Journal of Nonlinear and Convex Analysis
Volume 26, Number 4, 2025, 821–835

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

MODIFIED PARALLEL INERTIAL MONOTONE HYBRID THREE-STEP ITERATIONS FOR COMMON FIXED POINTS OF FINITE FAMILY OF G-NONEXPANSIVE MAPPINGS WITH AN APPLICATION

DAMRONGSAK YAMBANGWAI AND TANAKIT THIANWAN*

ABSTRACT. In this work, we aim to prove the convergence of the sequence generated by the modified parallel inertial monotone hybrid three-step iteration (abbreviated as MPIMHT) to find a common fixed point of a finite family of G-nonexpansive mappings in Hilbert spaces endowed with graphs. We obtain weak convergence results under some mild conditions. Furthermore, an application of the algorithm to a signal recovery problem with multiple blurring filters is presented. Consequently, numerical experiments of the proposed algorithms, defined by different types of blurred matrices on the algorithm, can show the efficiency and implementation of the LASSO problem in signal recovery.

1. Introduction

A fixed-point problem is a mathematical challenge with wide-ranging applications in real-world scenarios, such as signal recovery problems. It also holds significant importance in the field of graph theory. To begin, let's outline some foundational definitions within graph theory.

Let $\mathcal{C}$ be a nonempty subset of a real Hilbert space $\mathcal{H}$ with inner product $\langle ., . \rangle$ and the induced by norm $\|.\|$. We identify the graph G with the pair $(V(G), E(G))$, where the set $V(G)$ of its vertices coincide with set $\mathcal{C}$ and the set of edges $E(G)$ contains $\Delta = \{(x, x) : x \in \mathcal{C}\}$, where Δ denotes the diagonal of the cartesian product $\mathcal{C} \times \mathcal{C}$. Also, G is such that no two edges are parallel. A mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is said to be G-contraction if $\mathcal{T}$ preserves edges of G (or $\mathcal{T}$ is edge-preserving), i.e., $(x, y) \in E(G) \Rightarrow (\mathcal{T}x, \mathcal{T}y) \in E(G)$ and $\mathcal{T}$ decreases weights of edges of G in the following way: there exists $\alpha \in (0, 1)$ such that $(x, y) \in E(G) \Rightarrow \|\mathcal{T}x - \mathcal{T}y\| \leq \alpha\|x - y\|$. A mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is said to be G-nonexpansive (see [3], Definition 2.3 (iii)) if $\mathcal{T}$ preserves edges of G, i.e.,

$$(x, y) \in E(G) \Rightarrow (\mathcal{T}x, \mathcal{T}y) \in E(G),$$

and T non-increases weights of edges of G in the following way:

$$(x, y) \in E(G) \Rightarrow \|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|.$$

2020 *Mathematics Subject Classification.* 47H10, 65K05, 94A12.

Key words and phrases. Parallel inertial monotone hybrid iteration, G-nonexpansive mapping, Weak convergence, Signal recovery problem, Hilbert space.

*Corresponding author. This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2025, Grant No. 5074/2567).

The minimization problem of the sum of two functions is to find a solution of

$$\min_{\mathbf{x}\in\mathbb{R}^n}\{F(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})\}, \tag{1.1}$$

where $g : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is proper convex and lower semi-continuous function, and $f : \mathbb{R}^n \to \mathbb{R}$ is convex differentiable function with gradient ∇f being L-Lipschitz constant for some $L > 0$. The solution of (1.1) can be characterized by using Fermat's rule, Theorem 16.3 of Bauschke and Combettes [10] as follows: $\mathbf{x}^*$ is a minimizer of $(f+g) \Leftrightarrow 0 \in \partial g(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)$,
where ∂g is the subdifferential of g and ∇f is the gradient of f. The subdifferential of g at $\mathbf{x}^*$, denoted by $\partial g(\mathbf{x}^*)$, is defined by $\partial g(\mathbf{x}^*) := \{u : g(\mathbf{x}) - h(\mathbf{x}^*) \geq \langle u, \mathbf{x} - \mathbf{x}^*\rangle, \forall \mathbf{x}\}$. It is also well-known that the solution of (1.1) is characterized by the following fixed point problem:

