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APPROXIMATING TECHNIQUE INSPIRED BY THE CUTTING-PLANE METHOD FOR CONVEX MINIMIZATION PROBLEM OVER A FIXED POINT SET

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Abstract. The cutting-plane method is a technique for solving mixed integer linear programming problems and can be applied to continuous convex optimization. Motivated by this fact, we obtain an iterative sequence generated by a new method analogous to the cutting-plane method and prove its convergence to a solution to a convex minimization problem over a fixed point set of a given mapping defined on a complete geodesic space.

1. Introduction

Convex minimization problems over the set of fixed points of a specific mapping are hot topics in nonlinear analysis. It is formulated as follows: Let X be a space having a convexity structure and $f\colon X \to]-\infty, \infty]$ be a convex function. Find a point $x_0 \in X$ which minimizes the value of f on the set of fixed points of given mapping $T\colon X \to X$.

We often consider nonexpansive mappings for this problem because they have some advantages to study in nonlinear and convex analysis. Since the set of fixed points of such mappings is closed and convex, we may apply various techniques in convex analysis and the approximation theory of fixed points; see [4, 13, 17] for instance. We use many kinds of iterative schemes to generate an approximate sequence for solving these problems, such as the Mann type method, the Halpern type method, and several variations of projection methods; for the recent works, see [5, 10, 11, 15, 16, 19] and references therein.

The shrinking projection method, proposed as an approximation method for a family of nonexpansive mappings, strongly relates to the main result of this work. The following is a simplified version of this method.

Theorem 1.1 (Takahashi-Takeuchi-Kubota [18]). *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that the set $F(T)$ of fixed points of T is nonempty. Let $\{\alpha_n\}$ be a sequence in $[0, a]$, where $0 < a < 1$. For an arbitrarily chosen point $x \in H$, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, and*

$$\begin{aligned}
&y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n,\\
&C_{n+1} = \{z \in C_n : \|z - y_n\| \le \|z - x_n\|\},\\
&x_{n+1} = P_{C_{n+1}}x
\end{aligned}$$

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for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x \in C$, where P_K is the metric projection of H onto a nonempty closed convex subset K of H.

These techniques have been generalized to some complete geodesic spaces with particular geometrical structures. Among them, we adopt Hadamard spaces for this work.

The cutting-plane method [6] is mainly known as a technique for solving mixed integer linear programming problems, and this method has been widely applied to many kinds of optimization problems, including continuous convex optimization [9].

In this paper, we propose a new iterative method to generate a sequence approximating a solution to a convex minimization problem over the set of fixed points of nonexpansive mapping. This approach is inspired by the cutting-plane method, and the generating procedure of the sequence is original compared to the known algorithms. Moreover, from this result, we obtain some new results generalizing known theorems, such as a convergence theorem of a sequence generated by the shrinking projection method in the setting of Hadamard spaces. We also consider the case that the constraint set is the set of common fixed points of finitely many nonexpansive mappings.

2. Preliminaries

Let X be a metric space and $T\colon X \to X$. We call T a nonexpansive mapping if $d(Tx, Ty) \le d(x, y)$ for any $x, y \in X$. We denote by $\operatorname{Fix} T$ the set of all fixed points of T, that is,

$$\operatorname{Fix} T = \{z \in X \mid Tz = z\}.$$

Let X be a metric space. We say X is a geodesic space if for any $x, y \in X$ with $l = d(x, y)$, there exists an isometry $c_{xy}\colon [0, l] \to X$ such that $c_{xy}(0) = x$ and $c_{xy}(1) = y$. In what follows, we always assume that such a mapping c_{xy} is unique for each choice of $x, y \in X$. The image of c_{xy} is called a geodesic segment between x and y, and we denote it by $[x, y]$. For $x, y \in X$ with $l = d(x, y)$ and $t \in [0, 1]$, we define a convex combination $tx \oplus (1 - t)y$ of x and y with a coefficient t by

$$(1 - t)x \oplus ty = c_{xy}(tl),$$

that is, $(1 - t)x \oplus ty$ is a unique point z satisfying that $d(x, z) = tl$ and $d(z, y) = (1 - t)l$.

