Journal of Nonlinear and Convex Analysis
Volume 26, Number 4, 2025, 899–913

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

CONVERGENCE ANALYSIS OF A FIXED POINT ITERATION PROCESS

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Abstract. This paper introduces a novel fixed point iterative algorithm that demonstrates faster convergence compared to existing methods. We establish both strong and weak convergence theorems concerning Suzuki's generalized nonexpansive mappings, within the framework of uniformly Banach spaces. Additionally, we provide numerical examples and graphical representations to support our findings. These results generalize and refine several well-known results from the current literature.

1. Introduction

Fixed point theory is a fundamental area of mathematical analysis with profound implications and applications across various fields of science. Many nonlinear equations can be reformulated as fixed point problems, making fixed point theory a versatile and powerful tool in solving complex mathematical challenges.

A fundamental result in the theory was established by Banach [5], where he demonstrated that a contraction T defined on a complete metric space is guaranteed to have a unique fixed point. Furthermore, the iterative process known as Picard iteration converges to this fixed point. While the Picard iteration is effective for approximating the fixed points of contraction mappings, it may not converge to the fixed point when applied to nonexpansive mappings, even if the fixed point is unique. Mann [13], in 1953, filled this gap by introducing an iterative process to approximate fixed points of nonexpansive mappings. Further in 1974, Ishikawa [11] came up with a two step Mann iterative process for pseudo-contractive mappings. Numerous researchers have since explored the Mann and Ishikawa iterative methods for fixed point approximation of nonexpansive mappings (see, for example, [[6, 12, 17, 21, 24]]). Numerous researchers have since explored the Mann and Ishikawa iterative methods for fixed point approximation of nonexpansive mappings (see, for example, [[6, 12, 17, 21, 24]]). Some state-of-the-art iterative algorithms include the Noor iteration [14], which enhances convergence for mixed-type mappings; Agarwal et al. iteration [3], improving convergence for various mappings; Abbas and Nazir iteration [1], combining methods for robustness; Picard-S iteration [7], introducing faster convergence steps; Thakur et al. iterations [22], tailored for specific problems; and Chanchal et al. iterations [2, 8, 9], offering additional convergence guarantees.

Suzuki [19], in 2008, proposed a generalization of nonexpansive mappings, defined by a condition termed Condition (C), which is now known as Suzuki's generalized nonexpansive mapping.

2020 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. Suzuki nonexpansive mapping, fixed point, iteration process, strong and weak convergence.

A self map $\mathcal{T}$ defined on a subset A of a Banach space $\mathcal{X}$ is said to fulfill Condition (C) if

$$\frac{1}{2}\|x - \mathcal{T}x\| \leq \|x - y\| \implies \|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|.$$

Suzuki demonstrated that mappings satisfying Condition (C) are more general than nonexpansive mappings, and he also established several results concerning the existence of fixed points for such mappings.

Later, Ullah and Arshad [4] developed an iteration process called M^* iteration with faster convergence than most the schemes discussed above and is which is given as follows. For $a_0 \in \mathcal{X}$

$$\begin{cases} c_n = (1 - s_n)a_n + s_n\mathcal{T}a_n, \\ b_n = \mathcal{T}((1 - r_n)c_n + r_n\mathcal{T}c_n), \\ a_{n+1} = \mathcal{T}b_n, \end{cases} \tag{1.1}$$

for all $n \geq 0,$ where $\{r_n\}, \{s_n\}$ are in $(0, 1)$.

Taking inspiration from above mentioned iteration process, we propose a new iterative algorithm. For $a_0 \in \mathcal{X}$:

$$\begin{cases} d_n = \mathcal{T}((1 - \delta_n)a_n + \delta_n\mathcal{T}a_n), \\ c_n = \mathcal{T}((1 - t_n)\mathcal{T}a_n + t_n\mathcal{T}d_n), \\ b_n = \mathcal{T}((1 - s_n)\mathcal{T}d_n + s_n\mathcal{T}c_n), \\ a_{n+1} = \mathcal{T}((1 - r_n)b_n + r_n\mathcal{T}b_n). \end{cases} \tag{1.2}$$

Our proposed algorithm (1.2) introduces additional intermediate steps compared to existing methods, providing a more refined approximation process. This is designed to handle the complexities of generalized nonexpansive mappings more effectively. Comparing with iterative processes mentioned above, the novel algorithm is expected to provide faster convergence and improved accuracy in finding fixed points of Suzuki's generalized nonexpansive mappings.

2. Preliminaries

We now present some basic definitions and results that are utilized throughout this paper.

In uniformly convex Banach space $\mathcal{X}$, let E be a nonempty, closed, convex subset. We denote the fixed point set of the mapping $\mathcal{T}$ as $F(\mathcal{T}) = \{y \in \mathcal{X} : \mathcal{T}y = y\}$.

Definition 2.1 ([15])**.** A Banach space X satisfies Opial's condition if for every sequence $\{p_n\}$ in $\mathcal{X}$ that converges weakly to $p \in \mathcal{X}$, we have $\limsup_{n\to\infty}\|p_n - p\| < \limsup_{n\to\infty}\|p_n - q\|$ for all $q \in \mathcal{X}$ and $p \neq q$.

Definition 2.2. A mapping $\mathcal{T} : E \longrightarrow \mathcal{X}$ is said to be demiclosed at $p \in \mathcal{X}$ if for every sequence $\{q_n\}$ in E such that $q_n \rightharpoonup q$ in $\mathcal{X}$ and $\mathcal{T}q_n \to p$, it follows that $q \in E$ and $\mathcal{T}q = p$.

