem 2.16. *Let (X, p) be a complete PMS and let $T : X \to X$ be a mapping. Suppose that $x_0 \in X$ and consider $\{T^n(x_0)\}$ as the Picard iterative sequence such that*

$$\sum_{n=1}^{\infty} p(T^n(x_0), T^{n-1}(x_0)) < \infty. \tag{2.7}$$

Then there exist $c > 0$ and a mapping $F : \mathcal{O}_T(x_0) \times \mathcal{O}_T(x_0) \to [0, +\infty)$ such that for each $n \in \mathbb{N}$,

$$p(T^n(x_0), T^{n+1}(x_0)) \leq \frac{F(T^{n-1}(x_0), F(T^n(x_0)))}{c + F(T^n(x_0), T^{n+1}(x_0))} p(T^n(x_0), T^{n-1}(x_0)), \tag{2.8}$$

in which $\mathcal{O}_T(x_0) = \{T^n(x_0) : n \in \{0\} \cup \mathbb{N}\}$.

Proof. Put $a_n = p(T^n(x_0), T^{n-1}(x_0))$ and by (2.7), we have $\sum_{n=1}^{\infty} a_n < \infty$. By Lemma 2.15,

$$\exists\{\gamma_n\} \subseteq (0, +\infty) \quad \text{such that} \quad \lim_{n\to\infty} \gamma_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n a_n < \infty.$$

So

$$\exists b_1 \in \mathbb{R} \quad \text{such that} \quad \sum_{n=1}^{\infty} a_n\gamma_n = a_1 b_1.$$

By Lemma 2.15 and $\{a_n\}, \{\gamma_n\} \subset (0, +\infty)$, we want to define a positive sequence $\{b_n\}$ by the following way. If we take

$$b_n \geq \frac{\gamma_{n+1} a_{n+1}}{a_n} \geq 0.$$

Then

$$\frac{b_n a_n - \gamma_{n+1} a_{n+1}}{a_{n+1}} \geq 0.$$

Now, define $\{b_n\}$ by

$$b_{n+1} := \frac{b_n a_n - \gamma_{n+1} a_{n+1}}{a_{n+1}}.$$

Thus

$$\sum_{n=1}^{\infty} \gamma_{n+1} a_{n+1} = \sum_{n=1}^{\infty} (b_n a_n - b_{n+1} a_{n+1}) = a_1 b_1 - \lim_{n\to\infty} a_{n+1} b_{n+1} \tag{2.9}$$

which implies that,

$$\lim_{n\to\infty} a_{n+1} b_{n+1} = 0.$$

Define $F : \mathcal{O}_T(x_0) \times \mathcal{O}_T(x_0) \to [0, +\infty)$ such that for each $n \in \mathbb{N}$, $F(T^{n-1}(x_0), T^n(x_0)) = b_n$. Since

$$\lim_{n\to\infty} \gamma_n = \infty \iff \forall c > 0,\ \exists N > 0 \quad \text{such that} \quad \gamma_n > c.$$

Applying (2.9) for all $n \geq N$, we have

$$b_n a_n - b_{n+1} a_{n+1} = \gamma_{n+1} a_{n+1} > c a_{n+1}.$$

Thus,

$$a_{n+1} < \left(\frac{b_n}{b_{n+1} + c} \right) a_n. \tag{2.10}$$

Hence, we can rewrite (2.10) as follows

$$p(T^n(x_0), T^{n+1}(x_0)) \leq \frac{F(T^{n-1}(x_0), F(T^n(x_0)))}{c + F(T^n(x_0), T^{n+1}(x_0))} p(T^n(x_0), T^{n-1}(x_0)).$$

□

Corollary 2.17 ([3]). *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. Suppose that $x_0 \in X$ and consider $\{T^n(x_0)\}$ as the Picard iterative sequence such that*

$$\sum_{n=1}^{\infty} d(T^n(x_0), T^{n-1}(x_0)) < \infty. \tag{2.11}$$

Then there exist $c > 0$ and a mapping $F : \mathcal{O}_T(x_0) \times \mathcal{O}_T(x_0) \to [0, +\infty)$ such that for each $n \in \mathbb{N}$,

$$d(T^n(x_0), T^{n+1}(x_0)) \leq \frac{F(T^{n-1}(x_0), F(T^n(x_0)))}{c + F(T^n(x_0), T^{n+1}(x_0))} d(T^n(x_0), T^{n-1}(x_0)), \tag{2.12}$$

in which $\mathcal{O}_T(x_0) = \{T^n(x_0) : n \in \{0\} \cup \mathbb{N}\}$.

