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MODIFIED PROXIMAL POINT ALGORITHM FOR DEALING WITH FIXED POINT PROBLEMS AND CONVEX MINIMIZATION PROBLEMS IN CAT(0) SPACE

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Abstract. In this research, we provide a modified proximal point approach to estimate the shared component of the set of solutions of convex minimization problems and the fixed points in the framework of CAT(0) spaces. We also demonstrate that the suggested approach is Δ-convergent. Our findings expand upon and enhance the comparable recent findings in the literature.

1. Introduction

In this article, we suppose that $(\mathcal{Z}, d)$ is a geodesic metric space and $\mathfrak{G} : \mathcal{Z} \to (-\infty, \infty]$ is proper, lower-semicontinuous and convex functions.

In CAT(0) spaces, fixed point theory was brought in by Kirk [21], which caught the interest of numerous researchers and has been an exciting subject of study for the last several years. Kirk established that on a bounded convex closed subset of a complete CAT(0) space, one could construct a nonexpansive mapping with a fixed point.

A basic optimization task is to determine $u \in \mathcal{Z}$ in order that

$$\mathfrak{G}(u) = \min_{y \in \mathcal{Z}} \mathfrak{G}(v). \tag{1.1}$$

We indicate the solution of problem 1.1 by

$$\arg\min_{v \in \mathcal{Z}} \mathfrak{G}(v),$$

the set of a minimizer of a convex function. Proximal Point Algorithm (for short term, PPA) is among the most beneficial approaches for solving problem (1.1). After being employed for the first time by Martinet [24], Rockafellar [27] expanded the PPA in a Hilbert space and demonstrated the weak convergence of the sequence produced by the proximal point method to a zero of the maximum monotone operator in Hilbert spaces. In the domain of manifolds, which are extensions of Hilbert, Banach, and linear spaces, this topic has received a lot of attention lately for its potential to extend PPA for solving optimization problems ([13, 23, 26, 35]). Optimization problems on manifolds are solved by a wide range of applications in computer vision, machine learning, electronic structure computation, system balance, and robot manipulation (see [1, 29, 33, 34]).

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In 2014, Bačák [4] attained some results using the proximal point algorithm in CAT(0) spaces. Also, he extended the findings of Bertsekas [5] into Hadamard spaces by using a splitting version of the PPA to determine the minimizer of a sum of convex functions. Since then, a great deal of mathematician have produced a number of findings pretaining to the proximal point methods within the context of CAT(0) spaces (see [7, 8, 14, 16, 18–20, 31]).

In this study, captivated by recent research, we modify proximal point technique for locating a shared element between the set of minimizers of two lower semi-continuous functions and the set of fixed point mapings. It is demonstrate that the sequence $\{u_k\}$-converges to a shared element of the fixed point set of three single-valued nonexpansive mappings, the fixed point set of three multi-valued nonexpansive mappings, and the set of solutions of convex minimization problems within the context of CAT(0) spaces.

2. Preliminaries

This section reiterates a few commonly employed lemmas and ideas that are frequently used in our main findings.

If every geodesic triangle in a metric space $\mathcal{Z}$ is at least as thin as its corresponding triangle in the Euclidean plane and the space is geodesically connected, it is referred to as a CAT(0) space (more information may be found in [6]).

When a subset $\mathcal{D}$ of a CAT(0) space $\mathcal{Z}$ contains every geodesic segment linking two of its points, it is said to be convex; that is, for every pair of points $u, v \in \mathcal{D}$, we obtain $[u, v] \subset \mathcal{D}$, where $[u, v] := \{\varrho u \oplus (1 - \varrho)v : 0 \leq \varrho \leq 1\}$ is the unique geodesic joining u and v.

If, for every $u, v \in \mathcal{D}$, $d(\mathfrak{T}u, \mathfrak{T}v) \leq d(u, v)$, then single-valued mapping $\mathfrak{T} : \mathcal{D} \to \mathcal{D}$ is referred as nonexpansive mapping.

If any sequence $\{u_k\}$ in $\mathcal{D}$ fulfilling $\lim_{k\to\infty} d(\mathfrak{T}u_k, u_k) = 0$, has a convergent subsequence, then the single-valued mapping $\mathfrak{T} : \mathcal{D} \to \mathcal{D}$ is referred to as semi-compact. $\mathfrak{U}(\mathfrak{T})$ represents the set of all fixed points of $\mathfrak{T}$. Now, we talk about the following lemma, which turns out to be helpful later on.

Lemma 2.1 ([12])**.** *Given CAT(0) space $(\mathcal{Z}, d)$, the subsequent claims hold:*

(i) *A unique $z \in [u, v]$ exists for $u, v \in \mathcal{Z}$ and $p \in [0, 1]$ such that*

$$d(u, z) = pd(u, v) \ \text{ and } \ d(v, z) = (1 - p)d(u, v).$$

(ii) *For $u, v, z \in \mathcal{Z}$ and $p \in [0, 1]$, we have*

$$d((1 - p)u \oplus pv, z) \leq (1 - p)d(u, z) + pd(v, z)$$

and

$$d^2((1 - p)u \oplus pv, z) \leq (1 - p)d^2(u, z) + pd^2(v, z) - p(1 - p)d^2(u, v).$$

In the preceding Lemma, for the unique point z, we use the notation $(1-p)u \oplus py$.

Now, we gather some fundamental geometric properties that will be useful in the article.

Let $\{u_k\}$ be a bounded sequence in a complete CAT(0) space $\mathcal{Z}$. For $u \in \mathcal{Z}$ we write:

$$r(x, \{u_k\}) = \limsup_{k\to\infty} d(u, u_k).$$

The asymptotic radius $r(\{u_k\})$ is provided by

$$r(\{u_k\}) = \inf\{r(u, u_k) : x \in \mathcal{Z}\}$$

and the asymptotic center $A(\{u_k\})$ of $\{u_k\}$ is characterized as:

$$A(\{u_k\}) = \{x \in X : r(u, u_k) = r(\{u_k\})\}.$$

The fact that $A(\{u_k\})$ is composed of exactly one point in a complete CAT(0) space is widely known [11]. Now, in order to help with our explanation that follows, we provide the definition and some basic properties of the Δ-convergence.

Definition 2.2 ([21]). For every subsequence $\{s_k\}$ of $\{u_k\}$, if u is the unique asymptotic center of $\{s_k\}$, then $\{u_k\}$ in a CAT(0) space $\mathcal{Z}$ is said to be Δ-convergent to a point $u \in \mathcal{Z}$. In this instance, we denote u the Δ-limit of $\{u_k\}$ and write $\Delta - \lim_{k\to\infty} u_k = u$.