For a bounded sequence $\{p_n\}$ in $\mathcal{X}$ and each $p \in \mathcal{X}$, we define the distance to the sequence as

$$s(p, \{p_n\}) = \limsup_{n \to \infty} \|p - p_n\|.$$

The asymptotic radius of $\{p_n\}$ relative to E is given by

$$s(E, \{p_n\}) = \inf\{s(p, \{p_n\}) : p \in E\}.$$

The asymptotic center $Z(E, \{p_n\})$ of the sequence $\{p_n\}$ is defined as

$$Z(E, \{p_n\}) = \{p \in E : s(p, \{p_n\}) = s(E, \{p_n\})\}.$$

Uniformly convex Banach space is reflexive and $Z(E, \{p_n\})$ contains exactly one point. Further, a class of mappings that satisfy Condition (I) was introduced by Senter and Dotson [6]. A mapping $\mathcal{T} : E \longrightarrow E$ will satisfy the condition if there exists a nondecreasing function $g : [0, \infty) \longrightarrow [0, \infty)$ with the properties $g(0) = 0$ and $g(x) > 0$ for all $x \in (0, \infty)$ such that $\|p - \mathcal{T}p\| \geq g(d(p, F(\mathcal{T})))$ for all $p \in E$, where $d(p, F(\mathcal{T})) = \inf_{y \in F(\mathcal{T})} \|p - y\|$.

Definition 2.3. Consider two real sequences $\{a_n\}$ and $\{b_n\}$ that converge to the limits a and b, respectively. The sequence $\{a_n\}$ is said to exhibit a faster convergence rate than $\{b_n\}$ if $\lim_{n \to \infty} \frac{\|a_n - a\|}{\|b_n - b\|} = 0$.

Lemma 2.4 ([18]). *Let $\mathcal{X}$ be a uniformly convex Banach space, and consider a sequence $\{t_n\}$ such that $0 < s \leq t_n \leq t < 1$ for some $s, t \in \mathbb{R}$ and for all $n \geq 1$. If two sequences in $\mathcal{X}$, say $\{p_n\}$ and $\{q_n\}$ satisfy:* $\limsup_{n \to \infty} \|p_n\| \leq a, \limsup < n \to \infty \|q_n\| \leq b$, *and* $\limsup_{n \to \infty} \|t_n p_n + (1 - t_n) q_n\| = a$ *for some* $a \geq 0$. *Then, we have* $\lim_{n \to \infty} \|p_n - q_n\| = 0$.

Proposition 2.5 ([19]). *Assume any mapping $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}$. Then,*

(1) *$\mathcal{T}$ is a Suzuki generalized nonexpansive mapping, if $\mathcal{T}$ is nonexpansive.*
(2) *If $\mathcal{T}$ is a Suzuki generalized nonexpansive mapping, then:*
  (a) *If $F(\mathcal{T}) \neq \emptyset$, then $\mathcal{T}$ is a quasi-nonexpansive mapping.*
  (b) *For all $p, q \in \mathcal{X}$,*

$$\|p - \mathcal{T}q\| \leq 3\|\mathcal{T}p - p\| + \|p - q\|. \tag{2.1}$$

Lemma 2.6 ([20]). *Let $\mathcal{T}$ be a Suzuki generalized nonexpansive mapping defined on a subset E of a Banach space $\mathcal{X}$ that has the Opial property. Consider a sequence $\{p_n\}$ that converges weakly to p. If it holds that* $\lim_{n \to \infty} \|\mathcal{T}p_n - p_n\| = 0$, *then the mapping $I - \mathcal{T}$ is demiclosed at zero.*

Lemma 2.7 ([19]). *A Suzuki generalized nonexpansive mapping $\mathcal{T}$ defined on a compact convex subset E of a uniformly convex Banach space, then $\mathcal{T}$ possesses a fixed point.*

Definition 2.8 ([16]). A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is classified as a contractive-like mapping if there exists a continuous and strictly increasing function $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(0) = 0$ and a constant $b \in [0, 1)$. For every pair of points $p, q \in \mathcal{X}$, the following condition must be satisfied:

$$\|\mathcal{T}p - \mathcal{T}q\| \leq \kappa \|p - q\| + \psi(\|p - \mathcal{T}p\|).$$

3. Convergence analysis

In this section, we begin by showing that our algorithm (1.2) achieves a faster convergence rate than the iteration process described in (1.1).

Theorem 3.1. *Let E be a nonempty closed convex subset of a Banach space $\mathcal{X}$, and $\mathcal{T}$ a contractive-like mapping on E, where $F(\mathcal{T})$ the fixed point set, is not empty. Consider the sequence $\{a_n\}_{n\geq 1}$ generated by the iterative scheme described in (1.2). It follows that this sequence converges at a rate superior to that of the iterative algorithm presented in (1.1).*

Proof. For any $q \in F(T)$, using (1.1), we have

$$\begin{aligned}
\|c_n - q\| &= \|(1-s_n)a_n + s_n\mathcal{T}a_n - q\| \\
&\leq (1-s_n)\|a_n - q\| + s_n\kappa\|a_n - q\|, \\
\|c_n - q\| &= (1-(1-\kappa)s_n)\|a_n - q\|, \\
\|b_n - q\| &= \|(1-r_n)c_n + r_n\mathcal{T}c_n - q\| \\
&\leq (1-r_n)\|c_n - q\| + r_n\kappa\|c_n - q\| \\
&= (1-(1-\kappa)r_n)\|c_n - q\|, \\
\|b_n - q\| &= (1-(1-\kappa)r_n)(1-(1-\kappa)s_n)\|a_n - q\|, \\
\|a_{n+1} - q\| &= \|\mathcal{T}b_n - q\| \\
&\leq \kappa\|b_n - q\| \\
&\leq \kappa(1-(1-\kappa)r_n)(1-(1-\kappa)s_n)\|a_n - q\| \\
&\leq \kappa(1-(1-\kappa)r)(1-(1-\kappa)s)\|a_n - q\| \\
&\vdots \\
\|a_{n+1} - q\| &\leq \kappa^n(1-(1-\kappa)r)^n(1-(1-\kappa)s)^n\|a_1 - q\|.
\end{aligned}$$