For a geodesic space X, we say X to be a CAT(0) space if for every $x, y, z \in X$ and $t \in [0, 1]$, the inequality

$$d((1 - t)x \oplus ty, z)^2 \le td(x, z)^2 + (1 - t)d(y, z)^2 - t(1 - t)d(x, y)^2.$$

We note that we usually define a CAT(0) space by using the concept of model space. In this work, we adopt the equivalent definition mentioned above. In particular, a complete CAT(0) space is called a Hadamard space. A nonempty closed convex subset of a Hilbert space is a simple example of a Hadamard space. For more details of Hadamard space, see [1, 3] for instance.

Let $f\colon X \to \,]-\infty, \infty]$ be a function on a Hadamard space X. We say f is proper if the effective domain

$$\operatorname{Dom} f = \{x \in X \mid f(x) \in \mathbb{R}\}$$

of f is not empty. For such f, we may naturally define lower semicontinuity and convexity. Namely, f is said to be lower semicontinuous if

$$f(x_0) \le \liminf_{n\to\infty} f(x_n)$$

whenever a sequence $\{x_n\} \subset X$ is convergent to $x_0 \in X$; f is said to be convex if

$$f((1-t)x \oplus ty) \le (1-t)f(x) + tf(y)$$

for any $x, y \in X$ and $t \in \,]0,1[$. The set of all minimizers of f on X is denoted by $\operatorname{argmin}_{y\in X} f(y)$, or simply, $\operatorname{argmin}_X f$.

Let C be a nonempty closed convex subset of X. The indicator function $i_C\colon X \to \,]-\infty, \infty]$ for C is defined by

$$i_C(x) = \begin{cases} 0 & (x \in C), \\ \infty & (x \notin C) \end{cases}$$

for $x \in X$. From the assumptions of C, we easily see that i_C is proper, lower semicontinuous, and convex.

Let $\{x_n\}$ be a bounded sequence of a metric space X. Then, $x_0 \in X$ is called an asymptotic center of $\{x_n\}$ if x_0 is a minimizer of a function $g\colon X \to \mathbb{R}$ defined by $g(y) = \limsup_{n\to\infty} d(x_n, y)$ for $y \in X$. We know that any bounded sequence has a unique asymptotic center if X is a Hadamard space. We say $\{x_n\}$ to be Δ-convergent to $x_0 \in X$ if every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has an identical asymptotic center x_0.

The following lemma will be used in the main result.

Lemma 2.1. *Let X be a Hadamard space and let $f\colon X \to \,]-\infty, \infty]$ be a proper lower semicontinuous convex function. Then, for a bounded sequence $\{y_n\} \subset X$ with its asymptotic center y_0,*

$$f(y_0) \le \limsup_{n\to\infty} f(y_n).$$

Proof. Let $s = \limsup_{n\to\infty} f(y_n) \in \mathbb{R} \cup \{\pm\infty\}$. If $s = \infty$, then the inequality obviously holds. If $s = -\infty$, then $\{f(y_n)\}$ is divergent to $-\infty$. Considering that $f(y_0)$ may have the value ∞, we let

$$C = \{z \in X \mid f(z) \le \min\{f(y_0), 0\} - 1\}.$$

Then, $y_n \in C$ for sufficiently large $n \in \mathbb{N}$. Since C is closed and convex, the asymptotic center y_0 of $\{y_n\}$ belongs to C. It implies that

$$f(y_0) \le \min\{f(y_0), 0\} - 1 < \min\{f(y_0), 0\} \le f(y_0),$$

a contradiction. Thus we may assume $s \in \mathbb{R}$. For an arbitrary $\epsilon \in \,]0, \infty[$, let

