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COMMON FIXED POINT THEOREMS OF RAN-REURINGS TYPE IN b-METRIC SPACES

LILIANA GURAN AND MONICA-FELICIA BOTA

Abstract. The purpose of this paper is to give new common fixed point theorems of Ćirić type operators on complete b-metric space endowed with a partial order relation. Our results extend and generalize the results of Ran and Reurings and some other recent results in the literature. An example and an application are stated to sustain the main results.

1. Introduction and preliminaries

The multivalued extension of the well-known Banach-Cacciopoli contraction principle was established by Nadler in 1969, [19]. In the following year, Covitz and Nadler proved that this contraction principle remains valid even without the assumption that the values of the multivalued operator are bounded. Since then, numerous extensions of this principle have been developed in various directions.

The concept of metric space has many generalizations. One of them, which is quite essential for applications, is that of b-metric space. This notion can also be found in the literature as quasi-metric space. The contraction principle, a cornerstone of fixed point theory, has been extended to b-metric spaces too. This extension allows for the establishment of fixed point theorems in settings where traditional metrics may not apply. For instance, Czerwik's work on contraction mappings ([10]) in b-metric spaces has paved the way for further research in this area.

Research has shown that fixed point results can be derived in b-metric spaces, which are applicable to both single-valued and multivalued mappings. This has implications for solving equations and optimization problems in various fields, including economics and engineering. Further generalizations, such as strong partial b-metric spaces, have been introduced, enhancing the utility of b-metric spaces in fixed point theory. These generalizations facilitate the exploration of more complex mathematical structures and their properties.

We begin this section with the definition of the b-metric space.

Definition 1.1 (Bakhtin [3], Berinde [4], Czerwik [10]). Consider Ω be a nonempty set and let $b \geq 1$ be a given real number. The function $\widehat{\mathrm{d}} : \Omega \times \Omega \to \mathbb{R}_+$ is called a b-metric if and only if for all $\rho, \omega, \vartheta \in \Omega$ the following assertions hold:

(1) $\widehat{\mathrm{d}}(\rho, \omega) = 0$ if and only if $\rho = \omega$;
(2) $\widehat{\mathrm{d}}(\rho, \omega) = \widehat{\mathrm{d}}(\omega, \rho)$;
(3) $\widehat{\mathrm{d}}(\rho, \vartheta) \leq b \cdot [\widehat{\mathrm{d}}(\rho, \omega) + \widehat{\mathrm{d}}(\omega, \vartheta)]$.

The pair $(\Omega, \widehat{\mathrm{d}})$ is called a b-metric space.

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The notion of b-metric space is a generalization of traditional metric spaces that relax the triangle inequality, allowing for more flexibility in various mathematical contexts. They have emerged as significant tools in fixed point theory, which is crucial for solving existence problems in analysis and applied mathematics. Thus, the difference between the two notions: metric and b-metric appears in the third axiom. There the right hand side contains the given real number $b \geq 1$. Obviously, the b-metric is a usual metric by taking $b = 1$. However, it does not require the triangle inequality to hold strictly. This flexibility enables the exploration of new types of convergence and continuity, which are essential in various mathematical analyses.

Example 1.2 ([4])**.** The space $L^p[0,1]$ (where $0 < p < 1$) of all real functions $\rho(t)$, $t \in [0,1]$ such that $\int_0^1 |\rho(t)|^p \widehat{\mathrm{d}}t < \infty$, together with the functional $\widehat{\mathrm{d}}(\rho,\omega) := (\int_0^1 |\rho(t) - \omega(t)|^p \widehat{\mathrm{d}}t)^{1/p}$, is a b-metric space. Notice that $b = 2^{1/p}$.

Example 1.3 ([4])**.** For $0 < p < 1$, the set
$l^p(\mathbb{R}) := \{(\rho_n) \subset \mathbb{R} | \sum_{n=1}^{\infty} |\rho_n|^p < \infty\}$ together with the functional
$\widehat{\mathrm{d}} : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to \mathbb{R}$, $\widehat{\mathrm{d}}(\rho,\omega) := (\sum_{n=1}^{\infty} |\rho_n - \omega_n|^p)^{1/p}$, is a b-metric space with coefficient $b = 2^{1/p} > 1$. Notice that the above result holds for the general case $l^p(\Omega)$ with $0 < p < 1$, where Ω is a Banach space.

The classical notions of mathematical analysis are similar in this new context. For the next notions and related ones see Berinde [4], Czerwik [10], Miculescu-Mihail [18].

Let $(\Omega, \widehat{\mathrm{d}})$ be a metric space. We will use the following standard notations:
$\mathcal{P}(\Omega)$ - *the set of all nonempty subsets of* Ω;
$\mathcal{P}_{cl}(\Omega)$ - *the set of all nonempty closed subsets of* Ω;
$\mathcal{P}_{cp}(\Omega)$ - *the set of all nonempty compact subsets of* Ω;
$Fix(\Phi) := \{\rho \in \Omega \mid \rho \in \Phi(\rho)\}$ - *the set of the fixed points of* Φ;
$SFix(\Phi) := \{\rho \in \Omega \mid \{\rho\} = \Phi(\rho)\}$ - *the set of the strict fixed points of* Φ.
Denote by $\mathbb{N}$ be the set of all natural numbers and by $\mathbb{N}^* := \mathbb{N} \cup \{0\}$.
Let us introduce the following generalized functionals on a b-metric space $(\Omega, \widehat{\mathrm{d}})$.

The gap functional:

$$(1)\ \ \widehat{\mathcal{D}} : \mathcal{M}(\Omega) \times \mathcal{P}(\Omega) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$\widehat{\mathcal{D}}(L,P) = \begin{cases} \inf\{\widehat{\mathrm{d}}(\ell,\wp) |\ \ell \in L,\ \wp \in P\}, & L \neq \emptyset \neq P \\ 0, & L = \emptyset = P \\ +\infty, & \text{otherwise.} \end{cases}$$

In particular, if $\rho_0 \in \Omega$ then $\widehat{\mathcal{D}}(\rho_0, P) := \widehat{\mathcal{D}}(\{\rho_0\}, P)$.

The excess generalized functional:

$$(2)\ \ \alpha : \mathcal{M}(\Omega) \times \mathcal{M}(\Omega) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$\alpha(L,P) = \begin{cases} \sup\{\widehat{\mathcal{D}}(\ell,P) |\ \ell \in L\}, & L \neq \emptyset \neq P \\ 0, & L = \emptyset \\ +\infty, & P = \emptyset \neq L. \end{cases}$$

Pompeiu-Hausdorff generalized functional:

$$(3)\ \ \mathrm{H} : \mathcal{M}(\Omega) \times \mathcal{M}(\Omega) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$\mathrm{H}(L, P) = \begin{cases} \max\{\alpha(L, P), \alpha(P, L)\}, & L \neq \emptyset \neq P \\ 0, & L = \emptyset = P \\ +\infty, & \text{othewise.} \end{cases}$$

δ **functional:**

$$(4)\ \ \delta : \mathcal{M}(\Omega) \times \mathcal{M}(\Omega) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$\delta(L, P) = \begin{cases} \sup\{\widehat{\mathrm{d}}(\ell, \wp) |\ \ell \in L,\ \wp \in P\}, & L \neq \emptyset \neq P \\ 0, & L = \emptyset = P \\ +\infty, & \text{otherwise.} \end{cases}$$

If $L = P$ we have $\delta(L, L) := \delta(L)$.

It is known (Czerwik [9]) that $(\mathcal{M}_{\wp,\zeta_l}(\Omega), \mathrm{H})$ is a complete b-metric space provided $(\Omega, \widehat{\mathrm{d}})$ is a complete b-metric space.

The following lemmas are useful in the proof of main results (see [9]).

Lemma 1.4. *Let $(\Omega, \widehat{\mathrm{d}})$ be a b-metric space, $L \in \mathcal{M}(\Omega)$ and $\rho \in \Omega$. Then*

(i) $\widehat{\mathcal{D}}(\rho, L) = 0$ *if and only if* $\rho \in \overline{L}$,

(ii) $\widehat{\mathcal{D}}(\rho, L) \leq \sigma[\widehat{\mathrm{d}}(\rho, \omega) + \widehat{\mathcal{D}}(\omega, L)]$, *for all* $\rho, \omega \in \Omega$ *and* $L \subset \Omega$,

(iii) $\widehat{\mathrm{d}}(\rho_n, \rho_0) \leq \sigma\widehat{\mathrm{d}}(\rho_0, \rho_1) + \cdots + \sigma^{n-1}\widehat{\mathrm{d}}(\rho_{n-2}, \rho_{n-1}) + \sigma^{n-1}\widehat{\mathrm{d}}(\rho_{n-1}, \rho_n)$.

(iv) $\mathrm{H}(L, Z) \leq \sigma[\mathrm{H}(L, P) + \mathrm{H}(P, Z)]$, *for all* $L, P, Z \in \mathcal{M}(\Omega)$.