$$\mathbf{x}^* \text{ is a minimizer of } (f+g) \Leftrightarrow \mathbf{x}^* = prox_{\psi g}(I - \psi\nabla f)(\mathbf{x}^*),$$

where $\psi > 0, prox_g$ is the proximity operator of h defined by $prox_g := argmin\{g(\mathbf{y}) + \frac{1}{2}\|\mathbf{x}-\mathbf{y}\|_2^2\}$, see [23] for more details. It is also known that $prox_{\psi g}(I - \psi\nabla f)$ is a nonexpansive mapping when $\psi \in \left(0, \frac{2}{L}\right)$. We denote the fixed point set of a mapping $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ by $\mathcal{F}(\mathcal{T}) = \{x \in \mathcal{H} : \mathcal{T}x = x\}$. If we set $\mathcal{T}x = prox_{\psi g}(x - \psi\nabla f(x))$, where $\psi \in \left(0, \frac{2}{L}\right)$ and L is the Lipschitz constant of the gradient of functions f, then $\mathcal{T}$ is nonexpansive. It is known that if $\mathcal{T}$ is nonexpansive, then $\mathcal{T}$ is G-nonexpansive. This is the reason that why we interested in studying G-nonexpansive mapping.

In 1922, Banach [9] proved the existence of unique fixed point for contractions in a complete metric space. The most recent version of the theorem was proved in Banach spaces endowed with a graph G, where $G = (V(G), E(G))$ is a directed graph such that the set $V(G)$ of its vertices of a graph and the set $E(G)$ of its edges contains all loops. By combination of the concepts in fixed point theory and graph theory, Banach G-contraction was introduced by Jachymaski [18] in complete metric space accompanied with the graph G where the set of vertex matches with the metric space, also see e.g. [7, 11–13, 21, 24, 26, 35].

In the last few decades investigations of fixed points by some iterative schemes for G-contraction, G-nonexpansive and G-monotone nonexpansive mappings have been studied extensively by various authors (see [1–3, 34, 36] and the references cited therein). In 2017, Sridarat et al. [27] introduced the SP-iteration process and gave some weak and strong convergence theorems of such iterations for three G-nonexpansive mappings under proper conditions. The use of a three-step iterative process results in superior numerical results compared to estimations based on two-step or one-step iterations (see [16, 17]).

Inertial extrapolation, initially introduced by Polyak [25], is an acceleration technique for convex minimization inspired by the heavy ball method. This approach, involving two iterative steps derived from previous iterates, has proven effective in enhancing the convergence rates of various iterative algorithms, especially those using projection-based methods, as confirmed by multiple studies [4, 8, 14, 22, 32, 33, 37].

In a recent study, Suantai and colleagues [30, 31], building upon the work of Anh and Hieu (references [5, 6]), introduced a convergence analysis of an algorithm that combines the shrinking projection method with the parallel monotone hybrid method. This algorithm is devised to approximate common fixed points of a finite family of G-nonexpansive mappings. Moreover, they applied this algorithm to address signal recovery in scenarios where the noise type is unknown. This research contributes to advancing methods for signal processing under uncertain conditions.

The scheme is defined as follows: $x_1 \in C, C_0 = C$,

$$\begin{cases} v_n^i = \alpha_n^i x_n + (1-\alpha_n^i)\mathcal{T}_i x_n, i = 1,2,\ldots,N, \\ i_n = argmax\{\left\|v_n^i - x_n\right\| : i = 1,2,\ldots,N\}, \overline{v}_n := v_n^{i_n}, \\ C_{n+1} = \{v \in C_n : \|v - \overline{v}_n\| \leq \|v - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n^i\} \subset [0,1]$ and $\liminf_{n\to\infty} \alpha_n^i(1-\alpha_n^i) > 0$ for all i = 1, 2, …, N. $\overline{v}_n$ is chosen by the optimization all v_n^i with x_n. After that, the closed convex set C_{n+1} was constructed by $\overline{v}_n$. Finally, the next approximation x_{n+1} is defined as the projection of x_1 on to C_{n+1}. More recently, Cholamjiak et al. [15] proposed an inertial forward-backward splitting algorithm for finding the solution of common variational inclusion problems based on the inertial technique and parallel monotone hybrid methods. They proved strong convergence results under some suitable conditions in Hilbert spaces. Here in this paper, the algorithm was very useful in image restoration. For given initial points $x_0, x_1 \in C_1 = \mathcal{H}$,