Lemma 2.3 ([21]). *There exists a Δ-convergent subsequence for every bounded sequence in a complete CAT(0) space.*

Lemma 2.4 ([10]). *If $\mathcal{D}$ is a closed convex subset of a complete CAT(0) space $\mathcal{Z}$ and $\{u_k\}$ is a bounded sequence in $\mathcal{D}$, then the asymptotic center of $\{u_k\}$ is in $\mathcal{D}$.*

Lemma 2.5 ([12]). *In a complete CAT(0) space $(\mathcal{Z}, d)$, let $\mathcal{D}$ be a nonempty closed convex subset, and let $\mathfrak{T} : \mathcal{D} \to \mathcal{D}$ be a nonexpansive mapping. Then x is a fixed point of $\mathfrak{T}$ if $\{u_k\}$ is a bounded sequence in $\mathcal{D}$ such that $\Delta - \lim_k u_k = x$ and $\lim_{k\to\infty} d(\mathfrak{T}u_k, u_k) = 0$.*

Lemma 2.6 ([12]). *If $\{u_k\}$ is a bounded sequence in a complete CAT(0) space with $A(\{u_k\}) = \{x\}$, $\{\mathcal{S}_k\}$ is a subsequence of $\{u_k\}$ with $A(\{\mathcal{S}_k\}) = \{u\}$ and the sequence $\{d(u_k, u)\}$ converges, then $x = u$.*

Lemma 2.7 ([9, 32]). *In a complete CAT(0) space $(\mathcal{Z}, d)$, let $\mathcal{D}$ be a nonempty closed convex subset. Then, for any $\{u_i\}_{i=1}^k \in \mathcal{D}$ and $\varrho_i \in (0, 1)$, $i = 1, 2, ..., k$ with $\sum_{i=1}^k \varrho_i = 1$, we have the following inequalities:*

$$d(\oplus_{i=1}^k \varrho_i u_i, z) \le \sum_{i=1}^k \varrho_i d(u_i, z),\ \forall\ z \in \mathcal{D} \tag{2.1}$$

and

$$d^2(\oplus_{i=1}^k \varrho_i u_i, z) \le \sum_{i=1}^k \varrho_i d^2(u_i, z) - \sum_{i,j=1, i\ne j}^n \varrho_i \varrho_j d^2(u_i, u_j),\ \forall\ z \in \mathcal{D}. \tag{2.2}$$

In this paper, we mainly study lower semi-continuous and convex functions on CAT(0) spaces. Note that a function $\mathfrak{G} : \mathcal{D} \to (-\infty, \infty]$ defined on a convex subset $\mathcal{D}$ of a CAT(0) space is convex if and only if the function $\mathfrak{G} o \gamma$ is convex for any geodesic $\gamma : [a, b] \to \mathcal{D}$. In other words, $\mathfrak{G}(\varrho u \oplus (1 - \varrho)v) \le \varrho\mathfrak{G}(u) + (1 - \varrho)\mathfrak{G}(v)$ for all $u, v \in \mathcal{D}$. See [3] for a few noteworthy examples.

Further, a function $\mathfrak{G}$ defined on $\mathcal{D}$ is considered as lower semi-continuous at $u \in \mathcal{D}$ if

$$\mathfrak{G}(u) \leq \liminf_{k\to\infty} \mathfrak{G}(u_k)$$

for each sequence $\{u_k\}$ such that $u_k \to u$ as $k \to \infty$. A function $\mathfrak{G}$ is considered as a lower semi-continuous on $\mathcal{D}$ if it is lower semi-continuous at any point in $\mathcal{D}$.

For any $\lambda > 0$, define the Moreau-Yosida resolvent of $\mathfrak{G}$ in CAT(0) space as follows:

$$J_\lambda(u) = \arg\min_{v\in\mathcal{D}}[\mathfrak{G}(v) + \frac{1}{2\lambda}d^2(v,u)]$$

for all $u \in \mathcal{D}$. For any $\lambda \geq 0$, the mapping J_λ is clearly defined; see [15]. The set $\mathfrak{U}(J_\lambda)$ of the fixed point of the resolvent J_λ associated with $\mathfrak{G}$ coincides with the set $\arg\min_{v\in\mathcal{D}} \mathfrak{G}(v)$ of minimizers of $\mathfrak{G}$ if $\mathfrak{G}$ is a proper, convex, and lower semi-continuous function; see [3]. Moreover, the resolvent J_λ of $\mathfrak{G}$ is nonexpansive for every $\lambda > 0$; see [17].

Lemma 2.8 ([2]). *For a given complete CAT(0) space $(\mathcal{Z}, d)$, consider that $\mathfrak{G} : \mathcal{Z} \to (-\infty, \infty]$ is a proper, convex and lower semi-continuous function, then for all $u, v \in \mathcal{Z}$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda}d^2(J_\lambda u, v) - \frac{1}{2\lambda}d^2(u,v) + \frac{1}{2\lambda}d^2(y, J_\lambda u) + \mathfrak{G}(J_\lambda u) \leq \mathfrak{G}(v).$$

Lemma 2.9 ([17, 25]). *Assume that $(\mathcal{Z}, d)$ is a complete CAT(0) space and that $\mathfrak{G} : \mathcal{Z} \to (-\infty, \infty]$ is a lower semi-continuous, proper, convex function. Then, the subsequent identity is valid:*

$$J_\lambda u = J_\mu(\frac{\lambda-\mu}{\lambda}J_\lambda u \oplus \frac{\mu}{\lambda}u)$$

for all $u \in \mathcal{Z}$ and $\lambda > \mu > 0$.

The notations CB($\mathcal{D}$), CC($\mathcal{D}$), and KC($\mathcal{D}$) represent the families of nonempty closed bounded subsets, closed convex subsets, and compact convex subsets of $\mathcal{D}$, respectively.

$$H(A,B) = max\{\sup_{u\in A} dist(u,B), \sup_{v\in B} dist(v,A)\}$$

defines the Pompeiu-Hausdorff distance [28] on CB($\mathcal{D}$) for $A, B \in CB(\mathcal{D})$, where the distance between a point u and a subset $\mathcal{D}$ is denoted by $dist(u,\mathcal{D}) = inf\{d(u,v) : v \in \mathcal{D}\}$. The following defines a fixed point of a multi-valued mapping $\mathcal{S} : \mathcal{D} \to CB(\mathcal{D})$:

$$u \in \mathcal{D}\ if\ u \in \mathcal{S}u.$$

The set of all fixed points in $\mathcal{S}$ is represented by the notation $\mathfrak{U}(\mathcal{S})$.

A multi-valued mapping $\mathcal{S} : \mathcal{D} \to CB(\mathcal{D})$ is referred to as a nonexpansive mapping or Hemi-compact if, for every $u, v \in \mathcal{D}$, $H(\mathcal{S}u, \mathcal{S}v) \leq d(u,v)$, or if there is a subsequence $\{u_{k_i}\}$ of $\{u_k\}$ such that $\{u_{k_i}\}$ converges strongly to $u^* \in \mathcal{D}$ for each sequence $\{u_k\}$ in $\mathcal{D}$, with $\lim_{k\to\infty} dist(\mathcal{S}u_k, u_k) = 0$, respectively.