Lemma 1.5. *Let $(\Omega, \widehat{\mathrm{d}})$ be a b-metric space and $L, P \in M(\Omega)$. For each $\upsilon > 1$ and for all $\ell \in L$ there exists $\wp \in P$ such that*

$$\widehat{\mathrm{d}}(\ell, \wp) \leq \upsilon\, \mathrm{H}(L, P).$$

Definition 1.6. Let $(\Omega, \widehat{\mathrm{d}})$ be a b-metric space. Then a sequence $(\rho_n)_{n\in\mathbb{N}}$ in Ω is called:

(i) Cauchy if and only if for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for each $n, m \geq n(\varepsilon)$ we have $\widehat{\mathrm{d}}(\rho_n, \rho_m) < \varepsilon$.

(ii) convergent if and only if there exists $\rho \in \Omega$ such that for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$ we have $\widehat{\mathrm{d}}(\rho_n, \rho) < \varepsilon$. In this case we write $\lim\limits_{n\to\infty} \rho_n = \rho$.

Definition 1.7. Let $(\Omega, \widehat{\mathrm{d}})$ be a b-metric space. Then, a subset $\overline{\mathbb{Y}}$ of Ω is called:

(i) compact if and only if for every sequence of elements of $\overline{\mathbb{Y}}$ there exists a subsequence that converges to an element of $\overline{\mathbb{Y}}$.

(ii) closed if and only if for each sequence $(\rho_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{Y}}$ which converges to an element ρ, we have $\rho \in \overline{\mathbb{Y}}$

The b-metric space $(\Omega, \widehat{\mathrm{d}})$ is complete if every Cauchy sequence from Ω converges in Ω.

Lemma 1.8. *Notice that in a b-metric space $(\Omega, \widehat{\mathrm{d}})$ the following assertions hold:*

(i) *a convergent sequence has a unique limit;*

(ii) *each convergent sequence is Cauchy;*

In general, a b-metric is not continuous and the open ball $B(\rho_0; r) := \{\rho \in \Omega : \widehat{\mathrm{d}}(\rho_0, \rho) < r\}$ in a b-metric space $(\Omega, \widehat{\mathrm{d}})$ is not necessary an open set, while the closed ball $\tilde{B}(\rho_0; r) := \{\rho \in \Omega : \widehat{\mathrm{d}}(\rho_0, \rho) \leq r\}$ is not necessary a closed set. Related notions, results and examples can be found in [2,16] and [18].

Let Ω be a nonempty set, $\preceq$ be a partial order relation on Ω and $\widehat{\mathrm{d}}$ be a complete b-metric on Ω with the constant $b \geq 1$. Then, the triple $(\Omega, \widehat{\mathrm{d}}, \preceq)$ is said to be:

(1) i-regular if for any increasing sequence $(\rho_n)_{n\in\mathbb{N}}$ which is convergent to ρ^* as $n \to \infty$, we have that $\rho_n \preceq \rho^*$, for all $n \in \mathbb{N}$;
(2) d-regular if for any decreasing sequence $(\rho_n)_{n\in\mathbb{N}}$ which is convergent to ρ^* as $n \to \infty$, we have that $\rho_n \succeq \rho^*$, for all $n \in \mathbb{N}$.

Common fixed points are a significant concept in mathematical analysis, particularly in the study of mappings in metric spaces. When considering two or more mappings, a common fixed point is a point that serves as a fixed point for all the mappings involved. Various theorems have been developed to establish conditions involved, such as continuity and commutativity of the mappings, under which common fixed points exist for two or more mappings (see [1,7,11,26]). Common fixed point results are used in various fields, including differential equations, optimization problems, and even in the analysis of iterative methods used in numerical computations. The existence of common fixed points often leads to solutions of complex mathematical problems by ensuring that certain iterative processes converge to a stable solution.

The purpose of this paper is to give new common fixed point for Ćirić type operators where $(\Omega, \preceq, \widehat{\mathrm{d}})$ is a complete b-metric space endowed with a partial order relation. Our results extend and generalize Ran-Reurings results and some other results in the recent literature too.(see [6,20,21,29,32], etc.). Moreover, we give an example and an application to strength the main theorems of our paper.

2. Common fixed points results

Starting with theorem of Ran and Reurings (see [35]) a new research direction in the field of fixed point theory is given. The authors considered the following fixed point inclusion $\rho \in \Phi(\rho), \rho \in \Omega$, where the set Ω is endowed with a partial order relation $\preceq$ and the metric $\widehat{\mathrm{d}}$ is a complete metric. The function $\Phi : \Omega \to \mathcal{P}(\Omega)$ satisfies the contraction condition only for comparable elements (with respect to $\preceq$) of the space. For other results on this topic (see [36]).

For the convenience of the reader let us recall the following theorem.

Theorem 2.1 ([35]). *Let $(\Omega, \preceq)$ be a partially ordered set such that every pair $\rho, \omega \in \Omega$ has a lower bound and an upper bound. Furthermore, let $\widehat{\mathrm{d}}$ be a metric on Ω such that $(\Omega, \widehat{\mathrm{d}})$ is a complete metric space. Suppose that $\Phi : \Omega \to \Omega$ is a continuous and monotone (i.e., either increasing or decreasing) operator, for which there exists $c \in (0,1)$ such that the following conditions are satisfied:*

$$\widehat{\mathrm{d}}(\Phi(\rho), \Phi(\omega)) \leq c\widehat{\mathrm{d}}(\rho, \omega), \text{ for every } \rho \succeq \omega, \tag{2.1}$$

$$\text{there exists } \rho_0 \in \Omega \text{ such that } \rho_0 \preceq \Phi(\rho_0) \text{ or } \rho_0 \succeq \Phi(\rho_0). \tag{2.2}$$

Then Φ is a Picard operator, i.e., Φ has a unique fixed point $\rho^* \in \Omega$ and $\lim_{n\to\infty} \Phi^n(\rho) = \rho^*$, for every $\rho \in \Omega$.

Next, we recall the following lemma.

Lemma 2.2 ([18]). *Every sequence $(\rho_n)_{n\in\mathbb{N}}$ of elements from a b-metric space $(\Omega, \widehat{\mathrm{d}})$ with constant $b \geq 1$ having the property that there exists $\gamma \in [0,1)$ such that $\widehat{\mathrm{d}}(\rho_{n+1}, \rho_n) \leq \gamma\widehat{\mathrm{d}}(\rho_n, \rho_{n-1})$, $n \in \mathbb{N}$ is a Cauchy sequence. Moreover, the following estimation holds*

$$\widehat{\mathrm{d}}(\rho_{n+1}, \rho_{n+p}) \leq \frac{\gamma^n B}{1-\gamma}\widehat{\mathrm{d}}(\rho_0, \rho_1), \text{ for all } n, p \in \mathbb{N},$$

where $B := \sum_{i=1}^{\infty} \gamma^{2i\log_\gamma b + 2^{i-1}}$.

We give our first main result, a Ran-Reurings type theorem for Ćirić type operators.

Theorem 2.3. *Let Ω be a nonempty set, let $\preceq$ be a partial order on Ω and let $\widehat{\mathrm{d}}$ be a complete b-metric on Ω with the constant $b \geq 1$. Let $\Psi, \Phi : \Omega \to \mathcal{P}_{cl}(\Omega)$ be two increasing multivalued operators with respect to $\preceq$, for which there exists $q \in (0, \frac{1}{b})$ such that:*

(i) *there is $\rho_0, \rho_1 \in \rho$ with $\rho_0 \preceq \Psi(\rho_0)$, respectively $\rho_1 \preceq \Phi(\rho_1)$;*
(ii) *Ψ and Φ have closed graph with respect to $\widehat{\mathrm{d}}$ or the space $(\Omega, \widehat{\mathrm{d}}, \preceq)$ is i-regular;*
(iii) *$\mathrm{H}(\Psi(\rho), \Phi(\omega)) \leq q \max\{\widehat{\mathrm{d}}(\rho, \omega), \widehat{\mathcal{D}}(\rho, \Psi(\rho)), \widehat{\mathcal{D}}(\omega, \Phi(\omega)), \frac{1}{2}(\widehat{\mathcal{D}}(\rho, \Phi(\omega)) + \widehat{\mathcal{D}}(\omega, \Psi(\rho)))\}$ for all $\rho, \omega \in \Omega$ with $\rho \preceq \omega$.*

Then the mappings Ψ and Φ have a common fixed point.