$$C_\epsilon = \{z \in X \mid f(z) \le s + \epsilon\}.$$

Then, $y_n \in C_\epsilon$ for sufficiently large $n \in \mathbb{N}$. Thus we get $y_0 \in C_\epsilon$, that is,

$$f(y_0) \le s + \epsilon.$$

Since ϵ is arbitrary, we have $f(y_0) \le s$, the desired result. □

Let $f\colon X \to]-\infty, \infty]$ be a proper lower semicontinuous convex function on a Hadamard space X. Then, for $x \in X$, a function $g_x\colon X \to]-\infty, \infty]$ defined by

$$g_x(y) = f(y) + \frac{1}{2}d(y, x)^2$$

for $y \in X$ has a unique minimizer z_x. Using this point, we define a resolvent operator $R_f\colon X \to X$ by $R_f x = z_x$ for each $x \in X$; see [8, 14]. We know that R_f is a nonexpansive mapping and $\operatorname{Fix} R_f = \operatorname{argmin}_X f$.

In a Hadamard space, we can define a metric projection P_C as a single-valued mapping of X onto a nonempty closed convex subset C by $P_C = R_{i_C}$. Since $P_C x$ is a unique minimizer of the function $d(\cdot, x)$ on C, we have

$$\begin{aligned} R_{i_C} x &= \operatorname*{argmin}_{y \in X} \left(i_C(y) + \frac{1}{2}d(y, x)^2 \right) \\ &= \operatorname*{argmin}_{y \in C} \frac{1}{2}d(y, x)^2 \\ &= \operatorname*{argmin}_{y \in C} d(y, x) = P_C x. \end{aligned}$$

Therefore P_C is also nonexpansive and $\operatorname{Fix} P_C = C$.

3. Main result

The cutting-plane method can be regarded as an iterative process generating an approximating sequence of a solution to the problem. We first solve a problem without any constraint, and then we add a new constraint by using a hyperplane generated by the previous solution and define the next subproblem. Repeating this process, we obtain an approximating sequence converging to a solution to the original problem. In our method, we use a point defined by a resolvent operator instead of the solution to each subproblem.

Theorem 3.1. *Suppose that X is a Hadamard space and a subset $\{z \in X \mid d(z, y) \leq d(z, x)\}$ is convex for any $x, y \in X$. Let $f\colon X \to]-\infty, \infty]$ be a proper lower semicontinuous convex function, and let $T\colon X \to X$ be a nonexpansive mapping. Let $\{\lambda_n\} \subset]0, \infty[$ be a positive real sequence such that $\inf_{n \in \mathbb{N}} \lambda_n > 0$. Suppose that $\operatorname{argmin}_X f \cap \operatorname{Fix} T \neq \emptyset$. Generate a sequence $\{x_n\} \subset X$ as follows: $x_1 \in X$ is given, $f_1 = f$, and*

$$\begin{aligned} X_n &= \{z \in X \mid d(z, Tx_n) \leq d(z, x_n)\}, \\ f_{n+1} &= f_n + i_{X_n}, \\ x_{n+1} &= R_{\lambda_{n+1} f_{n+1}} x_n \end{aligned}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ is Δ-convergent to $x_0 \in \operatorname{argmin}_X f \cap \operatorname{Fix} T$.

Proof. Let $u \in \operatorname{argmin}_X f \cap \operatorname{Fix} T$. We know that $R_{\lambda_n f_n}$ is quasinonexpansive with $\operatorname{Fix} R_{\lambda_n f_n} = \operatorname{argmin}_X \lambda_n f_n = \operatorname{argmin}_X f_n$ for every $n \in \mathbb{N}$. On the other hand, since T is nonexpansive, we have $d(u, Tx_n) = d(Tu, Tx_n) \leq d(u, x_n)$, and thus $u \in X_n$ for every $n \in \mathbb{N}$. Therefore we have