Now from (1.2),

$$\begin{aligned}
\|d_n - q\| &= \|\mathcal{T}((1-\delta_n)a_n + \delta_n\mathcal{T}a_n) - q\| \\
&\leq \kappa(\|(1-\delta_n)a_n + \delta_n\mathcal{T}a_n - q\|) \\
&\leq \kappa((1-\delta_n)\|a_n - q\| + \delta_n\|\mathcal{T}a_n - q\|) \\
&\leq \kappa((1-\delta_n)\|a_n - q\| + \kappa\delta_n\|a_n - q\|) \\
&\leq \kappa(1-\delta_n(1-\kappa))\|a_n - q\|, \\
\|c_n - q\| &= \|\mathcal{T}((1-t_n)\mathcal{T}a_n + t_n\mathcal{T}d_n) - q\| \\
&\leq \kappa(\|(1-t_n)\mathcal{T}a_n + t_n\mathcal{T}d_n - q\|) \\
&\leq \kappa((1-t_n)\|\mathcal{T}a_n - \mathcal{T}q\| + t_n\|\mathcal{T}d_n - \mathcal{T}q\|) \\
&\leq \kappa((1-t_n)\kappa\|a_n - q\| + t_n\kappa\|d_n - q\|) \\
&\leq \kappa^2((1-t_n)\|a_n - q\| + t_n(\kappa((1-\delta_n(1-\kappa))\|a_n - q\|)),
\end{aligned}$$

$$
\begin{aligned}
\|c_n - q\| &\leq \kappa^2(1 - t_n(1 - \kappa(1 - \delta_n(1 - \kappa))))\|a_n - q\|, \\
\|b_n - q\| &= \|\mathcal{T}((1 - s_n)\mathcal{T}d_n + s_n\mathcal{T}c_n) - q\| \\
&\leq \kappa\|(1 - s_n)\mathcal{T}d_n + s_n\mathcal{T}c_n - q\| \\
&\leq \kappa((1 - s_n)\|\mathcal{T}d_n - \mathcal{T}q\| + s_n\|\mathcal{T}c_n - \mathcal{T}q\|) \\
&\leq \kappa^2((1 - s_n)\|d_n - q\| + s_n\|c_n - q\|) \\
&\leq \kappa^3((1 - s_n)(1 - \delta_n(1 - \kappa)) \\
&\quad + s_n\kappa(1 - t_n(1 - \kappa(1 - \delta_n(1 - \kappa)))))\|a_n - q\| \\
&\leq \kappa^3((1 - s_n((1 - \delta_n(1 - \kappa)) \\
&\quad - \kappa(1 - t_n(1 - \kappa(1 - \delta_n(1 - \kappa))))))\|a_n - q\|), \\
\|a_{n+1} - q\| &= \|\mathcal{T}((1 - r_n)b_n + r_n\mathcal{T}b_n) - q\| \\
&\leq \kappa\|(1 - r_n)b_n + r_n\mathcal{T}b_n - q\| \\
&\leq \kappa\left((1 - r_n)\|b_n - q\| + r_n\|\mathcal{T}b_n - \mathcal{T}q\|\right) \\
&\leq \kappa\left((1 - r_n)\|b_n - q\| + r_n\kappa\|b_n - q\|\right) \\
&\leq \kappa\left((1 - r_n(1 - \kappa))\|b_n - q\|\right) \\
&\leq \kappa^4((1 - r_n(1 - \kappa))((1 - s_n((1 - \delta_n(1 - \kappa)) \\
&\quad - \kappa(1 - t_n(1 - \kappa(1 - \delta_n(1 - \kappa))))))\|a_n - q\|)) \\
&\leq \kappa^4((1 - r(1 - \kappa))((1 - s((1 - \delta(1 - \kappa)) \\
&\quad - \kappa(1 - t(1 - \kappa(1 - \delta(1 - \kappa))))))\|a_n - q\|)) \\
&\;\vdots \\
&\leq \kappa^{4n}((1 - r(1 - \kappa))^n((1 - s((1 - \delta(1 - \kappa)) \\
&\quad - \kappa(1 - t(1 - \kappa(1 - \delta(1 - \kappa))))))^n\|a_1 - q\|.
\end{aligned}
$$

Assume that

$$
\begin{aligned}
x_n &= \kappa^n(1 - (1 - \kappa)r)^n(1 - (1 - \kappa)s)^n, \\
y_n &= \kappa^{4n}((1 - r(1 - \kappa))^n((1 - s((1 - \delta(1 - \kappa)) - \kappa(1 - t(1 - \kappa(1 - \delta(1 - \kappa))))))^n.
\end{aligned}
$$

Clearly,

$$
\frac{y_n}{x_n} = \frac{\kappa^{4n}((1 - r(1 - \kappa))^n((1 - s((1 - \delta(1 - \kappa)) - \kappa(1 - t(1 - \kappa(1 - \delta(1 - \kappa))))))^n}{\kappa^n(1 - (1 - \kappa)r)^n(1 - (1 - \kappa)s)^n},
$$

$$
\frac{y_n}{x_n} \longrightarrow 0, n \longrightarrow \infty.
$$

The sequence produced by algorithm (1.2) exhibits a faster convergence rate compared to the sequence generated by algorithm (1.1). □

Now, we prove both weak and strong convergence results for the sequence defined by equation (1.2) under Suzuki's generalized nonexpansive mappings. We start by presenting the following lemma.