Proof. Let $\rho_0, \rho_1 \in \Omega$ with $\rho_0 \preceq \Psi(\rho_0)$ and $\rho_1 \preceq \Phi(\rho_1)$. If $\rho_0 = \rho_1$ we obtain the conclusion. Let $\rho_0 \neq \rho_1$ and $\rho_1 \notin \Phi(\rho_1)$ we define a sequence $\{\rho_n\}$ as follows

$$\rho_{2n+1} \in \Psi(\rho_{2n}) \text{ and } \rho_{2n+2} \in \Phi(\rho_{2n+1}),\ n = 0, 1, 2, \dots. \tag{2.3}$$

By the monotonicity of the mappings Ψ and Φ, we get that $(\rho_n)_{n\in\mathbb{N}}$ is increasing. For $\rho_1 \in \Psi(\rho_0)$, by the properties of the functional H there exists $p > 1$ and $\rho_2 \in \Phi(\rho_1)$ such that:

$$\begin{aligned}
&\widehat{\mathrm{d}}(\rho_1, \rho_2) \leq p\mathrm{H}(\Psi(\rho_0), \Phi(\rho_1)) \leq pq \max\{\widehat{\mathrm{d}}(\rho_0, \rho_1), \widehat{\mathcal{D}}(\rho_0, \Psi(\rho_0)), \widehat{\mathcal{D}}(\rho_1, \Phi(\rho_1)),\\
&\frac{1}{2}(\widehat{\mathcal{D}}(\rho_0, \Phi(\rho_1) + \widehat{\mathcal{D}}(\rho_1, \Psi(\rho_0)))))\}\\
&\quad \leq pq \max\left\{\widehat{\mathrm{d}}(\rho_0, \rho_1), \widehat{\mathrm{d}}(\rho_0, \rho_1), \widehat{\mathrm{d}}(\rho_1, \rho_2), \frac{1}{2}(\widehat{\mathrm{d}}(\rho_0, \rho_2) + \widehat{\mathrm{d}}(\rho_1, \rho_1)))\right\}\\
&\quad \leq pq \max\left\{\widehat{\mathrm{d}}(\rho_0, \rho_1), \widehat{\mathrm{d}}(\rho_1, \rho_2), \frac{b}{2}(\widehat{\mathrm{d}}(\rho_0, \rho_1) + \widehat{\mathrm{d}}(\rho_1, \rho_2)))\right\}.
\end{aligned}$$

Further we shall prove that $(\rho_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Then for $p \in (1, b) \cup [b, \infty)$ we have $pq \in (0, 1) \cup [1, \infty)$.

(I) If $pq \in (0, 1)$ we have the following cases:

Case 1. If $\max\{\widehat{\mathrm{d}}(\rho_0, \rho_1), \widehat{\mathrm{d}}(\rho_1, \rho_2), \frac{b}{2}(\widehat{\mathrm{d}}(\rho_0, \rho_1) + \widehat{\mathrm{d}}(\rho_1, \rho_2)))\} = \widehat{\mathrm{d}}(\rho_1, \rho_2)$, then we have:

$$\widehat{\mathrm{d}}(\rho_1, \rho_2) \leq pq \cdot \widehat{\mathrm{d}}(\rho_1, \rho_2),$$

which contradicts $pq \in (0,1)$.

Case 2. If $\max\{\widehat{\mathrm{d}}(\rho_0,\rho_1), \widehat{\mathrm{d}}(\rho_1,\rho_2), \frac{b}{2}(\widehat{\mathrm{d}}(\rho_0,\rho_1) + \widehat{\mathrm{d}}(\rho_1,\rho_2)))\} = \widehat{\mathrm{d}}(\rho_0,\rho_1)$, then we have:

$$\widehat{\mathrm{d}}(\rho_1,\rho_2) \leq pq \cdot \widehat{\mathrm{d}}(\rho_0,\rho_1),$$

By Lemma 2.2 we get that the sequence $(\rho_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case 3. If $\max\{\widehat{\mathrm{d}}(\rho_0,\rho_1), \widehat{\mathrm{d}}(\rho_1,\rho_2), \frac{b}{2}(\widehat{\mathrm{d}}(\rho_0,\rho_1) + \widehat{\mathrm{d}}(\rho_1,\rho_2)))\} = \frac{b}{2}(\widehat{\mathrm{d}}(\rho_0,\rho_1) + \widehat{\mathrm{d}}(\rho_1,\rho_2))$, then we obtain:

$$\widehat{\mathrm{d}}(\rho_1,\rho_2) \leq \frac{qb}{2}(\widehat{\mathrm{d}}(\rho_0,\rho_1) + \widehat{\mathrm{d}}(\rho_1,\rho_2))$$

which is equivalent to

$$\widehat{\mathrm{d}}(\rho_1,\rho_2) \leq \frac{qb}{2-qb}\widehat{\mathrm{d}}(\rho_0,\rho_1).$$

Hence we have:

$$\widehat{\mathrm{d}}(\rho_1,\rho_2) \leq \gamma\widehat{\mathrm{d}}(\rho_0,\rho_1),$$

where $\gamma = \max\{q, \frac{qb}{2-qb}\} < 1$.

By induction we get that

$$\widehat{\mathrm{d}}(\rho_n,\rho_{n+1}) \leq \gamma\widehat{\mathrm{d}}(\rho_n,\rho_{n-1}), \forall n \in \mathbb{N}^*.$$

Applying Lemma 2.2 we obtain that the sequence $(\rho_n)_{n\in\mathbb{N}}$ is Cauchy.

(II) If $pq \in [1,\infty)$ we have the following cases.

Case 1. If $\max\{\widehat{\mathrm{d}}(\rho_0,\rho_1), \widehat{\mathrm{d}}(\rho_1,\rho_2), \frac{b}{2}(\widehat{\mathrm{d}}(\rho_0,\rho_1) + \widehat{\mathrm{d}}(\rho_1,\rho_2)))\} = \widehat{\mathrm{d}}(\rho_1,\rho_2)$, then we have:

$$\widehat{\mathrm{d}}(\rho_1,\rho_2) \leq pq \cdot \widehat{\mathrm{d}}(\rho_1,\rho_2).$$

If we let $p \to 1$ we get $q \geq 1$. Contradiction.

Case 2. If $\max\{\widehat{\mathrm{d}}(\rho_0,\rho_1), \widehat{\mathrm{d}}(\rho_1,\rho_2), \frac{b}{2}(\widehat{\mathrm{d}}(\rho_0,\rho_1) + \widehat{\mathrm{d}}(\rho_1,\rho_2)))\} = \widehat{\mathrm{d}}(\rho_0,\rho_1)$, then we have:

$$\widehat{\mathrm{d}}(\rho_1,\rho_2) \leq pq \cdot \widehat{\mathrm{d}}(\rho_0,\rho_1).$$

In this way we find that $\widehat{\mathrm{d}}(\rho_n,\rho_{n+1}) \leq (pq)^n\widehat{\mathrm{d}}(\rho_0,\rho_1)$.

For $m,n \in \mathbb{N}$, with $m > n$, we have

$$\begin{aligned}
\widehat{\mathrm{d}}(\rho_n,\rho_m) &\leq b[\widehat{\mathrm{d}}(\rho_n,\rho_{n+1}) + \widehat{\mathrm{d}}(\rho_{n+1},\rho_m)] \\
&\leq b(\widehat{\mathrm{d}}(\rho_n,\rho_{n+1})) + b^2[\widehat{\mathrm{d}}(\rho_{n+1},\rho_{n+2}) + \widehat{\mathrm{d}}(\rho_{n+2},\rho_m] \\
&\leq b(\widehat{\mathrm{d}}(\rho_n,\rho_{n+1})) + b^2(\widehat{\mathrm{d}}(\rho_{n+1},\rho_{n+2})) + b^3(\widehat{\mathrm{d}}(\rho_{n+2},\rho_{n+3})) \\
&\quad + \cdots + b^{m-n-1}(\widehat{\mathrm{d}}(\rho_{m-2},\rho_{m-1})) + b^{m-n}(\widehat{\mathrm{d}}(\rho_{m-1},\rho_m).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\widehat{\mathrm{d}}(\rho_n,\rho_m) &\leq b(pq)^n(\widehat{\mathrm{d}}(\rho_0,\rho_1)) + b^2(pq)^{n+1}(\widehat{\mathrm{d}}(\rho_0,\rho_1)) + b^3(pq)^{n+2}(\widehat{\mathrm{d}}(\rho_0,\rho_1)) \\
&\quad + \cdots + b^{m-n-1}(pq)^{m-2}(\widehat{\mathrm{d}}(\rho_0,\rho_1)) + b^{m-n}(pq)^{m-1}(\widehat{\mathrm{d}}(\rho_0,\rho_1)) \\
&= \sum_{i=1}^{m-n} b^i(pq)^{i+n-1}(\widehat{\mathrm{d}}(\rho_0,\rho_1)).
\end{aligned}$$

Therefore,

$$\begin{aligned}\widehat{\mathrm{d}}(\rho_n,\rho_m) &\leq \sum_{i=1}^{m-n} b^{i+n-1}(pq)^{i+n-1}\widehat{\mathrm{d}}(\rho_0,\rho_1) = \sum_{t=n}^{m-1} b^t(pq)^t\widehat{\mathrm{d}}(\rho_0,\rho_1)\\ &\leq \sum_{t=n}^{\infty}(bpq)^t\widehat{\mathrm{d}}(\rho_0,\rho_1) = \frac{(bpq)^n}{1-bpq}\widehat{\mathrm{d}}(\rho_0,\rho_1).\end{aligned}$$

If we let $p \to 1$ and $n \to \infty$ we obtain

$$\widehat{\mathrm{d}}(\rho_n,\rho_m) \leq \frac{(bq)^n}{1-bq}\widehat{\mathrm{d}}(\rho_0,\rho_1) \to 0.$$

Therefore $\{\rho_n\}$ is a Cauchy sequence in Ω.