$$f_{n+1}(u) = f_n(u) + i_{X_n}(u) = f_n(u) = \cdots = f_1(u)$$

$$= f(u) = \inf_{y\in X} f(y) = \inf_{y\in X} f_{n+1}(y),$$

which implies $u \in \operatorname{argmin}_X f_n = \operatorname{Fix} R_{\lambda_n f_n}$ for every $n \in \mathbb{N}$. It follows that

$$0 \le d(x_{n+1}, u) = d(R_{\lambda_{n+1} f_{n+1}} x_n, u) \le d(x_n, u),$$

and hence the sequence $\{d(x_n, u)\}$ has a limit $c_u \in [0, \infty[$. We also have $\{x_n\}$ is bounded. From the definition of resolvent, we have

$$\begin{aligned}
&\lambda_{n+1} f_{n+1}(x_{n+1}) + \frac{1}{2} d(x_{n+1}, x_n)^2 \\
&= \lambda_{n+1} f_{n+1}(R_{\lambda_{n+1} f_{n+1}} x_n) + \frac{1}{2} d(R_{\lambda_{n+1} f_{n+1}} x_n, x_n)^2 \\
&\le \lambda_{n+1} f_{n+1}(tu \oplus (1-t)x_{n+1}) + \frac{1}{2} d(tu \oplus (1-t)x_{n+1}, x_n)^2 \\
&\le t\lambda_{n+1} f_{n+1}(u) + (1-t)\lambda_{n+1} f_{n+1}(x_{n+1}) \\
&\quad + \frac{1}{2}\left(td(u, x_n)^2 + (1-t)d(x_{n+1}, x_n)^2 - t(1-t)d(u, x_{n+1})^2\right)
\end{aligned}$$

for $t \in]0, 1[$. Dividing by $t/2$ and letting $t \to 0$, we get

$$2\lambda_{n+1}(f_{n+1}(x_{n+1}) - f_{n+1}(u)) + d(x_{n+1}, x_n)^2 \le d(u, x_n)^2 - d(u, x_{n+1})^2. \tag{3.1}$$

From this inequality, we have

$$\begin{aligned}
0 \le d(x_{n+1}, x_n)^2 &\le 2\lambda_{n+1}(f_{n+1}(u) - f_{n+1}(x_{n+1})) + d(u, x_n)^2 - d(u, x_{n+1})^2 \\
&\le 2\lambda_{n+1}\left(f(u) - \inf_{y\in X} f_{n+1}(y)\right) + d(u, x_n)^2 - d(u, x_{n+1})^2 \\
&= d(u, x_n)^2 - d(u, x_{n+1})^2 \\
&\to c_u^2 - c_u^2 = 0.
\end{aligned}$$

Hence $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. Since $x_{n+1} = R_{\lambda_{n+1} f_{n+1}} x_n \in \operatorname{Dom} f_{n+1} = \operatorname{Dom} f_n \cap \operatorname{Dom} i_{X_n} \subset X_n$, we have $d(x_{n+1}, Tx_n) \le d(x_{n+1}, x_n)$ and thus

$$0 \le d(x_n, Tx_n) \le d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) \le 2d(x_{n+1}, x_n) \to 0.$$

It follows that $d(x_n, Tx_n) \to 0$ as $n \to \infty$.

Let $x_0 \in X$ be an asymptotic center of $\{x_n\}$, and let $\{x_{n_k}\}$ be an arbitrary subsequence of $\{x_n\}$ with its asymptotic center w. We show $w = x_0$. Let $\lambda_0 = \inf_{n\in\mathbb{N}} \lambda_n > 0$. By (3.1), we have

$$\begin{aligned}
d(u, x_{n_k-1})^2 - d(u, x_{n_k})^2 &\ge 2\lambda_{n_k}(f_{n_k}(x_{n_k}) - f_{n_k}(u)) + d(x_{n_k}, x_{n_k-1})^2 \\
&\ge 2\lambda_0(f_{n_k}(x_{n_k}) - f_{n_k}(u)) + d(x_{n_k}, x_{n_k-1})^2 \\
&\ge 2\lambda_0(f_{n_k}(x_{n_k}) - f(u)),
\end{aligned}$$

and thus

$$\frac{1}{2\lambda_0}\left(d(u, x_{n_k-1})^2 - d(u, x_{n_k})^2\right) \ge f(x_{n_k}) - f(u).$$