3. Main results

Lemma 3.1. *Assume that a complete CAT(0) space $\mathcal{Z}$ has a nonempty closed and convex subset $\mathcal{D}$. The single-valued nonexpansive mappings are denoted by $\mathfrak{T}_i : \mathcal{D} \to \mathcal{D}$; the multi-valued nonexpansive mappings are denoted by $\mathcal{S}_i : \mathcal{D} \to CB(\mathcal{D})$ for $i = 1 : 3$ and $\mathfrak{G}, h : \mathcal{D} \to (-\infty, \infty]$ are two proper convex and lower semi-continuous functions. Suppose that*

$$\Omega = \mathfrak{U}(\mathfrak{T}_1) \cap \mathfrak{U}(\mathfrak{T}_2) \cap \mathfrak{U}(\mathfrak{T}_3) \cap \mathfrak{U}(\mathcal{S}_1) \cap \mathfrak{U}(\mathcal{S}_2) \cap \mathfrak{U}(\mathcal{S}_3) \cap \arg\min_{y \in \mathcal{D}} \mathfrak{G}(y) \cap \arg\min_{\zeta \in \mathcal{D}} h(\zeta) \neq \emptyset$$

and $\mathcal{S}_i q = \{q\}$, $i = 1 : 3$ for $q \in \Omega$. For $u_1 \in \mathcal{D}$, let the sequence $\{u_k\}$ is generated in the following manner:

$$\begin{cases} s_k = \arg\min_{y \in \mathcal{D}} [\mathfrak{G}(y) + \frac{1}{2\lambda_k} d^2(y, u_k)], \\ v_k = \arg\min_{\zeta \in \mathcal{D}} [h(\zeta) + \frac{1}{2\sigma_k} d^2(\zeta, s_k)], \\ \varphi_k = \varrho_k u_k \oplus \varsigma_k v'_k \oplus \gamma_k v''_k, \\ \Psi_k = \psi_k u_k \oplus \kappa_k v'''_k \oplus \phi_k \mathfrak{T}_1 u_k, \\ u_{k+1} = \delta_k u_k \oplus \eta_k \mathfrak{T}_2 u_k \oplus \xi_k \mathfrak{T}_3 \Psi_k, \ \ for\ all\ k \in \mathbb{N}. \end{cases} \tag{3.1}$$

where $\{\varrho_k\}$, $\{\varsigma_k\}$, $\{\gamma_k\}$, $\{\psi_k\}$, $\{\kappa_k\}$, $\{\phi_k\}$, $\{\delta_k\}$, $\{\eta_k\}$ and $\{\xi_k\}$ are sequences in $(0, 1)$ such that

$$\begin{aligned} 0 < a \leq \{\varrho_k\}, \{\varsigma_k\}, \{\gamma_k\}, \{\psi_k\}, \{\kappa_k\}, \{\phi_k\}, \{\delta_k\}, \{\eta_k\}, \{\xi_k\} \leq b < 1, \\ \varrho_k + \varsigma_k + \gamma_k = 1, \psi_k + \kappa_k + \phi_k = 1, \delta_k + \eta_k + \xi_k = 1, \end{aligned} \tag{3.2}$$

for all $k \in \mathbb{N}$ and $\{\lambda_k\}$ is a sequence such that $\lambda_k \geq \lambda > 0$ for all $k \in \mathbb{N}$ and some λ. Then, the subsequent claims are true:

(i) $\lim_{k\to\infty} d(u_k, q)$ *exists for all $q \in \Omega$;*
(ii) $\lim_{k\to\infty} d(u_k, s_k) = 0$; $\lim_{k\to\infty} d(s_k, v_k) = 0$;
(iii) $\lim_{k\to\infty} dist(u_k, \mathcal{S}_i u_k) = 0, i = 1, 2, 3$;
(iv) $\lim_{k\to\infty} d(u_k, \mathfrak{T}_i u_k) = 0, i = 1, 2, 3$;
(v) $\lim_{k\to\infty} d(u_k, J_\lambda u_k) = 0$, $\lim_{k\to\infty} d(u_k, J_\sigma u_k) = 0$.

Proof. Let $q \in \Omega$, then

$$q = \mathfrak{T}_1 q = \mathfrak{T}_2 q = \mathfrak{T}_3 q \in (\mathcal{S}_1 q \cap \mathcal{S}_2 q \cap \mathcal{S}_3 q)$$

and

$$\mathfrak{G}(q) \leq \mathfrak{G}(y),\ and\ h(q) \leq h(\zeta),\ \forall y, \zeta \in \mathcal{D}.$$

Therefore, we have

$$\mathfrak{G}(q) + \frac{1}{2\lambda_k} d^2(q, q) \leq \mathfrak{G}(y) + \frac{1}{2\lambda_k} d^2(y, q),$$

and

$$h(q) + \frac{1}{2\sigma_k} d^2(q, q) \leq h(\zeta) + \frac{1}{2\sigma_k} d^2(\zeta, q),$$

for all $y, \zeta \in \mathcal{D}$ and therefore $q = J_\lambda q$ and $q = J_\sigma q$.

(i) Note that $w_k = J_{\lambda_k} u_k$, $v_k = J_{\sigma_k} w_k$ and J_{λ_k}, J_{σ_k} are nonexpansive map for each $k \in \mathbb{N}$. So, we have

$$d(w_k, q) = d(J_{\lambda_k} u_k, J_{\lambda_k} q) \leq d(u_k, q). \tag{3.3}$$

We also have

$$d(v_k, q) = d(J_{\sigma_k} w_k, J_{\sigma_k} q) \leq d(w_k, q). \tag{3.4}$$

Based on (3.3), (3.4), and Lemma 2.7, we obtain $q \in \mathcal{S}_i(q)$ for $i = 1 : 3$.

$$\begin{aligned} d(\varphi_k, q) &= d(\varrho_k u_k \oplus \varsigma_k v'_k \oplus \gamma_k v''_k, q) \\ &\leq \varrho_k d(u_k, q) + \varsigma_k d(v'_k, q) + \gamma_k d(v''_k, q) \\ &\leq \varrho_k d(u_k, q) + \varsigma_k d(\mathcal{S}_1 u_k, \mathcal{S}_1 q) + \gamma_k d(\mathcal{S}_2 v_k, \mathcal{S}_2 q) \\ &\leq d(u_k, q) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} d(\Psi_k, q) &= d(\psi_k u_k \oplus \kappa_k v'''_k \oplus \phi_k \mathfrak{T}_1 u_k, q) \\ &\leq \psi_k d(u_k, q) + \kappa_k d(v'''_k, q) + \phi_k d(\mathfrak{T}_1 u_k, q) \\ &\leq \psi_k d(u_k, q) + \kappa_k d(\mathcal{S}_3 \varphi_k, q) + \phi_k d(\mathfrak{T}_1 u_k, q) \\ &\leq d(u_k, q). \end{aligned} \tag{3.6}$$

Now, consider

$$\begin{aligned} d(u_{k+1}, q) &= d(\delta_k u_k \oplus \eta_k \mathfrak{T}_2 u_k \oplus \xi_k \mathfrak{T}_3 \Psi_k, q) \\ &\leq \delta_k d(u_k, q) + \eta_k d(\mathfrak{T}_2 u_k, q) + \xi_k d(\mathfrak{T}_3 \Psi_k) \\ &\leq d(u_k, q). \end{aligned} \tag{3.7}$$

This demonstrates the existence of $\lim_{k\to\infty} d(u_k, q)$.