Case 3. If $\max\{\widehat{\mathrm{d}}(\rho_0,\rho_1),\widehat{\mathrm{d}}(\rho_1,\rho_2),\frac{b}{2}(\widehat{\mathrm{d}}(\rho_0,\rho_1)+\widehat{\mathrm{d}}(\rho_1,\rho_2)))\} = \frac{b}{2}(\widehat{\mathrm{d}}(\rho_0,\rho_1)+\widehat{\mathrm{d}}(\rho_1,\rho_2))$, then we obtain:

$$\widehat{\mathrm{d}}(\rho_1,\rho_2) \leq \frac{pqb}{2}(\widehat{\mathrm{d}}(\rho_0,\rho_1)+\widehat{\mathrm{d}}(\rho_1,\rho_2))$$

which is equivalent to

$$\widehat{\mathrm{d}}(\rho_1,\rho_2) \leq \frac{pqb}{2-pqb}\widehat{\mathrm{d}}(\rho_0,\rho_1).$$

Hence we have:

$$\widehat{\mathrm{d}}(\rho_1,\rho_2) \leq \gamma\widehat{\mathrm{d}}(\rho_0,\rho_1),$$

where $\gamma = \max\{q, \frac{pqb}{2-pqb}\} < 1$.

By induction we get that

$$\widehat{\mathrm{d}}(\rho_n,\rho_{n+1}) \leq \gamma\widehat{\mathrm{d}}(\rho_n,\rho_{n-1}), \forall n \in \mathbb{N}^*.$$

For $m,n \in \mathbb{N}$, with $m > n$, we have

$$\begin{aligned}\widehat{\mathrm{d}}(\rho_n,\rho_m) &\leq b[\widehat{\mathrm{d}}(\rho_n,\rho_{n+1})+\widehat{\mathrm{d}}(\rho_{n+1},\rho_m)]\\ &\leq b(\widehat{\mathrm{d}}(\rho_n,\rho_{n+1}))+b^2[\widehat{\mathrm{d}}(\rho_{n+1},\rho_{n+2})+\widehat{\mathrm{d}}(\rho_{n+2},\rho_m]\\ &\leq b(\widehat{\mathrm{d}}(\rho_n,\rho_{n+1}))+b^2(\widehat{\mathrm{d}}(\rho_{n+1},\rho_{n+2}))+b^3(\widehat{\mathrm{d}}(\rho_{n+2},\rho_{n+3}))\\ &\quad+\cdots+b^{m-n-1}(\widehat{\mathrm{d}}(\rho_{m-2},\rho_{m-1}))+b^{m-n}(\widehat{\mathrm{d}}(\rho_{m-1},\rho_m).\end{aligned}$$

Moreover, we have

$$\begin{aligned}\widehat{\mathrm{d}}(\rho_n,\rho_m) &\leq b\gamma^n(\widehat{\mathrm{d}}(\rho_0,\rho_1))+b^2\gamma^{n+1}(\widehat{\mathrm{d}}(\rho_0,\rho_1))+b^3\gamma^{n+2}(\widehat{\mathrm{d}}(\rho_0,\rho_1))\\ &\quad+\cdots+b^{m-n-1}\gamma^{m-2}(\widehat{\mathrm{d}}(\rho_0,\rho_1))+b^{m-n}\gamma^{m-1}(\widehat{\mathrm{d}}(\rho_0,\rho_1))\\ &= \sum_{i=1}^{m-n} b^i\gamma^{i+n-1}(\widehat{\mathrm{d}}(\rho_0,\rho_1)).\end{aligned}$$

Therefore,

$$\widehat{\mathrm{d}}(\rho_n,\rho_m) \leq \sum_{i=1}^{m-n} b^{i+n-1}\gamma^{i+n-1}\widehat{\mathrm{d}}(\rho_0,\rho_1) = \sum_{t=n}^{m-1} b^t\gamma^t\widehat{\mathrm{d}}(\rho_0,\rho_1)$$

$$\leq \sum_{t=n}^{\infty}(b\gamma)^t\widehat{\mathrm{d}}(\rho_0,\rho_1)=\frac{(b\gamma)^n}{1-b\gamma}\widehat{\mathrm{d}}(\rho_0,\rho_1).$$

If we let $n\to\infty$ we obtain $\widehat{\mathrm{d}}(\rho_n,\rho_m)\to 0$.

Therefore $\{\rho_n\}$ is a Cauchy sequence in Ω.

Since $(\rho_n)_{n\in\mathbb{N}}$ is Cauchy in all the presented cases, by the completeness of the $b-$metric space, we have that the sequence converges in $(\Omega,\widehat{\mathrm{d}})$ to an element $\rho^*(\rho_0)\in\Omega$. We prove that $\rho^*(\rho_0)$ is a common fixed point for the mappings Ψ and Φ. For convenience we will write ρ^* instead of $\rho^*(\rho_0)$.

If Ψ and Φ have closed graph then the conclusion is obvious.

If the space $(\Omega,\widehat{\mathrm{d}},\preceq)$ is $i-$regular, for operator Ψ we have

$$\begin{aligned}
\widehat{\mathcal{D}}(\rho^*,\Psi(\rho^*)) &\leq b\widehat{\mathcal{D}}(\rho^*,\Psi(\rho_{2n}))+b\mathrm{H}(\Psi(\rho_{2n}),\Psi(\rho^*))\\
&\leq b\widehat{\mathcal{D}}(\rho^*,\Psi(\rho_{2n}))+bq\max\{\widehat{\mathrm{d}}(\rho_{2n},\rho^*),\widehat{\mathcal{D}}(\rho_{2n},\Psi(\rho_{2n})),\widehat{\mathcal{D}}(\rho^*,\Psi(\rho^*)),\\
&\frac{1}{2}(\widehat{\mathcal{D}}(\rho_{2n},\Psi(\rho^*))+\widehat{\mathcal{D}}(\rho^*,\Psi(\rho_{2n})))\}\\
&\leq b\widehat{\mathrm{d}}(\rho^*,\rho_{2n+1})+pbq\max\Big\{\widehat{\mathrm{d}}(\rho_{2n},\rho^*),\widehat{\mathrm{d}}(\rho_{2n},\rho_{2n+1}),\widehat{\mathcal{D}}(\rho^*,\Psi(\rho^*)),\\
&\qquad\frac{b}{2}(\widehat{\mathrm{d}}(\rho_{2n},\rho^*)+\widehat{\mathcal{D}}(\rho^*,\Psi(\rho^*))),\frac{1}{2}\widehat{\mathrm{d}}(\rho^*,\rho_{2n+1})\Big\}\\
&= b\widehat{\mathrm{d}}(\rho^*,\rho_{2n+1})+pbq\max\Big\{\widehat{\mathrm{d}}(\rho_{2n},\rho^*),\widehat{\mathrm{d}}(\rho_{2n},\rho_{2n+1}),\widehat{\mathcal{D}}(\rho^*,\Psi(\rho^*)),\\
&\qquad\frac{b}{2}(\widehat{\mathrm{d}}(\rho_{2n},\rho^*)+\widehat{\mathcal{D}}(\rho^*,\Psi(\rho^*))),\frac{1}{2}\widehat{\mathrm{d}}(\rho^*,\rho_{2n+1})\Big\}\\
&\leq b\widehat{\mathrm{d}}(\rho^*,\rho_{2n+1})+pbq\max\Big\{b\widehat{\mathrm{d}}(\rho_{2n},\rho^*),\widehat{\mathrm{d}}(\rho_{2n},\rho_{2n+1}),b\widehat{\mathcal{D}}(\rho^*,\Psi(\rho^*)),\\
&\qquad\frac{1}{2}\widehat{\mathrm{d}}(\rho^*,\rho_{2n+1})\Big\}.
\end{aligned}$$

By letting $n\to\infty$, we get

$$\widehat{\mathcal{D}}(\rho^*,S(\rho^*))\leq bq\widehat{\mathcal{D}}\widehat{\mathrm{d}}(\rho^*,\Psi(\rho^*)),$$

which implies that $\widehat{\mathcal{D}}(\rho^*,\Psi(\rho^*))=0$. Since Ψ is closed $\rho^*\in\Psi(\rho^*)$.