Lemma 3.2. *Let $\mathcal{T}$ denote Suzuki's generalized nonexpansive mapping defined on a nonempty closed convex subset E within a Banach space X, where $F(\mathcal{T}) \neq \emptyset$.*

Consider the iterative sequence $\{a_n\}$ generated by algorithm (1.2). Then, for every $q \in F(\mathcal{T})$, the $\lim_{n\to\infty}\|a_n - q\|$ exists.

Proof. Say $q \in F(T)$ and s $\in E$. Using definition of $\mathcal{T}$ we get,

$$\frac{1}{2}\|q - \mathcal{T}q\| = 0 \leq \|q - s\| \implies \|\mathcal{T}q - \mathcal{T}s\| \leq \|q - s\|.$$

We have from (1.2),

$$\begin{aligned}\|d_n - q\| &= \|\mathcal{T}((1-\delta_n)a_n + \delta_n\mathcal{T}a_n) - q\| \\ &\leq \|((1-\delta_n)a_n + \delta_n\mathcal{T}a_n) - q\| \\ &\leq ((1-\delta_n)\|a_n - q\| + \delta_n\|\mathcal{T}a_n - q\| \\ &\leq ((1-\delta_n)\|a_n - q\| + \delta_n\|a_n - q\| \\ &\leq \|a_n - q\|.\end{aligned} \tag{3.1}$$

Also,

$$\begin{aligned}\|c_n - q\| &= \|\mathcal{T}((1-t_n)\mathcal{T}a_n + t_n\mathcal{T}d_n) - q\| \\ &\leq \|((1-t_n)\mathcal{T}a_n + t_n\mathcal{T}d_n) - q\| \\ &\leq ((1-t_n)\|\mathcal{T}a_n - q\| + t_n\|\mathcal{T}d_n - q\| \\ &\leq ((1-t_n)\|a_n - q\| + t_n\|d_n - q\| \\ &\leq ((1-t_n)\|a_n - q\| + t_n\|a_n - q\| \\ &\leq \|a_n - q\|,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\|b_n - q\| &= \|\mathcal{T}((1-s_n)\mathcal{T}d_n + s_n\mathcal{T}c_n) - q\| \\ &\leq \|((1-s_n)\mathcal{T}d_n + s_n\mathcal{T}c_n) - q\| \\ &\leq ((1-s_n)\|\mathcal{T}d_n - q\| + s_n\|\mathcal{T}c_n - q\| \\ &\leq ((1-s_n)\|d_n - q\| + s_n\|c_n - q\| \\ &\leq ((1-s_n)\|a_n - q\| + s_n\|a_n - q\| \\ &\leq \|a_n - q\|.\end{aligned} \tag{3.3}$$

Now,

$$\begin{aligned}\|a_{n+1} - q\| &= \|\mathcal{T}((1-rn)b_n + r_n\mathcal{T}b_n) - q\| \\ &\leq \|((1-r_n)b_n + r_n\mathcal{T}b_n) - q\| \\ &\leq ((1-r_n)\|b_n - q\| + r_n\|\mathcal{T}b_n - q\| \\ &\leq ((1-r_n)\|b_n - q\| + r_n\|b_n - q\| \\ &\leq \|b_n - q\| \\ &\leq \|a_n - q\|.\end{aligned} \tag{3.4}$$

Clearly, $\{\|a_n - q\|\}$ is a decreasing sequence and bounded below. Thus, $\lim_{n\to\infty}\|a_n - q\|$ exists. ☐

Lemma 3.3. *Let E be a nonempty closed convex subset within a uniformly convex Banach space $\mathcal{X}$, and consider $\mathcal{T} : E \longrightarrow E$ as a Suzuki generalized nonexpansive mapping. The sequence $\{a_n\}$ produced by algorithm* (1.2) *will have a nonempty* *fixed point set $F(\mathcal{T})$ if and only if the sequence $\{a_n\}$ is bounded and the limit $\lim_{n\to\infty} \|\mathcal{T}a_n - a_n\|$ exists.*

Proof. Assume $F(\mathcal{T} \neq \emptyset$ with $q \in F(\mathcal{T}$, then by Lemma 3.2 $\lim_{n\to\infty} \|a_n - q\|$ exists. Say,

$$\lim_{n\to\infty} \|a_n - q\| = r. \tag{3.5}$$

As,

$$\|\mathcal{T}a_n - q\| \leq \|a_n - q\| \implies \limsup_{n\to\infty} \|\mathcal{T}a_n - q\| \leq r. \tag{3.6}$$