3. An application

Let Θ be the set of all functions $\theta : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(j1) *θ is a nondecreasing function, i.e., $t_1 < t_2$ implies $\theta(t_1) \leq \theta(t_2)$;*
(j2) *θ is continuous;*

Example 3.1. Let $x, y \in X = C([a, b], \mathbb{R})$. Put $\theta(t) = \frac{L(x,y)}{L(x,y)+\frac{1}{2}} t - \gamma$, where $\gamma \geq 0$ and $L \in \mathfrak{S}_G(X)$. So $\theta \in \Theta$.

Consider the following nonlinear integral equation:

$$x(t) = \phi(t) + \int_a^t K(t, s, x(s)) ds, \tag{3.1}$$

where $a \in \mathbb{R}$, $x \in C([a, b], \mathbb{R})$, $\phi[a, b] \to \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are two given functions.

Theorem 3.2. *Consider the nonlinear integral equation* (3.1). *Suppose that the following condition holds:*

(i) *K is continuous;*
(ii) *$\theta(t) = \frac{L(x,y)}{L(x,y)+\frac{1}{2}} t - \gamma$, where $L(x, y) = \alpha \max_{t \in [a,b]} |x(s) - y(s)| + \beta$ and $\gamma \geq \frac{\beta}{\alpha(2L(x,y)+1)}$, such that*

$$|K(t, s, x(s)) - K(t, s, y(s))| \leq \frac{\theta(|x(s) - y(s)|)}{b - a}$$

for all $x, y \in C([a, b], \mathbb{R})$, $\alpha > 0$, $\beta \geq 0$ and for $t, s \in [a, b]$. Then the nonlinear integral equation (3.1) *has a unique solution.*

Proof. Let $X := C([a, b], \mathbb{R})$, $T : X \to X$ defined by

$$(Tx)(t) = \phi(t) + \int_0^t K(t, s, x(s)) ds, \quad \forall x \in X.$$

The metric d given by $d(x, y) = \max_{t \in [a,b]} |x(s) - y(s)|$ for all $x, y \in X$. Thus X is a complete metric space. Now define p by $p(x, y) = \alpha d(x, y) + \beta$ for each $x, y \in X$ where $\alpha > 0$, $\beta \geq 0$. So (X, p) is a complete PMS.

Let $x, y \in X$ and $t \in [a, b]$. Therefore

$$\begin{aligned}
|Tx(t) - Ty(t)| &= \left| \int_a^t K(t, s, x(s)) ds - \int_a^t K(t, s, y(s)) ds \right| \\
&\leq \int_a^t |K(t, s, x(s)) - K(t, s, y(s))| ds \\
&\leq \int_a^t \frac{\theta(|x(s) - y(s)|)}{b - a} ds \\
&\leq \frac{1}{b - a} \int_a^t \theta(d(x, y)) ds \\
&\leq \theta(d(x, y))
\end{aligned}$$

$$\leq \frac{L(x,y)}{L(x,y)+\frac{1}{2}}d(x,y)-\gamma.$$

So

$$\begin{aligned}
d(Tx,Ty) &\leq \frac{L(x,y)}{L(x,y)+\frac{1}{2}}d(x,y)-\gamma. \\
\alpha d(Tx,Ty)+\beta &\leq \alpha\frac{L(x,y)}{L(x,y)+\frac{1}{2}}d(x,y)+\beta-\alpha\gamma \\
\alpha d(Tx,Ty)+\beta &\leq \alpha\frac{L(x,y)}{L(x,y)+\frac{1}{2}}d(x,y)+\frac{L(x,y)}{L(x,y)+\frac{1}{2}}\beta \\
p(Tx,Ty) &\leq \frac{L(x,y)}{L(x,y)+\frac{1}{2}}p(d(x,y))
\end{aligned}$$

for all $x,y\in X$. Hence it satisfies the contraction (2.1).

Thus all the conditions of Theorem 2.5 with $c=\frac{1}{2}$ are satisfied and hence T has a unique fixed point in X. This implies that there exists a unique solution of the nonlinear integral equation (3.1). □

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