In the same way we prove that ρ^* is a fixed point for Φ. Then we have:

$$\begin{aligned}
\widehat{\mathcal{D}}(\rho^*,\Phi(\rho^*)) &\leq b\widehat{\mathcal{D}}(\rho^*,\Phi(\rho_{2n+1}))+b\mathrm{H}(\Phi(\rho_{2n+1}),\Psi(\rho^*))\\
&\leq b\widehat{\mathcal{D}}(\rho^*,\Phi(\rho_{2n+1}))+bq\max\Big\{\widehat{\mathrm{d}}(\rho_{2n+1},\rho^*),\widehat{\mathcal{D}}(\rho_{2n+1},\Phi(\rho_{2n+1})),\widehat{\mathcal{D}}(\rho^*,\Phi(\rho^*)),\\
&\frac{1}{2}(\widehat{\mathcal{D}}(\rho_{2n+1},\Phi(\rho^*))+\widehat{\mathcal{D}}(\rho^*,\Phi(\rho_{2n+1})))\Big\}\\
&\leq b\widehat{\mathrm{d}}(\rho^*,\rho_{2n+2})+pbq\max\Big\{\widehat{\mathrm{d}}(\rho_{2n+1},\rho^*),\widehat{\mathrm{d}}(\rho_{2n+1},\rho_{2n+2}),\widehat{\mathcal{D}}(\rho^*,\Phi(\rho^*)),\\
&\qquad\frac{b}{2}(\widehat{\mathrm{d}}(\rho_{2n+1},\rho^*)+\widehat{\mathcal{D}}(\rho^*,\Phi(\rho^*))),\frac{1}{2}\widehat{\mathrm{d}}(\rho^*,\rho_{2n+2})\Big\}\\
&= b\widehat{\mathrm{d}}(\rho^*,\rho_{2n+2})+bq\max\Big\{\widehat{\mathrm{d}}(\rho_{2n+1},\rho^*),\widehat{\mathrm{d}}(\rho_{2n+1},\rho_{2n+2}),\widehat{\mathcal{D}}(\rho^*,\Phi(\rho^*)),
\end{aligned}$$

$$\frac{b}{2}(\widehat{\mathrm{d}}(\rho_{2n+1},\rho^*)+\widehat{\mathcal{D}}(\rho^*,\Phi(\rho^*))),\frac{1}{2}\widehat{\mathrm{d}}(\rho^*,\rho_{2n+2})\Big\}$$
$$\leq b\widehat{\mathrm{d}}(\rho^*,\rho_{2n+2})+pbq\max\Big\{b\widehat{\mathrm{d}}(\rho_{2n+1},\rho^*),\widehat{\mathrm{d}}(\rho_{2n+1},\rho_{2n+2}),$$
$$b\widehat{\mathrm{d}}(\rho^*,\Phi(\rho^*)),\frac{1}{2}\widehat{\mathrm{d}}(\rho^*,\rho_{2n+2})\Big\}.$$

Also, if $n\to\infty$, we get

$$\widehat{\mathcal{D}}(\rho^*,\Phi(\rho^*))\leq pbq\widehat{\mathcal{D}}(\rho^*,\Phi(\rho^*)),$$

which implies that $\widehat{\mathcal{D}}(\rho^*,\Phi(\rho^*))=0$. Since Φ is closed we get $\rho^*\in\Phi(\rho^*)$.

Then ρ^* is a common fixed point for the mappings Ψ and Φ. □

Remark 2.4. A similar result we can be obtained if we replace the condition $\rho_0\preceq\Phi(\rho_0)$ with $\Phi(\rho_0)\preceq\rho_0$, respectively $\rho_0\preceq\Psi(\rho_0)$ with $\Psi(\rho_0)\preceq\rho_0$ and the i-regularity of the space $(\Omega,\widehat{\mathrm{d}})$ with its d-regularity.

Corollary 2.5. *If all the conditions of Theorem 2.3 are satisfied then the common fixed point of the multivalued operators Ψ and Φ is unique.*

Proof. We assume that $\omega^*\in\Omega$ is another common fixed point for the mappings Ψ and Φ. Then we get

$$\widehat{\mathrm{d}}(\rho^*,\omega^*)\leq p\mathrm{H}(\Psi\rho^*,\Phi\omega^*)$$
$$\leq pq\max\Big\{\widehat{\mathrm{d}}(\rho^*,\omega^*),\widehat{\mathcal{D}}(\rho^*,\Psi\rho^*),\widehat{\mathcal{D}}(\omega^*,\Phi\omega^*),\frac{1}{2}(\widehat{\mathcal{D}}(\rho^*,\Phi\omega^*)+\widehat{\mathcal{D}}(\omega^*,\Psi\rho^*))\Big\}$$
$$\leq pq\max\Big\{\widehat{\mathrm{d}}(\rho^*,\omega^*),\widehat{\mathrm{d}}(\rho^*,\rho^*),\widehat{\mathrm{d}}(\omega^*,\omega^*),\frac{1}{2}(\widehat{\mathrm{d}}(\rho^*,\omega^*)+\widehat{\mathrm{d}}(\omega^*,\rho^*))\Big\}.$$
$$\leq pq\widehat{\mathrm{d}}(\rho^*,\omega^*).$$

This implies that $\rho^*=\omega^*$, which completes the proof. □

Remark 2.6. One can obtain a similar result by replacing the condition $\rho_0\preceq\Psi(\rho_0)$ with $\Psi(\rho_0)\preceq\rho_0$ and the i-regularity of the space with its d-regularity.

Example 2.7. Let $\Omega=[0,\infty)$ and let us define the order relation as follows:

$$\rho\leq\omega \text{ if and only if } \rho\leq\Phi\rho \text{ and } \omega\leq\Psi\omega.$$

Let us also define the b-metric $\widehat{\mathrm{d}}:\Omega\times\Omega\to R_+$ by $\widehat{\mathrm{d}}(\rho,\omega)=(\rho-\omega)^2$. We can notice that the constant from the definition of the b-metric is $b=2$.

It is obvious that $(\Omega,\widehat{\mathrm{d}},\leq)$ is a complete b-metric space.

Let us consider the following operators $\Psi,\Phi:\Omega\to P_{cl}\Omega$ defined by $\Psi\rho=\{\frac{1}{2}\rho\}$ and $\Phi\rho=\{\frac{1}{4}\rho\}$.

Next we must check the Ćirić-type contraction condition

$$\widehat{\mathrm{d}}(\rho,\omega)\leq p\mathrm{H}(\Phi\rho,\Psi\omega)$$
$$\leq pq\max\Big\{\widehat{\mathrm{d}}(\rho,\omega),\widehat{\mathcal{D}}(\rho,\Psi\rho),\widehat{\mathcal{D}}(\omega,\Phi\omega),\frac{1}{2}(\widehat{\mathcal{D}}(\rho,\Phi\omega)+\widehat{\mathcal{D}}(\omega,\Psi\rho))\Big\},$$

for all $\rho,\omega\in\rho$ with $\rho\leq\omega$.

For $q < \frac{1}{2}$ let $p \in (1, 2)$. For $\rho \le \omega$ we have:

$$\widehat{\mathrm{d}}\Big(\frac{1}{2}\rho, \frac{1}{4}\omega\Big) \le pq \max\Big\{\widehat{\mathrm{d}}(\rho,\omega), \widehat{\mathrm{d}}\Big(\rho, \frac{1}{2}\rho\Big), \widehat{\mathrm{d}}\Big(\omega, \frac{1}{4}\omega\Big), \frac{1}{2}\Big(\widehat{\mathrm{d}}\Big(\rho, \frac{1}{4}\omega\Big) + \widehat{\mathrm{d}}\Big(\omega, \frac{1}{2}\rho\Big)\Big)\Big\}.$$

Using the b-metric we obtain:

$$\Big(\frac{1}{2}\rho - \frac{1}{4}\omega\Big)^2 \le pq \max\Big\{(\rho-\omega)^2, \Big(\rho - \frac{1}{2}\rho\Big)^2, \Big(\omega - \frac{1}{4}\omega\Big)^2,$$
$$\frac{1}{2}\Big(\widehat{\mathrm{d}}\Big(\rho - \frac{1}{4}\omega\Big)^2 + \Big(\omega - \frac{1}{2}\rho\Big)^2\Big)\Big\}.$$

After computations we get:

$$\frac{1}{16}\omega^2 \le pq \max\Big\{0, \frac{1}{4}\omega^2, \frac{1}{4}\omega^2, \frac{13}{32}\omega^2\Big\}.$$

Then $\frac{1}{16} \le pq\frac{13}{32}$. This yields for $pq \ge \frac{2}{13}$.

Since all the hypothesis of Corollary 2.5 are accomplished we get that $\Psi(0) = \Phi(0) = 0$ is the unique fixed point of Ψ and Φ.

If we get Ψ=Φ in Theorem 2.3 we give the following Ran-Reurings type fixed points theorem for Ćirić type operators.