Letting $k \to \infty$, we have $0 \ge \limsup_{k\to\infty} f(x_{n_k}) - f(u)$ and it follows from Lemma 2.1 that

$$f(w) \le \limsup_{k\to\infty} f(x_{n_k}) \le f(u) = \inf_{y\in X} f(y).$$

Thus we have $w \in \operatorname{argmin}_X f$. We also have $w \in \operatorname{Fix} T$. Indeed, we have

$$\begin{aligned}\limsup_{k\to\infty} d(x_{n_k}, Tw) &\le \limsup_{k\to\infty}(d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k}, Tw))\\ &\le \limsup_{k\to\infty} d(x_{n_k}, Tx_{n_k}) + \limsup_{k\to\infty} d(Tx_{n_k}, Tw)\\ &\le \limsup_{k\to\infty} d(x_{n_k}, Tx_{n_k}) + \limsup_{k\to\infty} d(x_{n_k}, w)\\ &= \limsup_{k\to\infty} d(x_{n_k}, w).\end{aligned}$$

It shows that Tw is also an asymptotic center of $\{x_{n_k}\}$. Since the asymptotic center of a bounded sequence is unique, we get $Tw = w$ and thus $w \in \operatorname{argmin}_X f \cap \operatorname{Fix} T$. Then it follows that

$$\begin{aligned}\limsup_{k\to\infty} d(x_{n_k}, x_0) &\le \limsup_{n\to\infty} d(x_n, x_0)\\ &\le \lim_{n\to\infty} d(x_n, w) = c_w = \lim_{k\to\infty} d(x_{n_k}, w).\end{aligned}$$

This shows that x_0 is also an asymptotic center of $\{x_{n_k}\}$, and therefore $x_0 = w \in \operatorname{argmin}_X f \cap \operatorname{Fix} T$. Since every subsequence of $\{x_n\}$ has the identical asymptotic center x_0, $\{x_n\}$ is Δ-convergent to $x_0 \in \operatorname{argmin}_X f \cap \operatorname{Fix} T$, which is the desired result. □

Suppose T is the identity mapping on X. In that case, $\operatorname{Fix} T = X$, and each X_n in Theorem 3.3 coincides with X. Thus, the sequence in Theorem 3.3 is reduced to the proximal point algorithm, whose Δ-convergence was proved by Bačák [2].

As mentioned in the previous section, we know that the resolvent R_{i_C} of the indicator function i_C for a nonempty closed convex subset C of X coincides with the metric projection $P_C \colon X \to C$. Using this fact, we get the following result, which was obtained in [12].

Theorem 3.2 (Kimura [12]). *Suppose that X is a Hadamard space and a subset $\{z \in X \mid d(z, y) \le d(z, x)\}$ is convex for any $x, y \in X$. Let $T \colon X \to X$ a nonexpansive mapping with $\operatorname{Fix} T \ne \emptyset$. Generate a sequence $\{x_n\} \subset X$ as follows: $x_1 \in X$ is given, $C_1 = X$, and*

$$\begin{aligned}X_n &= \{z \in X \mid d(z, Tx_n) \le d(z, x_n)\},\\ C_{n+1} &= C_n \cap X_n,\\ x_{n+1} &= P_{C_{n+1}} x_n\end{aligned}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ is Δ-convergent to $x_0 \in \operatorname{Fix} T$.

At the end of this section, we consider the case that the constraint set is the set of common fixed points of finitely many nonexpansive mappings.