(ii) Next, we will prove that $\lim_{k\to\infty} d(u_k, w_k) = 0$ and $\lim_{k\to\infty} d(w_k, v_k) = 0$. Assume that

$$\lim_{k\to\infty} d(u_k, q) = r \tag{3.8}$$

for some $r \geq 0$. Based on Lemma 2.8, we possess

$$\frac{1}{2\lambda_k}\{d^2(w_k, q) - d^2(u_k, q) + d^2(u_k, w_k)\} \leq \mathfrak{G}(q) - f(w_k).$$

Given that for every $k \in \mathbb{N}$, $f(p) \leq f(w_k)$, it follows that

$$d^2(u_k, w_k) \leq d^2(u_k, q) - d^2(w_k, q). \tag{3.9}$$

and

$$\frac{1}{2\sigma_k}\{d^2(v_k, q) - d^2(w_k, q) + d^2(w_k, v_k)\} \leq h(q) - h(v_k).$$

Since $h(p) \leq h(v_k)$ for all $k \in \mathbb{N}$, It thus follows that

$$d^2(w_k, v_k) \leq d^2(w_k, q) - d^2(v_k, q). \tag{3.10}$$

Using (3.7) along with the fact that $\delta_k + \eta_k + \xi_k = 1$ for all $k \geq 1$, we obtain

$$\begin{aligned} d(u_{k+1}, q) &\leq \delta_k d(u_k, q) + \eta_k d(\mathfrak{T}_2 u_k, q) + \xi_k d(\mathfrak{T}_3 \Psi_k, q) \\ &\leq (1 - \xi_k) d(u_k, q) + \xi_k d(\Psi_k, q), \end{aligned}$$

That is the same as

$$\begin{aligned} d(u_k, q) &\leq \frac{1}{\xi_k}[d(u_k, q) - d(u_{k+1}, q)] + d(\Psi_k, q) \\ &\leq \frac{1}{a}[d(u_k, q) - d(u_{k+1}, q)] + d(\Psi_k, q), \end{aligned}$$

that provides

$$\liminf_{k\to\infty} d(u_k, q) \leq \liminf_{k\to\infty}\{\frac{1}{a}[d(u_k, q) - d(u_{k+1}, q)] + d(\Psi_k, q)\}.$$

Assuming our hypothesis and considering $\liminf_{k\to\infty}$ on both sides, we come up with

$$\liminf_{k\to\infty} d(u_k, q) \leq \liminf_{k\to\infty} d(\Psi_k, q) = r. \tag{3.11}$$

Using (3.6), we possess

$$\limsup_{k\to\infty} d(\Psi_k, q) \leq \limsup_{k\to\infty} d(u_k, q) = r. \tag{3.12}$$

Combining (3.11) and (3.12), we attain

$$\lim_{k\to\infty} d(\Psi_k, q) = r. \tag{3.13}$$

From (3.5), we obtain

$$\limsup_{k\to\infty} d(\varphi_k, q) \leq \limsup_{k\to\infty} d(u_k, q) = r. \tag{3.14}$$

Similarly, (3.6) yields

$$\begin{aligned} d(\Psi_k, q) &\leq \psi_k d(u_k, q) + \kappa_k d(\varphi_k, q) + \phi_k d(u_k, q) \\ &\leq d(u_k, q) - \kappa_k d(u_k, q) + \kappa_k d(\varphi_k, q), \end{aligned}$$

which results into

$$\begin{aligned} d(u_k, q) &\leq \frac{1}{\kappa_k}[d(u_k, q) - d(\Psi_k, q)] + d(\varphi_k, q) \\ &\leq \frac{1}{a}[d(u_k, q) - d(\Psi_k, q)] + d(\varphi_k, q), \end{aligned}$$

This, when combined with (3.8) and (3.13), yields

$$r \leq \liminf_{k\to\infty} d(\varphi_k, q). \tag{3.15}$$

From (3.14) and (3.15), we acquire

$$\lim_{k\to\infty} d(\varphi_k, q) = r. \tag{3.16}$$

Now, on using (3.5), we derive

$$d(u_k, q) \leq \frac{1}{a}[d(u_k, q) - d(\varphi_k, q)] + d(s_k, q),$$

which along with (3.8) and (3.16) gives

$$r \leq \liminf_{k\to\infty} d(s_k, q). \tag{3.17}$$

Also, (3.3) results into

$$\limsup_{k\to\infty} d(s_k, q) \leq \limsup_{k\to\infty} d(u_k, q) = r. \tag{3.18}$$

Utilizing (3.17) and (3.18), we obtain

$$\lim_{k\to\infty} d(s_k, q) = r. \tag{3.19}$$

From (3.8), (3.9) and (3.19), we attain

$$\lim_{k\to\infty} d(u_k, s_k) = 0. \tag{3.20}$$

With reference to (3.5), we have

$$\begin{aligned} d(\varphi_k, q) &= d(\varrho_k u_k \oplus \varsigma_k v'_k \oplus \gamma_k v''_k, q) \\ &\leq \varrho_k d(u_k, q) + \varsigma_k d(v'_k, q) + \gamma_k d(v''_k, q) \\ &\leq (1-\gamma_k) d(u_k, q) + \gamma_k d(v_k, q), \end{aligned}$$

which suggests that

$$d(u_k, q) \leq \frac{1}{a}[d(u_k, q) - d(\varphi_k, q)] + d(v_k, q),$$

We obtain $r = \liminf_{k\to\infty} d(u_k, p) \leq \liminf_{k\to\infty} d(v_k, p)$ using (3.8) and (3.16). This, along with $\limsup_{k\to\infty} d(v_k, p) \leq \limsup_{k\to\infty} d(u_k, p) = r$, indicates that

$$\lim_{k\to\infty} d(v_k, p) = r. \tag{3.21}$$

By utilizing (3.10) and (3.21), we can ensure

$$\lim_{k\to\infty} d(u_k, v_k) = 0 \tag{3.22}$$

(iii) Now, we prove $\lim_{k\to\infty} d(u_k, \mathcal{S}_i u_k) = 0$ for $i = 1 : 3$.
Consider

$$\begin{aligned} d^2(\varphi_k, q) &= d^2(\varrho_k u_k \oplus \varsigma_k v'_k \oplus \gamma_k v''_k, q) \\ &\leq \varrho_k d^2(u_k, q) + \varsigma_k d^2(v'_k, q) + \gamma_k d^2(v''_k, q) \\ &\quad -\varrho_k \varsigma_k d^2(u_k, v'_k) - \varrho_k \gamma_k d^2(u_k, v''_k) - \varsigma_k \gamma_k d^2(v'_k, v''_k) \\ &\leq d^2(u_k, q) - \varrho_k \varsigma_k d^2(u_k, v'_k) - \varrho_k \gamma_k d^2(u_k, v''_k) - \varsigma_k \gamma_k d^2(v'_k, v''_k), \end{aligned}$$

which is equivalent to

$$\varrho_k \varsigma_k d^2(u_k, v'_k) + \varrho_k \gamma_k d^2(u_k, v''_k) + \varsigma_k \gamma_k d^2(v'_k, v''_k) \leq d^2(u_k, q) - d^2(\varphi_k, q).$$