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n^i = (1-\alpha_n^i)y_n + \alpha_n^i J_{r_n}^B(I - r_n A_i)y_n, i = 1,2,\ldots,N, \\ i = argmax\{\left\|z_n^i - x_n\right\| : i = 1,2,\ldots,N\}, \overline{z}_n := z_n^i, \\ C_{n+1} = \{v \in C_n : \|\overline{z}_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 \\ \qquad - 2\theta_n \langle x_n - v, x_{n-1} - x_n\rangle\}, \\ x_{n+1} = P_{C_{n+1}} x_1, n \geq 1, \end{cases} \tag{1.3}$$

where $A_i : \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{H} \to 2^{\mathcal{H}}$ are monotone operator with $J_{r_n}^B = (I + r_n B)^{-1}$, $\{r_n\} \subset (0, 2\alpha)$, $\{\theta_n\} \subset [0,\theta]$ for some $\theta \in [0,1]$ and $\left\{\alpha_n^i\right\}$ is a sequence in $[0,1]$ for all $i = 1,2,\ldots,N$. It has been notable that if $\{r_n\} \subset (0,2\alpha)$, where α is a constant of inverse strongly monotone operator A, then the mapping $J_{r_n}^B(I - r_n A)$ is nonexpansive.

Jun-on et al. [20] proposed the inertial parallel algorithm for finding a common fixed point of a finite family of G-nonexpansive mappings as follows:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n^i = (1-\beta_n^i)w_n + \beta_n^i \mathcal{T}_i w_n, \\ z_n^i = (1-\alpha_n^i)\mathcal{T}_i w_n + \alpha_n^i \mathcal{T}_i y_n^i, \\ x_{n+1} = argmax\{\|z_n^i - w_n\|, i = 1,2,\ldots,N\}, \end{cases} \tag{1.4}$$

where $\{\theta_n\} \subset [0,\theta]$ for each $\theta \in (0,1]$ and $\{\alpha_n^i\}$, $\{\beta_n^i\} \subset [0,1]$. They proved a weak convergence theorem under some suitable conditions in the setting of Hilbert space endowed with graphs. Also, they applied the inertial parallel algorithm (1.4) for solving signal recovery problems.

This paper aims to develop a modified parallel inertial monotone hybrid three-step algorithm to approximate the common fixed points of a finite family of G-nonexpansive mappings within a Hilbert space endowed with a graph. We also apply our algorithm to a signal recovery problem with multiple blurring filters.

2. Graph basic definitions

In this section, we recall a few basic notions concerning the connectivity of graphs. All of these notions can be found, e.g., in [19].

Suppose that x and y are vertices in a graph G. A path in G from x to y of length N ($N \in \mathbb{N} \cup \{0\}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_i, x_{i+1}) \in E(G)$ for $i = 0, 1, \dots N-1$. A graph G is connected if there is a path between any two vertices. A directed graph $G = (V(G), E(G))$ is said to be transitive if, for any $x, y, z \in V(G)$ such that (x, y) and (y, z) are in $E(G)$, we have $(x, z) \in E(G)$. The set of edges $E(G)$ is said to be convex if $(x_i, y_i) \in E(G)$ for all $i = 1, 2, \dots, N$ and $\alpha_i \in (0, 1)$ such that $\sum_{i=1}^{N} \alpha_i = 1$, then $(\sum_{i=1}^{N} \alpha_i x_i, \sum_{i=1}^{N} \alpha_i y_i) \in E(G)$. We denote G^{-1} the conversion of a graph G and $E(G^{-1}) = \{(x, y) \in \mathcal{X} \times \mathcal{X} : (y, x) \in E(G)\}$.

Let $x_0 \in V(G)$ and $\mathcal{A}$ a subset of $V(G)$. We say that $\mathcal{A}$ is dominated by x_0 if $(x_0, x) \in E(G)$ for all $x \in \mathcal{A}$. $\mathcal{A}$ dominates x_0 if for each $x \in \mathcal{A}$, $(x, x_0) \in E(G)$.

In this manuscript, we utilize $\rightharpoonup$ to represent weak convergence. We will require the following lemmas in the sequel to prove our main results.