For two nonexpansive mappings S_1 and S_2 of X into itself such that $\operatorname{Fix} S_1 \cap \operatorname{Fix} S_2 \ne \emptyset$, let $T \colon X \to X$ by

$$Tx = tS_1 \oplus (1-t)S_2$$

for $x \in X$, where $t \in \,]0, 1[$. Then, we may prove the following in the same way in the case of Hilbert spaces: T is also nonexpansive and $\operatorname{Fix} T = \operatorname{Fix} S_1 \cap \operatorname{Fix} S_2$.

Let $\{S_1, S_2, \dots, S_m\}$ be a finite family of nonexpansive mappings of X into itself, and suppose that $\bigcap_{k=1}^m \operatorname{Fix} S_k \neq \emptyset$. Define a mapping $T\colon X \to X$ by the following way: let $T_1 = S_1$,

$$T_{k+1} = \frac{1}{k} S_{k+1} \oplus \left(1 - \frac{1}{k}\right) T_k$$

for $k = 1, 2, \dots, m-1$, and $T = T_m$. Then, from the fact mentioned above, we have T is nonexpansive and

$$\begin{aligned}
\operatorname{Fix} T = \operatorname{Fix} T_m &= \operatorname{Fix} S_m \cap \operatorname{Fix} T_{m-1} \\
&= \operatorname{Fix} S_m \cap \operatorname{Fix} S_{m-1} \cap \operatorname{Fix} T_{m-2} \\
&= \operatorname{Fix} S_m \cap \operatorname{Fix} S_{m-1} \cap \operatorname{Fix} S_{m-2} \cap \operatorname{Fix} T_{m-3} \\
&= \cdots \\
&= \bigcap_{k=1}^m \operatorname{Fix} S_k.
\end{aligned}$$

Using this mapping T with Theorem 3.3, we obtain an iterative sequence $\{x_n\}$ which is Δ-convergent to $x_0 \in \operatorname{argmin} f \cap \bigcap_{k=1}^m \operatorname{Fix} S_k$.

We can also use the balanced mapping [7] to define a nonexpansive mapping U such that the set $\operatorname{Fix} U$ of its fixed points coincides with $\bigcap_{k=1}^m \operatorname{Fix} S_k$. For a finite family of nonexpansive mappings $\{S_1, S_2, \dots, S_m\}$ with $\bigcap_{k=1}^m \operatorname{Fix} S_k \neq \emptyset$, their balanced mapping $U\colon X \to X$ is defined by

$$Ux = \operatorname*{argmin}_{y \in X} \frac{1}{m} \sum_{k=1}^m d(y, S_k x)^2$$

for $x \in X$. Then, we know from [7] that U is also a nonexpansive mapping and $\operatorname{Fix} U = \bigcap_{k=1}^m \operatorname{Fix} S_k$.

We remark that if the underlying space is a nonempty closed convex subset of a Hilbert space, then both mappings coincide with each other, and can be expressed by a convex combination of given mappings $\{S_1, S_2, \dots, S_m\}$, that is,

$$T = U = \frac{1}{m} \sum_{k=1}^m S_k.$$

Consequently, we obtain the following new result in the setting of Hilbert spaces.

Theorem 3.3. *Let H is a Hilbert space and let $f\colon H \to \,]-\infty, \infty]$ be a proper lower semicontinuous convex function. Let $\{S_1, S_2, \dots, S_m\}$ be a finite family of nonexpansive mappings on H, and let*

$$T = \frac{1}{m} \sum_{k=1}^m S_k.$$

Let $\{\lambda_n\} \subset \,]0, \infty[$ be a positive real sequence such that $\inf_{n \in \mathbb{N}} \lambda_n > 0$. Suppose that $\operatorname{argmin}_H f \cap \bigcap_{k=1}^m \operatorname{Fix} S_k \neq \emptyset$. Generate a sequence $\{x_n\} \subset H$ as follows: $x_1 \in H$ is given, $f_1 = f$, and

$$\begin{aligned}
X_n &= \{z \in H \mid \|z - Tx_n\| \le \|z - x_n\|\}, \\
f_{n+1} &= f_n + i_{X_n},
\end{aligned}$$

$$x_{n+1} = R_{\lambda_{n+1} f_{n+1}} x_n$$

for $n \in \mathbb{N}$. *Then* $\{x_n\}$ *is* Δ*-convergent to* $x_0 \in \operatorname{argmin}_H f \cap \bigcap_{k=1}^m \operatorname{Fix} S_k$.