With the use of (3.8) and (3.14), we yield

$$\lim_{k\to\infty} d(u_k, v'_k) = 0, \tag{3.23}$$

$$\lim_{k\to\infty} d(u_k, v''_k) = 0 \tag{3.24}$$

and

$$\lim_{k\to\infty} d(v'_k, v''_k) = 0. \tag{3.25}$$

Now, triangle inequality gives

$$dist(u_k, \mathcal{S}_1 u_k) \leq d(u_k, v'_k) + dist(v'_k, \mathcal{S}_1 u_k),$$

which on using (3.23) results into

$$\lim_{k\to\infty} dist(u_k, \mathcal{S}_1 u_k) = 0. \tag{3.26}$$

Again, consider

$$\begin{aligned} dist(u_k, \mathcal{S}_2 u_k) &\leq d(u_k, v''_k) + dist(v''_k, \mathcal{S}_2 u_k) \\ &\leq d(u_k, v''_k) + d(v_k, u_k), \end{aligned}$$

which on using (3.20) and (3.24) gives

$$\lim_{k\to\infty} dist(u_k, \mathcal{S}_2 u_k) = 0. \tag{3.27}$$

Now, we have

$$\begin{aligned} d^2(\Psi_k, q) &\leq \psi_k d^2(u_k, q) + \kappa_k d^2(v_k''', q) + \phi_k d^2(\mathfrak{T}_1 u_k, q) \\ &\quad -\psi_k\kappa_k d^2(u_k, v_k''') - \psi_k\phi_k d^2(u_k, \mathfrak{T}_1 u_k) - \kappa_k\phi_k d^2(v_k''', \mathfrak{T}_1 u_k) \\ &\leq d^2(u_k, q) - \psi_k\kappa_k d^2(u_k, v_k''') - \psi_k\phi_k d^2(u_k, \mathfrak{T}_1 u_k) - \kappa_k\phi_k d^2(v_k''', \mathfrak{T}_1 u_k), \end{aligned}$$

which is equivalent to

$$\psi_k\kappa_k d^2(u_k, v_k''') + \psi_k\phi_k d^2(u_k, \mathfrak{T}_1 u_k) + \kappa_k\phi_k d^2(v_k''', \mathfrak{T}_1 u_k) \leq d^2(u_k, q) - d^2(\Psi_k, q),$$

By using (3.8) and (3.13), this results in

$$\lim_{k\to\infty} d(u_k, v_k''') = 0, \tag{3.28}$$

$$\lim_{k\to\infty} d(u_k, \mathfrak{T}_1 u_k) = 0 \tag{3.29}$$

and

$$\lim_{k\to\infty} d(v_k''', \mathfrak{T}_1 u_k) = 0. \tag{3.30}$$

On using (3.23) and (3.24), we have

$$\begin{aligned} d(\varphi_k, u_k) &\leq \varrho_k d(u_k, u_k) + \varsigma_k d(v_k', u_k) + \gamma_k d(v_k'', u_k) \\ &\to 0 \ as \ k \to \infty. \end{aligned} \tag{3.31}$$

Thus, with the help of (3.28) and (3.31), we obtain

$$\begin{aligned} dist(u_k, \mathcal{S}_3 u_k) &\leq d(u_k, v_k''') + dist(v_k''', \mathcal{S}_3 u_k) \\ &\leq d(u_k, v_k''') + d(\varphi_k, u_k) \to 0 \\ &\quad as \ k \to \infty. \end{aligned} \tag{3.32}$$

(iv) Afterwards, we demonstrate that $\lim_{k\to\infty} d(u_k, \mathfrak{T}_1 u_k) = \lim_{k\to\infty} d(u_k, \mathfrak{T}_2 u_k) = \lim_{k\to\infty} d(u_k, \mathfrak{T}_3 u_k) = 0$.

We have previously demonstrated in (3.29) that $\lim_{k\to\infty} d(u_k, \mathfrak{T}_1 u_k) = 0$.

Thus,

$$\begin{aligned} d^2(u_{k+1}, q) &\leq d^2(u_k, q) - \delta_k\eta_k d^2(u_k, \mathfrak{T}_2 u_k) \\ &\quad -\delta_k\xi_k d^2(u_k, \mathfrak{T}_3\Psi_k) - \eta_k\xi_k d^2(\mathfrak{T}_2 u_k, \mathfrak{T}_3\Psi_k), \end{aligned}$$

which results into

$$\lim_{k\to\infty} d(u_k, \mathfrak{T}_2 u_k) = 0, \tag{3.33}$$

$$\lim_{k\to\infty} d(u_k, \mathfrak{T}_3\Psi_k) = 0 \tag{3.34}$$

and

$$\lim_{k\to\infty} d(\mathfrak{T}_2 u_k, \mathfrak{T}_3\Psi_k) = 0. \tag{3.35}$$

On using (3.28) and (3.29), we obtain

$$\begin{aligned} d(\Psi_k, u_k) &\leq \psi_k d(u_k, u_k) + \kappa_k d(v_k''', u_k) + \phi_k d(\mathfrak{T}_1 u_k, u_k) \\ &\to 0 \ as \ k \to \infty. \end{aligned} \tag{3.36}$$

Now, (3.33), (3.35) and (3.36) yields

$$\begin{aligned} d(u_k, \mathfrak{T}_3 u_k) &\leq d(u_k, \mathfrak{T}_2 u_k) + d(\mathfrak{T}_2 u_k, \mathfrak{T}_3 \Psi_k) + d(\mathfrak{T}_3 \Psi_k, \mathfrak{T}_3 u_k) \to 0 \\ & \quad \textit{as } k \to \infty. \end{aligned} \tag{3.37}$$