Theorem 2.8. *Let Ω be a nonempty set, $\preceq$ be a partial order on Ω and $\widehat{\mathrm{d}}$ be a complete b-metric on Ω with the constant $b \ge 1$. Let $\Phi : \Omega \to \mathcal{P}_{cl}(\Omega)$ be an increasing mapping with respect to $\preceq$, for which there exists $q \in (0, \frac{1}{b})$ such that:*

(i) *there is $\rho_0 \in \Omega$ with $\rho_0 \preceq \Phi(\rho_0)$;*

(ii) *Φ has closed graph with respect to $\widehat{\mathrm{d}}$ or the space $(\Omega, \widehat{\mathrm{d}}, \preceq)$ is i-regular;*

(iii) $\mathrm{H}(\Phi\rho, \Phi\omega) \le q\max\{\widehat{\mathrm{d}}(\rho,\omega), \widehat{\mathcal{D}}(\rho,\Phi\rho), \widehat{\mathcal{D}}(\omega,\Phi\omega), \frac{1}{2}(\widehat{\mathcal{D}}(\rho,\Phi\omega) + \widehat{\mathcal{D}}(\omega,\Phi\rho))\}$ *for all $\rho, \omega \in \Omega$ with $\rho \preceq \omega$.*

Then the multivalued operator Φ has a unique fixed point.

Remark 2.9. A similar result holds if we replace the condition $\rho_0 \preceq \Phi(\rho_0)$ with $\Phi(\rho_0) \preceq \rho_0$ and the i-regularity of the space with its d-regularity.

Let us give the singlevalued version of the previous result.

Corollary 2.10. *Let Ω be a nonempty set, $\preceq$ be a partial order on Ω and $\widehat{\mathrm{d}}$ be a complete b-metric space on Ω with the constant $b \ge 1$. Let $\Phi : \Omega \to \Omega$ be an increasing mapping with respect to $\preceq$, for which there exists $q \in (0, \frac{1}{b})$ such that*

(i) *there exists $\rho_0 \in \Omega$ with $\rho_0 \preceq \Phi(\rho_0)$;*

(ii) *Φ have closed graph with respect to $\widehat{\mathrm{d}}$ or the space $(\Omega, \widehat{\mathrm{d}}, \preceq)$ is i-regular;*

(iii) $\widehat{\mathrm{d}}(\Phi(\rho), \Phi(\omega)) \le q\max\{\widehat{\mathrm{d}}(\rho,\omega), \widehat{\mathrm{d}}(\rho,\Phi\rho), \widehat{\mathrm{d}}(\omega,\Phi\omega), \frac{1}{2}(\widehat{\mathrm{d}}(\rho,\Phi\omega) + \widehat{\mathrm{d}}(\omega,\Phi\rho))\}$, *for all $\rho, \omega \in \Omega$ with $\rho \preceq \omega$.*

Then the mapping Φ has a unique fixed point.

Another fixed point result for multivalued operators case is the following.

Theorem 2.11. *Let Ω be a nonempty set, $\preceq$ be a partial order on Ω and $\widehat{\mathrm{d}}$ be a complete b-metric on Ω with the constant $b \ge 1$. Let $\Phi : \Omega \to \mathcal{P}_{cl}(\Omega)$ be an increasing multivalued operator with respect to $\preceq$, for which there exists $q \in (0, \frac{1}{b})$ such that:*

(i) *there is* $\rho_0 \in \Omega$ *with* $\rho_0 \preceq \Phi(\rho_0)$*;*

(ii) Φ *has closed graph with respect to* $\widehat{\mathrm{d}}$ *or the space* $(\rho, \widehat{\mathrm{d}}, \preceq)$ *is i-regular;*

(iii) $\mathrm{H}(\Phi^n\Omega, \Phi^n\omega)$
$\leq q\max\{\widehat{\mathrm{d}}(\rho,\omega), \widehat{\mathcal{D}}(\rho, \Phi^n\rho), \widehat{\mathcal{D}}(\omega, \Phi^n\omega), \frac{1}{2}(\widehat{\mathcal{D}}(\rho, \Phi^n\omega) + \widehat{\mathcal{D}}(\omega, \Phi^n\rho))\}$
for all $\rho, \omega \in \Omega$ *with* $\rho \preceq \omega$*.*

Then the operator Φ *has a unique fixed point.*

Proof. By Theorem 2.8 we obtain $\rho^* \in \Omega$ such that

$$\rho^* \in \Phi^n\rho^*.$$

There exists p¿1 such that:

$$\begin{aligned}
\widehat{\mathcal{D}}(\Phi\rho^*, \rho^*) &\leq p\mathrm{H}(\Phi(\Phi^n(\rho^*)), \Phi^n(\rho^*)) = p\mathrm{H}(\Phi^n(\Phi(\rho^*)), \Phi^n(\rho^*)) \\
&\leq pq\max\Big\{\widehat{\mathcal{D}}(\Phi(\rho^*), \rho^*), \mathrm{H}(\Phi(\rho^*), \Phi^n(\Phi(\rho^*))), \widehat{\mathcal{D}}(\rho^*, \Phi^n(\rho^*)), \\
&\qquad \frac{1}{2}(\mathrm{H}(\Phi(\rho^*), \Phi^n(\rho^*)) + \widehat{\mathcal{D}}(\rho^*, \Phi^n(\Phi(\rho^*))))\Big\} \\
&\leq pq\max\Big\{\widehat{\mathcal{D}}(\Phi\rho^*, \rho^*), \mathrm{H}(\Phi(\rho^*), \Phi(\rho^*)), \widehat{\mathrm{d}}(\rho^*, \rho^*), \\
&\qquad \frac{1}{2}(\widehat{\mathcal{D}}(\Phi(\rho^*), \rho^*) + \widehat{\mathcal{D}}(\rho^*, \Phi(\rho^*)))\Big\} \\
&\leq pq\widehat{\mathcal{D}}(\Phi\rho^*, \rho^*).
\end{aligned}$$

Then $\Phi^n(\rho^*) \in \Phi(\rho^*)$ and $\rho^* \in \Phi(\rho^*)$. □

The next result represents the relation between the fixed point and strict fixed point sets. Also the well-posedness of the fixed point problem with respect to the functionals $\widehat{\mathcal{D}}$, respectively H will be given.

Theorem 2.12. *Let* $(\Omega, \widehat{\mathrm{d}})$ *be a complete b-metric space with constant* $b > 1$*. Suppose that all the hypotheses of Theorem 2.8. Suppose also that* $SFix(\Phi) \neq \emptyset$ *then we have:*

(i) $Fix(\Phi) = SFix(\Phi) = \{\rho^*\}$*.*

(ii) *(Well-posedness of the fixed point problem with respect to* $\widehat{\mathcal{D}}$*)*
If $(\rho_n)_{n\in(N)}$ *is a sequence in* Ω *such that* $\widehat{\mathcal{D}}(\rho_n, \Phi(\rho_n)) \to 0$ *as* $n \to \infty$*, then* $\rho_n \overset{\widehat{\mathrm{d}}}{\to} \rho^*$ *as* $n \to \infty$*.*

(iii) *(Well-posedness of the fixed point problem with respect to* H*)*
If $(\rho_n)_{n\in(N)}$ *is a sequence in* Ω *such that* $\mathrm{H}(\rho_n, \Phi(\rho_n)) \to 0$ *as* $n \to \infty$*, then* $\rho_n \overset{\widehat{\mathrm{d}}}{\to} \rho^*$ *as* $n \to \infty$*.*

Proof. (ii) From Theorem 2.8, we get that $Fix(\Phi) \neq \emptyset$. Let $\rho^* \in SFix(\Phi)$. Notice first that $SFix(\Phi) = \{\rho^*\}$. We will prove that $Fix(\Phi) = \{\rho^*\}$. Let $\omega \in Fix(\Phi)$, i.e. $\omega \in \Phi(\omega)$ with $\omega \preceq \Phi(\omega)$ and $\omega \neq \rho^*$. We estimate the following distance

$$\begin{aligned}
&\widehat{\mathrm{d}}(\rho^*, \omega) = \widehat{\mathcal{D}}(\Phi(\rho^*), \omega) \leq \mathrm{H}(\Phi(\rho^*), \Phi(\omega)) \\
&\quad \leq q\max\Big\{\widehat{\mathrm{d}}(\rho^*, \omega), \widehat{\mathcal{D}}(\rho^*, \Phi(\rho^*)), \widehat{\mathcal{D}}(\omega, \Phi(\omega)), \frac{1}{2}(\widehat{\mathcal{D}}(\rho^*, \Phi(\omega)) + \widehat{\mathcal{D}}(\omega, \Phi(\rho*)))\Big\}
\end{aligned}$$

$$\leq q\widehat{\mathrm{d}}(\rho^*, \omega).$$

Contradiction. Then $\widehat{\mathrm{d}}(\rho^*, \omega) = 0$, so $\rho^* = \omega$. Hence $Fix(\Phi) \subset SFix(\Phi)$. Since $SFix(\Phi) \subset Fix(\Phi)$ we obtain that $SFix(\Phi) = Fix(\Phi)$.