Since $\|d_n - q\| \leq \|a_n - q\|$, it follows that

$$\limsup_{n\to\infty} \|d_n - q\| \leq r.$$

Also, $\|c_n - q\| \leq \|a_n - q\|$, which gives

$$\limsup_{n\to\infty} \|c_n - q\| \leq r.$$

Moreover, $\|b_n - q\| \leq \|a_n - q\|$, which implies

$$\limsup_{n\to\infty} \|b_n - q\| \leq r.$$

Now,

$$\begin{aligned}
\|a_{n+1} - q\| &= \|\mathcal{T}((1 - r_n)b_n + r_n\mathcal{T}b_n) - q\| \\
&\leq \|(1 - r_n)b_n + r_n\mathcal{T}b_n - q\| \\
&\leq \|b_n - q\| \\
&\leq \|\mathcal{T}((1 - s_n)d_n + s_n\mathcal{T}c_n) - q\| \\
&\leq \|c_n - q\| \\
&\leq \|\mathcal{T}((1 - t_n)a_n + t_n\mathcal{T}d_n) - q\| \\
&\leq \|(1 - t_n)\mathcal{T}a_n + t_n\mathcal{T}d_n - q\| \\
&\leq (1 - t_n)\|\mathcal{T}a_n - q\| + t_n\|\mathcal{T}d_n - q\| \\
&\leq (1 - tn)\|a_n - q\| + t_n\|d_n - q\| \\
&\leq \|a_n - q\| - t_n\|a_n - q\| + t_n\|d_n - q\|,
\end{aligned}$$

$$\begin{aligned}
\|a_{n+1} - q\| - \|a_n - q\| &\leq\ t_n \left(\|d_n - q\| - \|a_n - q\|\right) \\
\frac{\|a_{n+1} - q\| - \|a_n - q\|}{t_n} &\leq\ \|d_n - q\| - \|a_n - q\| \\
\|a_{n+1} - q\| - \|a_n - q\| \leq\ \frac{\|a_{n+1} - q\| - \|a_n - q\|}{t_n} &\leq\ \|d_n - q\| - \|a_n - q\| \\
\|a_{n+1} - q\| - \|a_n - q\| &\leq \|d_n - q\| - \|a_n - q\| \\
\|a_{n+1} - q\| &\leq \|d_n - q\| \\
r &\leq \|d_n - q\| \\
r &\leq \liminf_{n\to\infty} \|d_n - q\|.
\end{aligned}$$

Hence,

$$r = \lim_{n\to\infty} \|d_n - q\|. \tag{3.7}$$

From equation (3.1),

$$\|d_n - q\| \leq (1-\delta_n)\|a_n - q\| + \delta_n\|\mathcal{T}a_n - q\| \leq \|a_n - q\|.$$

Now by (3.5) and (3.7), we have

$$\limsup_{n\to\infty}[(1-\delta_n)\|a_n - q\| + \delta_n\|\mathcal{T}a_n - q\|] = r. \tag{3.8}$$

Finally by Lemma 2.4 with (3.5), (3.6) and (3.8),

$$\lim_{n\to\infty} \|\mathcal{T}a_n - a_n\| = 0.$$

On the other hand, let $\{a_n\}$ be bounded and $\lim_{n\to\infty} \|\mathcal{T}a_n - a_n\| = 0$. If $q \in Z(C, \{a_n\})$, then

$$\begin{aligned}
s(\mathcal{T}q, \{a_n\}) &= \lim_{n\to\infty} \limsup \|a_n - \mathcal{T}q\| \\
&\leq \lim_{n\to\infty} \limsup [3\|\mathcal{T}a_n - a_n\| + \|a_n - q\|] \\
&= \lim_{n\to\infty} \limsup \|a_n - q\| \\
&= s(q, \{a_n\}).
\end{aligned}$$

It shows that $Tq \in Z(C, \{a_n\})$. As $\mathcal{X}$ is uniformly convex, so $Z(C, \{a_n\})$ is singleton. This implies that $Tq = q$ and that $F(\mathcal{T}) \neq \emptyset$. □

Theorem 3.4 (Weak convergence theorem)**.** *Let E be a nonempty closed convex subset of a uniformly convex Banach space $\mathcal{X}$ that satisfies Opial's condition. Suppose $\mathcal{T} : E \longrightarrow E$ be a Suzuki generalized nonexpansive mapping with $F(\mathcal{T}) \neq \emptyset$. If $\{a_n\}$ is the sequence produced by algorithm (1.2), then $\{a_n\}$ weakly converges to a fixed point of $\mathcal{T}$.*

Proof. Since $F(\mathcal{T}) \neq \emptyset$, let $q \in F(\mathcal{T})$. By Lemma 3.2, $\lim_{n\to\infty} \|a_n - q\|$ exists. As $\mathcal{X}$ is uniformly convex, there exists a subsequence $\{a_{n_i}\}$ that converges weakly to some $u \in \mathcal{X}$. Next, we will demonstrate that $\{a_n\}$ has a unique subsequential limit. Assume, for contradiction, that there exist subsequences $\{a_{n_i}\}$ and $\{a_{n_j}\}$ converging to u and v respectively. Since E is a closed convex subset of $\mathcal{X}$, we have $u, v \in E$. By Lemma 3.3, the sequence $\{a_n\}$ is bounded, and $\lim_{n\to\infty} \|\mathcal{T}a_n - a_n\| = 0$. Furthermore, as established in Lemma 2.6, $(I - \mathcal{T})$ is demiclosed at zero, implying that $u, v \in F(\mathcal{T})$. Thus, both $\lim_{n\to\infty} \|a_n - u\|$ and $\lim_{n\to\infty} \|a_n - v\|$ exist. If $u \neq v$, then by Opial's condition,

$$\begin{aligned}\lim_{n\to\infty}\|a_n-u\| &= \lim_{i\to\infty}\|a_{n_i}-u\|\\ &< \lim_{i\to\infty}\|a_{n_i}-v\|\\ &= \lim_{n\to\infty}\|a_n-v\|\\ &= \lim_{j\to\infty}\|a_{n_j}-v\|\\ &< \lim_{j\to\infty}\|a_{n_j}-v\|\\ &= \lim_{n\to\infty}\|a_n-u\|,\end{aligned}$$

this leads to a contradiction, implying $u=v$. Hence, $\{a_n\}$ weakly converges to a fixed point of T. □