(v) Now, as $s_k = J_{\lambda_k} u_k$, from Lemma 2.9 we have

$$\begin{aligned} d(J_\lambda u_k, u_k) &\leq d(J_\lambda u_k, w_k) + d(w_k, u_k) \\ &= d(J_\lambda u_k, J_{\lambda_k} u_k) + d(w_k, u_k) \\ &= d(J_\lambda u_k, J_\lambda(\frac{\lambda_k - \lambda}{\lambda_k} J_{\lambda_k} u_k \oplus \frac{\lambda}{\lambda_k} u_k)) + d(w_k, u_k) \\ &\leq d(u_k, (1 - \frac{\lambda}{\lambda_k}) J_{\lambda_k} u_k \oplus \frac{\lambda}{\lambda_k} u_k) + d(w_k, u_k) \\ &\leq (1 - \frac{\lambda}{\lambda_k}) d(u_k, J_{\lambda_k} u_k) + \frac{\lambda}{\lambda_k} d(u_k, u_k) + d(w_k, u_k) \\ &= (1 - \frac{\lambda}{\lambda_k}) d(u_k, w_k) + d(w_k, u_k) \to 0 \\ & \quad \textit{as } k \to \infty. \end{aligned}$$

Similarly by using Lemma 2.9 and using $v_k = J_{\sigma_k} w_k$, we obtain

$$\begin{aligned} d(J_\sigma u_k, u_k) &\leq d(J_\sigma u_k, v_k) + d(v_k, w_k) + d(w_k, u_k) \\ &= d(J_\sigma u_k, J_{\sigma_k} w_k) + d(v_k, w_k) + d(w_k, u_k) \\ &= d\big(J_\sigma u_k, J_\sigma(\frac{\sigma_k - \sigma}{\sigma_k} J_{\sigma_k} w_k \oplus \frac{\sigma}{\sigma_k} w_k)\big) + d(v_k, w_k) + d(w_k, u_k) \\ &\leq d(u_k, (1 - \frac{\sigma}{\sigma_k}) J_{\sigma_k} w_k \oplus \frac{\sigma}{\sigma_k} w_k) + d(v_k, w_k) + d(w_k, u_k) \\ &\leq (1 - \frac{\sigma}{\sigma_k}) d(u_k, J_{\sigma_k} w_k) + \frac{\sigma}{\sigma_k} d(u_k, w_k) + d(v_k, w_k) + d(w_k, u_k) \\ &= (1 - \frac{\sigma}{\sigma_k}) d(u_k, v_k) + (1 + \frac{\sigma}{\sigma_k}) d(w_k, u_k) + d(v_k, w_k) \to 0 \\ & \quad \textit{as } k \to \infty. \end{aligned}$$

□

We now present the Δ-convergence result in CAT(0) spaces.

Theorem 3.2. *Assume that a complete CAT(0) space $\mathcal{Z}$ has a nonempty closed and convex subset $\mathcal{D}$. The single-valued nonexpansive mappings are denoted by $\mathfrak{T}_i : \mathcal{D} \to \mathcal{D}$; the multi-valued nonexpansive mappings are denoted by $\mathcal{S}_i : \mathcal{D} \to CB(\mathcal{D})$ for $i = 1 : 3$ and $\mathfrak{G}, h : \mathcal{D} \to (-\infty, \infty]$ are two proper convex and lower semi-continuous functions. Consider that*

$$\Omega = \mathfrak{U}(\mathfrak{T}_1) \cap \mathfrak{U}(\mathfrak{T}_2) \cap \mathfrak{U}(\mathfrak{T}_3) \cap \mathfrak{U}(\mathcal{S}_1) \cap \mathfrak{U}(\mathcal{S}_2) \cap \mathfrak{U}(\mathcal{S}_3) \cap \arg\min_{y \in \mathcal{D}} \mathfrak{G}(y) \cap \arg\min_{\zeta \in \mathcal{D}} h(\zeta) \neq \emptyset$$

and $\mathcal{S}_i q = \{q\}$, $i = 1 : 3$ for $q \in \Omega$. For $u_1 \in \mathcal{D}$, let the sequence $\{u_k\}$ is generated by (3.1), where $\{\varrho_k\}$, $\{\varsigma_k\}$, $\{\gamma_k\}$, $\{\psi_k\}$, $\{\kappa_k\}$, $\{\phi_k\}$, $\{\delta_k\}$, $\{\eta_k\}$ and $\{\xi_k\}$ are sequences in $(0, 1)$ such that it satisfies (3.2) and $\{\lambda_k\}$ is a sequence such that $\lambda_k \geq \lambda > 0$ for all $n \in \mathbb{N}$ and some λ. In turn, the sequence $\{u_k\}$ Δ-converges to a point in Ω.

Proof. Let $W_\omega(\{u_k\}) = \cup_{\Lambda_k \subset \{u_k\}} A(\{\Lambda_k\}) \subset \Omega$. Let $\Lambda \in W_\omega(u_k)$. Then a subsequence Λ_k of $\{u_k\}$ occurs such that $A(\Lambda_k) = \Lambda$. Consequently, for any $\nu \in \Omega$, there exists a subsequence $\{\nu_k\}$ of Λ_k such that $\Delta - \lim_{k\to\infty} \nu_k = \nu$. In view of Theorem 3.1, we obtain

$$\lim_{k\to\infty} d(\nu_k, \mathfrak{T}_i \nu_k) = 0,\ i = 1:3$$

and

$$\lim_{k\to\infty} d(\nu_k, J_\lambda \nu_k) = 0, \quad \lim_{k\to\infty} d(\nu_k, J_\sigma \nu_k) = 0.$$

Given that $\mathfrak{T}_i$, $i = 1 : 3$, J_λ, and J_σ are nonexpansive mappings, we may utilize Lemma 2.5 to acquire

$$\nu = \mathfrak{T}_1 \nu = \mathfrak{T}_2 \nu = \mathfrak{T}_3 \nu = J_\lambda \nu = J_\sigma \nu.$$

So, we have

$$\nu \in \mathfrak{U}(\mathfrak{T}_1) \cap \mathfrak{U}(\mathfrak{T}_2) \cap \mathfrak{U}(\mathfrak{T}_3) \cap \arg\min_{y\in\mathcal{D}} \mathfrak{G}(y) \cap \arg\min_{\zeta\in\mathcal{D}} h(\zeta). \tag{3.38}$$

Since $\mathcal{S}_i$, is compact valued for $i = 1 : 3$, then for every $k \in \mathbb{N}$, there exist $r_k^i \in \mathcal{S}_i \nu_k$ and $p_k^i \in \mathcal{S}_i \nu$ for $i = 1 : 3$ such that

$$d(\nu_k, r_k^i) = dist(\nu_k, \mathcal{S}_i \nu_k),\ i = 1:3$$

and

$$d(r_k^i, p_k^i) = dist(r_k^i, \mathcal{S}_i \nu),\ i = 1:3.$$

Theorem 3.1 allows us to obtain

$$\lim_{k\to\infty} d(\nu_k, r_k^i) = 0,\ i = 1:3.$$

Utilizing the compactness of $\mathcal{S}_i \nu$ for $i = 1 : 3$, we can deduce the existence of a subsequence $\{p_{n_j}^i\}$ of $\{p_k^i\}$, such that $\lim_{j\to\infty} p_{n_j}^i = p^i \in \mathcal{S}_i \nu$. With the aid of the Opial condition, we are able to get

$$\begin{aligned}
\limsup_{j\to\infty} d(\nu_{n_j}, p^i) &\le \limsup_{j\to\infty} (d(\nu_{n_j}, r_{n_j}^i) + d(r_{n_j}^i, p_{n_j}^i) + d(p_{n_j}^i, p^i)) \\
&\le \limsup_{j\to\infty} (d(\nu_{n_j}, r_{n_j}^i) + dist(r_{n_j}^i, \mathcal{S}_i \nu) + d(p_{n_j}^i, p^i)) \\
&\le \limsup_{j\to\infty} (d(\nu_{n_j}, r_{n_j}^i) + H(\mathcal{S}_i \nu_{n_j}, \mathcal{S}_i \nu) + d(p_{n_j}^i, p^i)) \\
&\le \limsup_{j\to\infty} (d(\nu_{n_j}, r_{n_j}^i) + d(\nu_{n_j}, \nu) + d(p_{n_j}^i, p^i)) \\
&= \limsup_{j\to\infty} d(\nu_{n_j}, \nu).
\end{aligned}$$