The uniqueness condition for the strict fixed point can be proved using same method.

(ii) Let $\rho^* \in SFix(\Phi)$ and let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence in Ω such that $\widehat{\mathcal{D}}(\rho_n, \Phi(\rho_n)) \to 0$, as $n \to \infty$. Then for $v_n \in \Phi(\rho_n)$ with $v_n \preceq \Phi(\rho_n)$, such that $\widehat{\mathrm{d}}(\rho_n, v_n) = \widehat{\mathcal{D}}(\rho_n, \Phi(\rho_n))$, $n \in \mathbb{N}$, we have:

$$\begin{aligned}
\widehat{\mathrm{d}}(\rho_n, \rho^*) &\leq b[\widehat{\mathrm{d}}(\rho_n, v_n) + \widehat{\mathrm{d}}(v_n, \rho^*)] = b[\widehat{\mathrm{d}}(\rho_n, v_n) + \widehat{\mathcal{D}}(v_n, \Phi(\rho^*))] \\
&\leq b[\widehat{\mathrm{d}}(\rho_n, v_n) + \mathrm{H}(\Phi(\rho_n), \Phi(\rho^*))] \\
&\leq b\Big[\widehat{\mathcal{D}}(\rho_n, \Phi(\rho_n)) + q\max\Big\{\widehat{\mathrm{d}}(\rho_n, \rho^*), \widehat{\mathcal{D}}(\rho_n, \Phi(\rho_n)), \widehat{\mathcal{D}}(\rho^*, \Phi(\rho^*)), \\
&\qquad \frac{1}{2}(\widehat{\mathcal{D}}(\rho^*, \Phi(\rho_n)) + \widehat{\mathcal{D}}(\rho_n, \Phi(\rho^*)))\Big\}\Big] \\
&\leq b[\widehat{\mathcal{D}}(\rho_n, \Phi(\rho_n)) + q\widehat{\mathrm{d}}(\rho_n, \rho^*)].
\end{aligned}$$

Then we obtain $\widehat{\mathrm{d}}(\rho_n, \rho^*) \leq \frac{b}{1-bq}\widehat{\mathcal{D}}(\rho_n, \Phi(\rho_n)) \to 0$ as $n \to \infty$.

Then $\widehat{\mathrm{d}}(\rho_n, \rho^*) \to 0$ as $n \to \infty$ which means the fixed point problem is well posed with respect to $\widehat{\mathcal{D}}$.

(iii) Let $\rho^* \in SFix(\Phi)$ and let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence in Ω such that $\mathrm{H}(\rho_n, \Phi(\rho_n)) \to 0$, as $n \to \infty$. Then for $v_n \in \Phi(\rho_n)$ with $v_n \preceq \Phi(\rho_n)$, such that $\widehat{\mathrm{d}}(\rho_n, v_n) = p\mathrm{H}(\rho_n, \Phi(\rho_n))$, for $p > 1$ and $n \in \mathbb{N}$, we have:

$$\begin{aligned}
\widehat{\mathrm{d}}(\rho_n, \rho^*) &\leq b[\widehat{\mathrm{d}}(\rho_n, v_n) + \widehat{\mathrm{d}}(v_n, \rho^*)] = b[\widehat{\mathrm{d}}(\rho_n, v_n) + \widehat{\mathcal{D}}(v_n, \Phi(\rho^*))] \\
&\leq b[pH(\rho_n, \Phi(v_n)) + \mathrm{H}(\Phi(\rho_n), \Phi(\rho^*))] \\
&\leq b\Big[p\mathrm{H}(\rho_n, \Phi(\rho_n)) + q\max\Big\{\widehat{\mathrm{d}}(\rho_n, \rho^*), \widehat{\mathcal{D}}(\rho_n, \Phi(\rho_n)), \widehat{\mathcal{D}}(\rho^*, \Phi(\rho^*)), \\
&\qquad \frac{1}{2}(\widehat{\mathcal{D}}(\rho^*, \Phi(\rho_n)) + \widehat{\mathcal{D}}(\rho_n, \Phi(\rho^*)))\Big\}\Big] \\
&\leq b[p\mathrm{H}(\rho_n, \Phi(\rho_n)) + q\widehat{\mathrm{d}}(\rho_n, \rho^*)].
\end{aligned}$$

Then we obtain $\widehat{\mathrm{d}}(\rho_n, \rho^*) \leq \frac{bp}{1-qb}\mathrm{H}(\rho_n, \Phi(\rho_n)) \to 0$ as $n \to \infty$.

Then $\widehat{\mathrm{d}}(\rho_n, \rho^*) \to 0$ as $n \to \infty$ which means the fixed point problem is well posed with respect to H. □

3. Applications in integral type contractions

Further, let us give some applications of our results in integral type contractions. Then, we recall next the definition for altering distance function.

Definition 3.1. The function $\varphi : [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following assertions hold:

(i) φ is continuous and nondecreasing,
(ii) $\varphi(\mathrm{t}) = 0$ if and only if $\mathrm{t} = 0$.

Let us give the following definition.

Definition 3.2. Let $\mathfrak{F}$ be the set of the function $f : [0, \infty) \to [0, \infty)$ that satisfies the following conditions:

(i) f is Lebesgue Integrable on each compact subset of $[0, \infty)$;
(ii) $\int_0^{\varepsilon} f(\mathrm{t})d\mathrm{t} > 0$ for every $\varepsilon > 0$.

Remark 3.3. It is obvious that a mapping $\Psi : [0, \infty) \to [0, \infty)$ given by

$$\Psi(\mathrm{t}) = \int_0^{\varepsilon} f(\mathrm{t})d\mathrm{t} > 0,$$

is an altering distance function.

Our first integral type theorem is the following.

Theorem 3.4. *Let Ω be a nonempty set, $\preceq$ be a partial order on Ω, and $\widehat{\mathrm{d}}$ be a complete b-metric on Ω with the constant $b \geq 1$. Let $\Psi, \Phi : \Omega \to P_{cl}(\Omega)$ be two increasing mappings with respect to $\preceq$, for which there exists $q \in (0, \frac{1}{b})$ such that:*

(i) *there is $\rho_0, \rho_1 \in \Omega$ with $\rho_0 \preceq \Psi(\rho_0)$, respectively $\rho_1 \preceq \Phi(\rho_1)$;*
(ii) *Ψ and Φ have closed graph with respect to $\widehat{\mathrm{d}}$ or the space $(\Omega, \widehat{\mathrm{d}}, \preceq)$ is i-regular;*
(iii) *$\int_0^{\widehat{\mathrm{d}}(\Psi\rho,\Phi\omega)} f(\mathrm{t})dt \leq q \int_0^{\mathcal{M}(\rho,\omega)} f(\mathrm{t})d\mathrm{t}$, for all $\rho, \omega \in \Omega$ with $\rho \preceq \omega$, $0 < q < 1$ and $f \in \mathfrak{F}$ with*

$$\mathcal{M}(\rho, \omega) = \max\{\widehat{\mathrm{d}}(\rho, \omega), \widehat{\mathcal{D}}(\rho, \Psi\rho), \widehat{\mathcal{D}}(\omega, \Phi\omega), \frac{1}{2}(\widehat{\mathcal{D}}(\rho, \Phi\omega) + \widehat{\mathcal{D}}(\omega, \Psi\rho))\}.$$

Then the mappings Ψ and Φ have a unique common fixed point.

Proof. Using Theorem 2.3 for $\Psi(\mathrm{t}) = \int_0^{\mathrm{t}} f(u)du$ we get the conclusion. □

For $\Psi = \Phi$ we get another fixed point of integral type result.

Theorem 3.5. *Let Ω be a nonempty set, $\preceq$ be a partial order on Ω, and $\widehat{\mathrm{d}}$ be a complete b-metric on v with the constant $b \geq 1$. Let $\Psi, \Phi : \Omega \to \mathcal{P}_{cl}(\Omega)$ be two increasing mappings with respect to $\preceq$, for which there exists $q \in (0, \frac{1}{b})$ such that:*

(i) *there is $\rho_0 \in \Omega$ with $\rho_0 \preceq \Phi(\rho_0)$;*
(ii) *Φ have closed graph with respect to $\widehat{\mathrm{d}}$ or the space $(\Omega, \widehat{\mathrm{d}}, \preceq)$ is i-regular;*
(iii) *$\int_0^{\widehat{\mathrm{d}}(\Phi\rho,\Phi\omega)} f(\mathrm{t})d\mathrm{t} \leq q \int_0^{\mathcal{N}(\rho,\omega)} f(\mathrm{t})d\mathrm{t}$, for all $\rho, \omega \in \Omega$ with $\rho \preceq \omega$, $0 < q < 1$ and $f \in \mathfrak{F}$ with*

$$\mathcal{N}(\rho, \omega) = \max\left\{\widehat{\mathrm{d}}(\rho, \omega), \widehat{\widehat{\mathcal{D}}}(\rho, \Phi\rho), \widehat{\mathcal{D}}(\omega, \Phi\omega), \frac{1}{2}(\widehat{\mathcal{D}}(\rho, \Phi\omega) + \widehat{\mathcal{D}}(\omega, \Phi\rho))\right\}.$$

Then Φ have a unique fixed point.