Theorem 3.5 (Strong convergence theorem)**.** *Let E be a nonempty closed convex subset of a Banach space $\mathcal{X}$ that satisfies Opial's condition. Let $\mathcal{T}: E \longrightarrow E$ be a Suzuki generalized nonexpansive mapping with $F(\mathcal{T})\neq\emptyset$. Then, the sequence $\{a_n\}$ generated by the algorithm in (1.2) strongly converges to a fixed point of $\mathcal{T}$.*

Proof. By Lemma 2.7, since $F(\mathcal{T})\neq\emptyset$ and according to Lemma 3.3, we have $\lim_{n\to\infty}\|\mathcal{T}a_n-a_n\|=0$. Given that E is compact, there exists a subsequence of $\{a_n\}$, denoted as $\{a_{n_j}\}$, which converges strongly to some $q\in\mathcal{X}$. Now, by Proposition 2.5(iii), we can express the following inequality:

$$\|a_{n_j}-\mathcal{T}q\|\leq\|\mathcal{T}a_{n_j}-a_{n_j}\|+\|a_{n_j}-q\|.$$

As j approaches infinity, this implies that $\{a_{n_j}\}$ converges to q, leading to $\mathcal{T}q=q$, which means $p\in F(\mathcal{T})$. Moreover, since $\lim_{n\to\infty}\|a_n-p\|$ exists for every $q\in F(\mathcal{T})$, it follows from Lemma 3.2 that $\{a_n\}$ converges strongly to q. □

Theorem 3.6. *Suppose E is a nonempty closed convex subset of a Banach space $\mathcal{X}$ that satisfies Opial's condition, and let $\mathcal{T}: E\longrightarrow E$ be a Suzuki's generalized nonexpansive mapping with $F(\mathcal{T})\neq\emptyset$. A sequence $\{a_n\}$ generated by the Algorithm in (1.2) converges strongly to a fixed point of $\mathcal{T}$ if $\mathcal{T}$ satisfies condition (I).*

Proof. According to Lemma 3.2, $\lim_{n\to\infty}\|a_n-q\|$ exists, and we have $\|a_{n+1}-q\|\leq\|a_n-q\|$ for all $q\in F(\mathcal{T})$. Thus, we obtain:

$$\inf_{q\in F(\mathcal{T})}\|a_{n+1}-q\|\leq\inf_{q\in F(\mathcal{T})}\|a_n-q\|,$$

which implies that

$$d(a_{n+1},F(\mathcal{T}))\leq d(a_n,F(\mathcal{T})).$$

This indicates that $\{d(a_n,F(\mathcal{T}))\}$ constitutes a decreasing sequence bounded below, hence $\lim_{n\to\infty}d(a_n,F(\mathcal{T}))$ exists. Let $\lim_{n\to\infty}\|a_n-q\|=r$ for some $r\geq0$. If $r=0$, then the conclusion follows immediately.

Assuming $r>0$, by condition (I), we have:

$$g(d(a_n,F(\mathcal{T})))\leq\|a_n-\mathcal{T}a_n\|.$$

Thus, it follows that:

$$\lim_{n\to\infty}\|\mathcal{T}a_n-a_n\|=0\implies\lim_{n\to\infty}g(d(a_n,F(\mathcal{T})))=0.$$

Since g is a nondecreasing function, this implies $\lim_{n\to\infty} d(a_n, F(\mathcal{T})) = 0$. Therefore, we can find a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ and a sequence $\{b_{n_j}\}$ in $F(\mathcal{T})$ such that

$$\|a_{n_j} - b_j\| < \frac{1}{2^j} \tag{3.9}$$

for all $j \in \mathbb{N}$. By the inequality in (3.9), we have:

$$\begin{aligned} \|a_{n_{j+1}} - b_j\| &\leq \|a_{n_j} - b_j\| < \frac{1}{2^j}, \\ \|b_{j+1} - b_j\| &\leq \|b_{n_j} - a_{n_j}\| + \|a_{n_j} - b_j\| \\ &< \frac{1}{2^{j+1}} + \frac{1}{2^j} \\ &< \frac{1}{2^{j-1}} \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \end{aligned}$$

Thus, $\{b_j\}$ is a Cauchy sequence in $F(\mathcal{T})$, and since $F(\mathcal{T})$ is closed, it converges to $\{b_j\}$ converges to a point $q \in F(\mathcal{T})$. Consequently, $\{a_n\}$ strongly converges to q, and since $\lim_{n\to\infty} \|a_n - q\|$ exists, we conclude that $a_n \longrightarrow q \in F(\mathcal{T})$. □

4. Numerical example

In this section, we numerically compare the convergence rates of the previously discussed iterative schemes with our algorithm in (1.2). We also construct an example of a Suzuki generalized nonexpansive mapping that is not nonexpansive to highlight the superior convergence rate of our algorithm over existing methods.

Example 4.1. Let $\mathcal{T} : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\mathcal{T}(x) = \frac{x}{3}$. Clearly $\mathcal{T}$ is a nonexpansive mapping with $x = 0$ as the fixed point of $\mathcal{T}$.

Table 1 and 2 shows the influence of various parameters on the number of steps different iterative processes required to converge to the fixed point.