We obtain $\nu = p^i \in \mathcal{S}_i \nu$ for $i = 1 : 3$, because the asymptotic center is unique. Equation (3.38) is used to get

$$\nu \in \mathfrak{U}(\mathfrak{T}_1) \cap \mathfrak{U}(\mathfrak{T}_2) \cap \mathfrak{U}(\mathfrak{T}_3) \cap \mathfrak{U}(\mathcal{S}_1) \cap \mathfrak{U}(\mathcal{S}_2) \cap \mathfrak{U}(\mathcal{S}_3) \cap \arg\min_{y\in\mathcal{D}} \mathfrak{G}(y) \cap \arg\min_{\sigma\in\mathcal{D}} h(\zeta) = \Omega.$$

We derive $q = \nu$ and $W_\omega(\{u_k\}) \subset \Omega$ from Theorem 3.1 and Lemma 2.6.

Ultimately, it is sufficient to demonstrate that $W_\omega(\{u_k\})$ is composed of a single element. For this, let $\{\Lambda_k\}$ be a subsequence of $\{u_k\}$ and let $A(\{u_k\}) = u$. Since $\Lambda \in W_\omega(u_k) \subset \Omega$ and $d(u_k, \Lambda)$ converges, we have $\nu = u$. This indicates that $W_\omega(\{u_k\}) = \{u\}$. □

Strong convergence theorems for the suggested approach in CAT(0) spaces are shown in the following results.

Theorem 3.3. *According to Theorem 3.2, if* J_λ*, or* J_σ*, or* $\mathfrak{T}_1$*, or* $\mathfrak{T}_2$*, or* $\mathfrak{T}_3$ *are semi-compact, or* $\mathcal{S}_1$*, or* $\mathcal{S}_2$*, or* $\mathcal{S}_3$*, are hemi-compact then the sequence* $\{u_k\}$ *converges to an element of* Ω *in the given scenario.*

Proof. Assuming $\mathcal{S}_1$ to be hemi-compact, we can proceed without losing generality. Consequently, a subsequence $\{v_k\}$ of $\{u_k\}$ exists, and it has a strong limit p in $\mathcal{D}$. Theorem 3.1 provide us

$$\lim_{k\to\infty} d(\mathfrak{T}_i s_k, s_k) = 0,\ i = 1, 2, 3,$$

$$\lim_{k\to\infty} d(J_\lambda s_k, s_k) = 0$$

$$\lim_{k\to\infty} d(J_\sigma s_k, s_k) = 0$$

and

$$\lim_{k\to\infty} dist(\mathcal{S}_i s_k, s_k) = 0,\ i = 1, 2, 3.$$

Lemma 2.5 allows us to gain

$$p \in \mathfrak{U}(\mathfrak{T}_1) \cap \mathfrak{U}(\mathfrak{T}_2) \cap \mathfrak{U}(\mathfrak{T}_3) \cap \arg\min_{y\in\mathcal{D}} \mathfrak{G}(y) \cap \arg\min_{\zeta\in\mathcal{D}} h(\zeta). \tag{3.39}$$

By using nonexpansiveness of $\mathcal{S}_1$, we have

$$\begin{aligned} dist(p, \mathcal{S}_1 p) &\le d(p, s_k) + dist(s_k, \mathcal{S}_1 s_k) + H(\mathcal{S}_1 s_k, \mathcal{S}_1 p) \\ &\le 2d(p, s_k) + dist(s_k, \mathcal{S}_1 s_k) \\ &\to 0 \ as\ k \to \infty. \end{aligned}$$

$dist(p, \mathcal{S}_1 p) = 0$ is the outcome of this, and it is equivalent to $p \in \mathcal{S}_1 p$. Therefore, $p \in \mathfrak{U}(\mathcal{S}_1)$. Likewise, it is possible to demonstrate that $p \in \mathfrak{U}(\mathcal{S}_2)$ and $p \in \mathfrak{U}(\mathcal{S}_3)$. Consequently, from (3.39), we attain

$$p \in \mathfrak{U}(\mathfrak{T}_1) \cap \mathfrak{U}(\mathfrak{T}_2) \cap \mathfrak{U}(\mathfrak{T}_3) \cap \mathfrak{U}(\mathcal{S}_1) \cap \mathfrak{U}(\mathcal{S}_2) \cap \mathfrak{U}(\mathcal{S}_3) \cap \arg\min_{y\in\mathcal{D}} \mathfrak{G}(y) \cap \arg\min_{\zeta\in\mathcal{D}} h(\zeta) = \Omega.$$

By using double extract subsequence principle, the sequence $\{u_k\}$ is found to have a strong convergence to $p \in \Omega$. □

Every multi-valued mapping $\mathcal{S} : \mathcal{D} \to CB(\mathcal{D})$ is hemi-compact for a compact subset $\mathcal{D}$ of $\mathcal{Z}$. Thus, Theorem 3.3 can be used to quickly arrive at the following result.

Theorem 3.4. *Suppose that* $\mathcal{Z}$ *is a complete CAT(0) space and* $\mathcal{D}$ *is a nonempty closed and convex subset of* $\mathcal{Z}$*. The single-valued nonexpansive mappings are denoted by* $\mathfrak{T}_i : \mathcal{D} \to \mathcal{D}$*; the multi-valued nonexpansive mappings are denoted by* $\mathcal{S}_i : \mathcal{D} \to KC(\mathcal{D})$ *for* $i = 1 : 3$ *and* $\mathfrak{G}, h : \mathcal{D} \to (-\infty, \infty]$ *are two proper convex and lower semi-continuous functions. Suppose that*

$$\Omega = \mathfrak{U}(\mathfrak{T}_1) \cap \mathfrak{U}(\mathfrak{T}_2) \cap \mathfrak{U}(\mathfrak{T}_3) \cap \mathfrak{U}(\mathcal{S}_1) \cap \mathfrak{U}(\mathcal{S}_2) \cap \mathfrak{U}(\mathcal{S}_3) \cap \arg\min_{y\in\mathcal{D}} \mathfrak{G}(y) \cap \arg\min_{\zeta\in\mathcal{D}} h(\zeta) \neq \emptyset$$

and $\mathcal{S}_i q = \{q\}$ *for* $q \in \Omega$ *and* $i = 1 : 3$*. For* $u_1 \in \mathcal{D}$*, let the sequence* $\{u_k\}$ *is generated by (3.1), where* $\{\varrho_k\}$*,* $\{\varsigma_k\}$*,* $\{\gamma_k\}$*,* $\{\psi_k\}$*,* $\{\kappa_k\}$*,* $\{\phi_k\}$*,* $\{\delta_k\}$*,* $\{\eta_k\}$ *and* $\{\xi_k\}$ *are sequences in* $(0, 1)$ *such that it satisfies* (3.2) *and* $\{\lambda_k\}$ *is a sequence such that*

$\lambda_k \geq \lambda > 0$ for all $k \in \mathbb{N}$ and some λ. Then, there is a strong convergence of the sequence $\{u_k\}$ to a point in Ω.