Proof. Using Theorem 2.8 for $\Psi(\mathrm{t}) = \int_0^{\mathrm{t}} f(u)du$ we get the conclusion. □

Remark 3.6. Similar results with the previous integral type fixed point results, Theorem 3.4, respectively Theorem 3.5, can be obtain if we replace the condition $\rho_0 \preceq \Phi(\rho_0)$ with $\Phi(\rho_0) \preceq \rho_0$, respectively $\rho_0 \preceq \Psi(\rho_0)$ with $\Psi(\rho_0) \preceq \rho_0$, and the space $(\Omega, \widehat{\mathrm{d}}, \preceq)$ is d-regular.

4. Existence result of a solution of an integral equation

In this section we will give an application of our fixed points results to integral equations. then, let us consider the following integral equation.

$$\rho(\mathrm{t}) = \int_{\ell}^{\mathrm{t}} \mathcal{K}(\mathrm{t}, s, \rho(s))ds + g(\mathrm{t}),\ \mathrm{t} \in [\ell, \wp]. \tag{4.1}$$

The aim of this section is to give an existence result for the equation (4.1), applying Corollary 2.10.

Theorem 4.1. *Let us consider the integral equation (4.1). Assume the following conditions are satisfied:*

(i) $\mathcal{K} : [\ell, \wp] \times [\ell, \wp] \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $g : [\ell, \wp] \to \mathbb{R}^n$ *are continuous.*

(ii) $\mathcal{K}(\mathrm{t}, s, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ *is increasing for each* $\mathrm{t}, s \in [\ell, \wp]$

(iii) *there exists* $q \in (0, \frac{1}{b})$ *such that*

$$\mathcal{K}(\mathrm{t}, s, u) - \mathcal{K}(\mathrm{t}, s, v) \leq q\mathcal{Q}(\ell, \wp),$$

where $\mathcal{Q}(\ell, \wp) = \max\{|\ell - \wp|, |\ell - \Phi\ell|, |\wp - \Phi\wp|, \frac{1}{2}(|\ell - \Phi\wp| + |\wp - \Phi\ell|)\}$ *for each* $\mathrm{t}, s \in [\ell, \wp]$, $u, v \in \mathbb{R}^n$, $u \preceq v$.

(iv) *there exists* $\rho_0 \in C([\ell, \wp], \mathbb{R}^n)$ *such that* $\rho_0 \leq \int_{\ell}^{\mathrm{t}} \mathcal{K}(\mathrm{t}, s, \rho(s))ds + g(\mathrm{t})$ *for any* $\mathrm{t} \in [\ell, \wp]$.

Then the integral equation (4.1) has a unique solution in $C([\ell, \wp], \mathbb{R}^n)$.

Proof. Let $\rho = C([\ell, \wp], \mathbb{R}^n)$ be a b-metric space endowed with the b-metric

$$\widehat{\mathrm{d}}(\rho, \omega) = \| \rho - \omega \|_{\infty} = \sup_{\mathrm{t} \in [\ell, \wp]} |\rho(\mathrm{t}) - \omega(\mathrm{t})|^p,$$

with $\wp = 2^{p-1}$.

Consider on Ω the partial order defined by $\rho, \omega \in C([\ell, \wp], \mathbb{R}^n)$, $\rho \leq \omega$ if and only if $\rho(\mathrm{t}) \leq \omega(\mathrm{t})$ for any $\mathrm{t} \in [\ell, \wp]$.

Then $(\Omega, \| \cdot \|_{\infty}, \leq)$ is an ordered and complete b-metric space. Then, for any increasing sequence $\{\rho_n\}$ in Ω converging to an $\rho^* \in \Omega$, we have $\rho_n(\mathrm{t}) \leq \rho^*(\mathrm{t})$, for any $\mathrm{t} \in [\ell, \wp]$. Then, the space $(\Omega, \| \cdot \|_{\infty}, \leq)$ is i-regular. Let us define the following function:

$$\Phi : C([\ell, \wp], \mathbb{R}^n) \to C([\ell, \wp], \mathbb{R}^n)$$

by the formula

$$\Phi\rho(\mathrm{t}) = \int_{\ell}^{\mathrm{t}} \mathcal{K}(\mathrm{t}, s, \rho(s))ds + g(\mathrm{t}),\ \mathrm{t} \in [\ell, \wp].$$

From (i) we get that Φ has a closed graph.

From (ii) we have that Φ is increasing. Also, for each $\Omega, \omega \in \rho$ with $\rho \leq \omega$ we get

$$\begin{aligned} |\Phi\rho(\mathrm{t}) - \Phi\omega(\mathrm{t})|^p &\leq \int_{\ell}^{\mathrm{t}} |\mathcal{K}(\mathrm{t}, s, \rho(s)) - \mathcal{K}(\mathrm{t}, s, \omega(s))|^p ds \\ &\leq q^p \int_{\ell}^{\mathrm{t}} |\mathcal{Q}(\rho(\mathrm{t}), \omega(\mathrm{t}))|^p ds \end{aligned}$$

$$\le q^p \int_{\ell}^{\mathrm{t}} |\max\{|\rho(s)-\omega(s)|, |\rho(s)-\Phi\rho(s)|, |\omega-\Phi\omega(s)|, \frac{1}{2}(|\rho-\Phi\omega(s)|+|\omega(s)-\Phi\rho(s)|)|^p\}ds$$

$$\le q^p \int_{\ell}^{\mathrm{t}} \max\{|\rho(s)-\omega(s)|^p, |\rho(s)-\Phi\rho(s)|^p, |\omega-\Phi\omega(s)|^p, \frac{1}{2}(|\rho-\Phi\omega(s)|^p+|\omega(s)-\Phi\rho(s)|^p)\}ds$$

$$\le q^p \int_{\ell}^{\mathrm{t}} \max\{\sup_{\mathrm{t}\in[\ell,\wp]} |\rho(s)-\omega(s)|^p, \sup_{\mathrm{t}\in[\ell,\wp]} |\rho(s)-\Phi\rho(s)|^p, \sup_{\mathrm{t}\in[\ell,\wp]} |\omega-\Phi\omega(s)|^p, \frac{1}{2}(\sup_{\mathrm{t}\in[\ell,\wp]} |\rho-\Phi\omega(s)|^p+\sup_{\mathrm{t}\in[\ell,\wp]} |\omega(s)-\Phi\rho(s)|^p)\}ds$$

$$\le q^p \max\{\widehat{\mathrm{d}}(\rho,\omega), \widehat{\mathrm{d}}(\rho,\Phi\rho), \widehat{\mathrm{d}}(\omega,\Phi\omega), \frac{1}{2}(\widehat{\mathrm{d}}(\rho,\Phi\omega)+\widehat{\mathrm{d}}(\omega,\Phi\rho))\}$$

$$\le q^p\mathcal{M}(\rho,\omega), \text{ for any } \mathrm{t}\in[\ell,\wp].$$

Hence for $0<\alpha=q^p<\frac{1}{b}$ we obtain:

$$\| \Phi\rho-\Phi\omega \|_\infty \le \alpha\mathcal{M}(\rho,\omega),$$

for each $\rho,\omega\in\Omega$ with $\rho\le\omega$.

From (iv) we get that $\rho_0\le\Phi\rho_0$. The conclusion follows from Corollary 2.10. □

5. Conclusions

In [6] Bota at al. gave some interesting results of fixed points theorems Ran-Reurings type in b-metric space. Our paper extend and improve these results, giving new aspects considering the case of multivalued operators. The b-metric spaces are instrumental in establishing fixed point theorems, which are crucial for proving the existence and uniqueness of solutions to equations. This is particularly relevant in applied mathematics, where such solutions are often needed in modeling and problem-solving scenarios.

Also, the fixed point results in the case of b-metric spaces are also applied in solving integral equations, where the properties of these spaces facilitate the derivation of existence and uniqueness results for solutions. This has implications in mathematical modeling and simulations in various scientific fields. In this view, we gave here an application of our result for the case of an integral equation.

As future research directions, will be interesting to find applications of our results in the field of ordinary or partial differential equations, and respectively, for the case of fractional differential equations.

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L. Guran
Department of Hospitality Sevices, Babeş-Bolyai University Cluj-Napoca, Horea Street, no. 7, 400174, Cluj-Napoca, Romania
E-mail address: liliana.guran@ubbcluj.ro

M.-F. Bota
Department of Mathematics, Babeş-Bolyai University Cluj-Napoca, Kogălniceanu Street, No. 1, 400084, Cluj-Napoca, Romania
E-mail address: monica.bota@ubbcluj.ro