Example 4.2. Define a map $\mathcal{T} : [0, 1] \longrightarrow [0, 1]$ as

$$\mathcal{T}(a) = \begin{cases} 1 - a, & \text{if } a \in \left[0, \frac{1}{11}\right) \\ \frac{a+10}{11}, & \text{if } a \in \left[\frac{1}{11}, 1\right] \end{cases}$$

We begin by proving that $\mathcal{T}$ is not a nonexpansive mapping. To do this, let $a = \frac{8}{100}$ and $b = \frac{1}{11}$. Then,

$$\|\mathcal{T}a - \mathcal{T}b\| = \|(1 - a) - \left(\frac{b + 10}{11}\right)\| = 0.00264462$$

and

$$\|a - b\| = \left|\frac{8}{100} - \frac{1}{11}\right| = 0.01090909.$$

Clearly, $\|\mathcal{T}a - \mathcal{T}b\| > \|a - b\|$, so $\mathcal{T}$ is not a nonexpansive mapping. Now, we show that $\mathcal{T}$ satisfies condition (C). We will consider the following cases:

Case 1. Let $a \in \left[0, \frac{1}{11}\right)$. Then $\frac{1}{2}\|a - \mathcal{T}a\| = \frac{1}{2}|a - (1 - a)| = \frac{1}{2}|2a - 1|$. For $\frac{1}{2}\|a - \mathcal{T}a\| \leq \|a - b\|$, we must have $\frac{1}{2}|2a - 1| \leq \|a - b\|$ i.e., $\frac{1}{2}|2a - 1| \leq |a - b|$. Here, we see that $b < a$ is not possible, so we have only one choice $a < b$, which gives

TABLE 1. Influence of initial value and constant parameter

Algorithm	Initial	Points	($r_n = 0.95$,	$s_n = 0.95$,	$t_n = 0.95$)
	5	50	500	5000	10000
Mann	10	12	15	17	18
Ishikawa	6	7	8	9	10
Noor	5	6	7	9	9
Agarwal	5	6	7	9	9
Abbas	4	6	7	8	8
Thakur	4	5	5	6	6
Thakur New	3	4	5	6	6
M^*	3	4	4	5	5
New	2	2	3	3	3
		$r_n = 0.25$,	$s_n = 0.25$,	$t_n = 0.25$	
Mann	47	60	72	85	89
Ishikawa	25	30	36	43	45
Noor	45	53	66	77	81
Agarwal	8	10	12	15	16
Abbas	5	6	7	9	10
Thakur	7	9	10	12	13
Thakur New	4	5	6	7	7
M^*	4	5	6	7	7
New	2	3	3	4	4
		$r_n = 0.1$,	$s_n = 0.1$,	$t_n = 0.1$	
Mann	125	158	192	225	235
Ishikawa	63	80	97	113	120
Noor	121	153	185	217	227
Agarwal	8	10	12	14	15
Abbas	5	6	7	8	8
Thakur	8	10	12	14	14
Thakur New	4	5	6	7	8
M^*	4	5	6	7	7
New	2	3	3	4	4

$\frac{1}{2}\left|2a-1\right| \leq (b-a)$, leading to $b \geq \frac{1}{2}$. Thus, $b \in \left[\frac{1}{2}, 1\right]$. Now we have $a \in \left[0, \frac{1}{11}\right)$ and $b \in \left[\frac{1}{2}, 1\right]$. We find

$$\|\mathcal{T}a - \mathcal{T}b\| = \|(1-a) - \left(\frac{b+10}{11}\right)\| = \left|\frac{11a-b+1}{11}\right| < \frac{1}{11}$$

and

$$\|a-b\| = |a-b| > \frac{9}{22}.$$

Hence,

$$\frac{1}{2}\|a - \mathcal{T}a\| \leq \|a-b\| \implies \|\mathcal{T}a - \mathcal{T}b\| \leq \|a-b\|.$$

Case 2. Let $a \in \left[\frac{1}{11}, 1\right]$. Then $\frac{1}{2}\|a - \mathcal{T}a\| = \frac{1}{2}\left|a - \frac{a+10}{11}\right| = \frac{1}{2}\left|\frac{10-10a}{11}\right|$. For $\frac{1}{2}\|a-\mathcal{T}a\| \leq \|a-b\|$, we must have $\frac{1}{2}\left|\frac{10-10a}{11}\right| \leq \|a-b\|$ i.e., $\frac{1}{2}\left|\frac{10-10a}{11}\right| \leq |a-b|$. Here, we have two choices: **A.** When $a < b$, we get $\left(\frac{10-10a}{22}\right) \leq (b-a)$ which results

Table 2. Influence of initial value and variable parameter

Algorithm	Initial	Points	$(r_n = \frac{n}{n+20}$,	$s_n = \frac{1}{(n+4)^{\frac{2}{3}}}$,	$t_n = \delta_n = \frac{n}{4n+1})$
	10	10^2	10^3	10^4	10^5
Mann	53	66	80	91	104
Ishikawa	34	43	53	65	73
Noor	15	63	75	88	100
Agarwal	9	11	13	15	17
Abbas	5	6	8	9	10
Thakur New	8	10	11	13	15
Thakur	4	5	6	7	8
M^*	4	5	6	7	8
New	3	3	4	4	5
		$r_n = \frac{2n}{7n+6}$,	$s_n = (1 - \frac{1}{(n+6)^2})$,	$t_n = \delta_n = (1 - \frac{1}{2n+5})$	
Mann	48	60	71	82	105
Ishikawa	9	11	13	14	17
Noor	33	41	48	55	63
Agarwal	8	10	12	13	15
Abbas	4	5	6	7	7
Thakur	4	5	6	7	8
Thakur New	4	5	5	6	6
M^*	4	4	5	6	6
New	2	3	3	3	4
		$r_n = \frac{n}{n+5}$,	$s_n = \frac{n}{(36n^2+1)^{\frac{1}{2}}}$,	$t_n = \delta_n = (\frac{2n}{4n+5})^{\frac{1}{2}}$	
Mann	19	22	25	28	31
Ishikawa	17	19	22	25	27
Noor	18	21	24	27	29
Agarwal	9	11	13	14	16
Abbas	5	6	7	9	10
Thakur	6	8	9	10	12
Thakur New	4	5	6	7	8
M^*	4	5	6	6	7
New	2	3	3	4	4

in $b \geq \frac{11+12a}{22}$. Thus, $b \in \left[\frac{133}{242}, 1\right] \subset \left[\frac{1}{11}, 1\right]$, which implies $\|\mathcal{T}a - \mathcal{T}b\| = \frac{1}{11}\|a-b\| < \|a-b\|$. Hence,

$$\frac{1}{2}\|a - \mathcal{T}a\| \leq \|a-b\| \implies \|\mathcal{T}a - \mathcal{T}b\| \leq \|a-b\|.$$