References

[1] M. Bačák, *Convex analysis and optimization in Hadamard spaces*, De Gruyter Series in Nonlinear Analysis and Applications, vol. 22, De Gruyter, Berlin, 2014.

[2] M. Bačák, *The proximal point algorithm in metric spaces*, Israel J. Math. **194** (2013), 689–701.

[3] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.

[4] F. Deutsch and I. Yamada, *Minimizing certain convex functions over the intersection of the fixed point sets of nonexpansive mappings*, Numer. Funct. Anal. Optim. **19** (1998), 33–56.

[5] M. Dilshad, D. Filal, S. Chandok, and M. Akram, *Iterative approximate solutions for variational problems in Hadamard manifold*, Axioms **11** (2022): 352.

[6] R. E. Gomory, *Outline of an algorithm for integer solutions to linear programs*, Bull. Amer. Math. Soc. **64** (1958), 275–278.

[7] T. Hasegawa and Y. Kimura, *Convergence to a fixed point of a balanced mapping by the Mann algorithm in a Hadamard space*, Linear Nonlinear Anal. **4** (2018), 405–412.

[8] J. Jost, *Convex functionals and generalized harmonic maps into spaces of nonpositive curvature*, Comment. Math. Helv. **70** (1995), 659–673.

[9] J. E. Kelley, Jr., *The cutting-plane method for solving convex programs*, J. Soc. Indust. Appl. Math. **8** (1960), 703–712.

[10] S. Khatoon, W. Cholamjiak, and I. Uddin, *A modified proximal point algorithm involving nearly asymptotically quasi-nonexpansive mappings*, J. Inequal. Appl. **2021** (2021): Paper No. 83.

[11] S. Khatoon, I. Uddin, and M. Basarir, *A modified proximal point algorithm for a nearly asymptotically quasi-nonexpansive mapping with an application*, Comput. Appl. Math. **40** (2021): Paper No. 250.

[12] Y. Kimura, *Comparison of convergence theorems for a complete geodesic space*, in: RIMS Kôkyûroku (Study on Nonlinear Analysis and Convex Analysis), no. 2240, Kyoto University, 2023, pp. 122–128.

[13] P.-E. Maingé, *Convex minimization over the fixed point set of demicontractive mappings*, Positivity **12** (2008), 269–280.

[14] U. F. Mayer, *Gradient flows on nonpositively curved metric spaces and harmonic maps*, Comm. Anal. Geom. **6** (1998), 199–253.

[15] W. Singh and S. Chandok, *Mann-type extragradient algorithm for solving variational inequality and fixed point problems*, Comput. Appl. Math. **43** (2024): Paper No. 259.

[16] W. Singh and S. Chandok, *A modified form of inertial viscosity projection methods for variational inequality and fixed point problems*, J. Math. **2024** (2024): Art. ID 9509788.

[17] A. Tada and W. Takahashi, *Strong convergence theorem for an equilibrium problem and a nonexpansive mapping*, in: Proceedings of the Fourth International Conference on Nonlinear Analysis and Convex Analysis (Tokyo, Japan) W. Takahashi and T. Tanaka (eds.), Yokohama Publishers, 2007, pp. 609–617.

[18] W. Takahashi, Y. Takeuchi, and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), 276–286.

[19] I. Uddin, J. J. Nieto, and J. Ali, *One-step iteration scheme for multivalued nonexpansive mappings in CAT(0) spaces*, Mediterr. J. Math. **13** (2016), 1211–1225.

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