Remarks.

(i) As per [6], any CAT(k) space with $k \leq 0$ is a CAT(k') space for $k' \geq k$. Therefore, our findings are instantly applicable to any CAT(k) space.

(ii) We may obtain the following convergence conclusions from Theorems 3.2 and 3.3 since $\mathcal{H}$ is a complete CAT(0) space for all real Hilbert spaces.

Corollary 3.5. *Let $\mathcal{Z}$ be a real Hilbert space and $\mathcal{D}$ be a nonempty closed and convex subset of $\mathcal{Z}$. The single-valued nonexpansive mappings are denoted by $\mathfrak{T}_i : \mathcal{D} \to \mathcal{D}$; the multi-valued nonexpansive mappings are denoted by $\mathcal{S}_i : \mathcal{D} \to CB(\mathcal{D})$ for $i = 1 : 3$ and $\mathfrak{G}, h : \mathcal{D} \to (-\infty, \infty]$ are two proper convex and lower semi-continuous functions. Suppose that $\Omega = \mathfrak{U}(\mathfrak{T}_1) \cap \mathfrak{U}(\mathfrak{T}_2) \cap \mathfrak{U}(\mathfrak{T}_3) \cap \mathfrak{U}(\mathcal{S}_1) \cap \mathfrak{U}(\mathcal{S}_2) \cap \mathfrak{U}(\mathcal{S}_3) \cap \arg\min_{y\in\mathcal{D}} \mathfrak{G}(y) \cap \arg\min_{\zeta\in\mathcal{D}} h(\zeta) \neq \emptyset$ and $\mathcal{S}_i q = \{q\}$, $i = 1 : 3$ for $q \in \Omega$. For $u_1 \in \mathcal{D}$, let us assume that the sequence $\{u_k\}$ is produced as follows:*

$$\begin{cases} s_k = \arg\min_{y\in\mathcal{Z}}[\mathfrak{G}(y) + \frac{1}{2\lambda_k}\|y - u_k\|^2], \\ v_k = \arg\min_{\zeta\in\mathcal{Z}}[h(\zeta) + \frac{1}{2\sigma_k}\|\zeta - s_k\|^2], \\ \varphi_k = \varrho_k u_k + \varsigma_k w'_k + \gamma_k w''_k, \\ \Psi_k = \psi_k u_k + \kappa_k w'''_k + \phi_k \mathfrak{T}_1 u_k, \\ u_{k+1} = \delta_k u_k + \eta_k \mathfrak{T}_2 u_k + \xi_k \mathfrak{T}_3 \Psi_k, \ \textit{for all } k \in \mathbb{N}. \end{cases} \tag{3.40}$$

where $\{\varrho_k\}$, $\{\varsigma_k\}$, $\{\gamma_k\}$, $\{\psi_k\}$, $\{\kappa_k\}$, $\{\phi_k\}$, $\{\delta_k\}$, $\{\eta_k\}$ and $\{\xi_k\}$ are sequences in $(0,1)$ such that it satisfies (3.2) and $\{\lambda_k\}$ and $\{\sigma_k\}$ are sequences such that $\lambda_k \geq \lambda > 0$ $\sigma_k \geq \sigma > 0$ for all $k \in \mathbb{N}$ and some λ, σ. Then, the sequence $\{u_k\}$ Δ-converges to a point in Ω.

Corollary 3.6. *Let $\mathcal{Z}$ be a real Hilbert space and $\mathcal{D}$ be a nonempty closed and convex subset of $\mathcal{Z}$. The single-valued nonexpansive mappings are denoted by $\mathfrak{T}_i : \mathcal{D} \to \mathcal{D}$; the multi-valued nonexpansive mappings are denoted by $\mathcal{S}_i : \mathcal{D} \to CB(\mathcal{D})$ for $i = 1 : 3$ and $\mathfrak{G}, h : \mathcal{D} \to (-\infty, \infty]$ are two proper convex and lower semi-continuous functions. Also, assume that $\Omega = \mathfrak{U}(\mathfrak{T}_1) \cap \mathfrak{U}(\mathfrak{T}_2) \cap \mathfrak{U}(\mathfrak{T}_3) \cap \mathfrak{U}(\mathcal{S}_1) \cap \mathfrak{U}(\mathcal{S}_2) \cap \mathfrak{U}(\mathcal{S}_3) \cap \arg\min_{y\in\mathcal{D}} \mathfrak{G}(y) \cap \arg\min_{\zeta\in\mathcal{D}} h(\zeta) \neq \emptyset$ and $\mathcal{S}_i q = \{q\}$, $i = 1 : 3$ for $q \in \Omega$. For $u_1 \in \mathcal{D}$, let the sequence $\{u_k\}$ is generated by (3.40), where $\{\varrho_k\}$, $\{\varsigma_k\}$, $\{\gamma_k\}$, $\{\psi_k\}$, $\{\kappa_k\}$, $\{\phi_k\}$, $\{\delta_k\}$, $\{\eta_k\}$ and $\{\xi_k\}$ are sequences in $(0,1)$ such that it satisifes (3.2) and $\{\lambda_k\}$ is a sequence such that $\lambda_k \geq \lambda > 0$ for all $k \in \mathbb{N}$ and some λ. Then, the sequence $\{u_k\}$ converges to an element of Ω if J_λ or $\mathfrak{T}_1$ or $\mathfrak{T}_2$ or $\mathfrak{T}_3$ is semi-compact or $\mathcal{S}_1$ or $\mathcal{S}_2$ or $\mathcal{S}_3$ is hemi-compact.*

4. Conclusion

This study aimed to offer a modified proximal point technique for addressing the fixed point problem of nonexpansive single-valued and multi-valued mappings in CAT(0) spaces, as well as the constrained convex minimization problem. The corresponding findings of Cholamjiak [7], Suantai and Phuengrattana [30], Kumam et al. [22], Weng et al. [37], Weng et al. [36], and Garodia et al. [14] are all expanded upon by our findings.

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