B. When $a > b$, we get $\left(\frac{10-10a}{22}\right) \leq (a-b)$ which results in $b \leq \frac{32a-10}{22}$. Thus, $b \in [0,1]$. Also, $a \geq \frac{22b+10}{32}$ which yields $a \in \left[\frac{10}{32}, 1\right]$. For $a \in \left[\frac{10}{32}, 1\right]$ and $b \in \left[\frac{1}{11}, 1\right]$, we can use Case 2A. Therefore, we only verify for $a \in \left[\frac{10}{32}, 1\right]$ and $b \in \left[0, \frac{1}{11}\right)$. For this,

$$\|\mathcal{T}a - \mathcal{T}b\| = \left|\frac{a+10}{11} - (1-b)\right| = \left|\frac{11b+a-1}{11}\right| \leq \frac{1}{11}$$

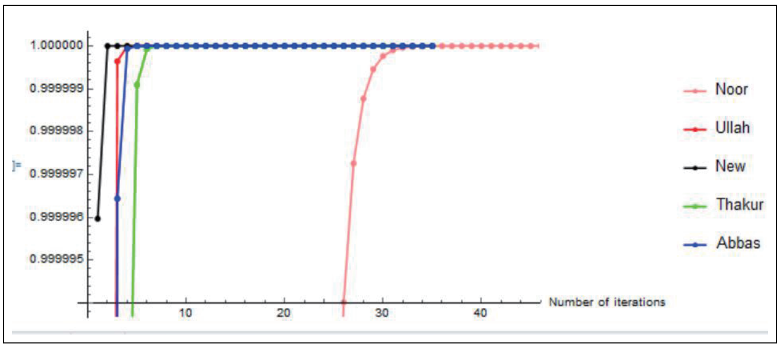
and

$$\|a-b\| = |a-b| > \frac{78}{363}.$$

TABLE 3. Comparison with various standard algorithms

Steps	Noor	Thakur	Abbas	Ullah	New
1	0.5854161983471075	0.975292561983471	0.9777013492620351	0.993219113448535	0.9999947966177338
2	0.6034415795653273	0.9978555341669916	0.9997324988487486	0.9999487065753968	0.999999999688
3	0.6364880959476426	0.9998225038993521	0.9999959772775701	0.9999996432602183	0.9999999999999836
4	0.6801094517058519	0.9999859973272771	0.9999999294937671	0.999999997709476	1
5	0.7293232110313737	9999989471791831	0.9999999986202659	0.9999999999863718	1
6	0. 7794482410426337	0.9999999245386132	0.9999999999706847	0.9999999999999246	1
7	0.8267088958732511	0.9999999948415058	0.9999999999993366	0.9999999999999997	1
8	0.8685372354287928	0.9999999996634754	0.9999999999999842	1	1
9	0.9035933583353488	0.9999999999790337	0.9999999999999996	1	1
10	0.9315817547798162	0.9999999999987517	1	1	1
11	0.9529618614615832	0.9999999999987517	1	1	1
12	0.9686407146617604	0. 9999999999999962	1	1	1
13	0.9797082592075379	0.9999999999999998	1	1	1
14	0.9872448453196314	1	1	1	1
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
27	0.9999999816474701	1	1	1	1
28	0.9999999926569909	1	1	1	1
29	0.9999999971144102	1	1	1	1
30	1	1	1	1	1

FIGURE 1. Comparing our algorithm with other standard algorithms



Hence,

$$\frac{1}{2}\|a - \mathcal{T}a\| \leq \|a - b\| \implies \|\mathcal{T}a - \mathcal{T}b\| \leq \|a - b\|.$$

Thus, the mapping $\mathcal{T}$ satisfies condition (C) for all possible cases. Applying the mapping outlined above, we aim to demonstrate that the algorithm (1.2) achieves faster convergence than Noor iteration, Thakur iteration, Abbas and Nazir iteration, and Ullah and Arshad iteration. Set $r_n = s_n = t_n = \delta_n = \frac{1}{n+4}$ and $a_0 = 0.02$.

5. CONCLUSION

In this study a new fixed iteration process (1.2) has been obtained which is utilized to approximate fixed point of Suzuki generalized nonexpansive mappings. Further, we establish that the proposed algorithm (1.2) achieves faster convergence compared to the recent Ullah and Arshad iteration process. Next we performed weak and strong convergence analysis of our algorithm (1.2). At last we perform numerical examples to illustrate the convergence behaviour and comparison with other iterative methods. Additionally, it is important to note that the computational time of the convergence is higher than that of other algorithms due to its enhanced structure and additional intermediate steps designed to improve convergence accuracy. .

Acknowledgments

The authors express their heartfelt gratitude to the learned referees for identifying numerous omissions and mistakes, as well as for their valuable insights into the future scope of this work.

The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA, for funding this research work through the project number NBU-FFR-2025-2182-02.

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Manuscript received March 6, 2024
revised June 19, 2024

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