Lemma 2.1 ([4])**.** *Let $\{\psi_n\}$, $\{\delta_n\}$ and $\{\alpha_n\}$ be the sequences in $[0, +\infty)$ such that $\psi_{n+1} \leq \psi_n + \alpha_n(\psi_n - \psi_{n-1}) + \delta_n$, for all $n \geq 1$, $\sum_{n=1}^{\infty} \delta_n < +\infty$ and there exists a real number a with $0 \leq \alpha_n \leq \alpha < 1$ for all $n \geq 1$. Then the followings hold:*

(i) $\sum_{n\geq 1}[\psi_n - \psi_{n-1}] < +\infty$ *where* $[t] = max\{t, 0\}$*;*

(ii) *There exists $\psi^* \in [0, +\infty)$ such that* $\lim_{n\to+\infty} \psi_n = \psi^*$.

Lemma 2.2 ([28])**.** *Let $\mathcal{X}$ be a Banach space satisfying Opial's condition and let $\{x_n\}$ be a sequence in $\mathcal{X}$. Let $u, v \in \mathcal{X}$ be such that* $\lim_{n\to\infty} \|x_n - u\|$ *and* $\lim_{n\to\infty} \|x_n - v\|$ *exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

Lemma 2.3 ([29])**.** *Let $\mathcal{C}$ be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$ and $G = (V(G), E(G))$ a directed graph such that $V(G) = \mathcal{C}$. Let $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be a G-nonexpansive mapping and $\{u_n\}$ be a sequence in $\mathcal{C}$ such that $u_n \rightharpoonup u$ for some $u \in \mathcal{C}$. If there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $(u_{n_k}, u) \in E(G)$ for all $k \in N$ and $\{u_n - \mathcal{T}u_n\} \to v$ for some $v \in \mathcal{H}$. Then $(I - \mathcal{T})u = v$.*

3. Main results

In this section, we are now ready to prove the theorem of weak convergence of a modified parallel inertial monotone hybrid three-step iteration (MPIMHT) to a common fixed point for a finite family of G-nonexpansive mappings in Hilbert spaces endowed with a graph.

Theorem 3.1. *Let $\mathcal{H}$ be a real Hilbert space and $G = (V(G), E(G))$ a transitive directed graph such that $E(G)$ is convex. Let $\mathcal{T}_i : \mathcal{H} \to \mathcal{H}$ be a family of*

G-nonexpansive mappings for all $i = 1, 2, \ldots, N$ such that $F = \cap_{i=1}^{N} F(\mathcal{T}_i) \neq \emptyset$. Suppose that $\{\theta_n\} \subset [0, \theta]$ for each $\theta \in (0, 1]$ and $\{\alpha_n^i\}$, $\{\beta_n^i\}$, $\{\gamma_n^i\} \subset [0, 1]$.

Algorithm 1: *Modified parallel inertial monotone hybrid three-step iteration (MPIMHT)*

---

initialization: *Take $x_0, x_1 \in \mathcal{H}$.* ***For*** *$n \geq 1$:*
Compute

$$
\begin{aligned}
w_n &= x_n + \theta_n(x_n - x_{n-1}), \\
z_n^i &= (1 - \gamma_n^i)w_n + \gamma_n^i \mathcal{T}_i w_n, \\
y_n^i &= (1 - \beta_n^i)w_n^i + \beta_n^i \mathcal{T}_i z_n^i, \\
h_n^i &= (1 - \alpha_n^i)\mathcal{T}_i z_n^i + \alpha_n^i \mathcal{T}_i y_n^i, \\
x_{n+1} &= argmax\{\|h_n^i - w_n\|, i = 1, 2, \ldots, N\}.
\end{aligned}
$$

Let $\{x_n\}$ and $\{w_n\}$ be the sequences generated by Algorithm 1 such that the following additional conditions hold:

(i) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$;
(ii) *$\{w_n\}$ is dominated by t and $\{w_n\}$ dominates t for all $t \in F$, and if there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\{w_{n_k}\} \rightharpoonup u \in \mathcal{H}$, then $(\{w_{n_k}\}, u) \in E(G)$;*
(iii) $\limsup_{n\to\infty} \alpha_n^i < 1$;
(iv) $0 < \liminf_{n\to\infty} \gamma_n^i \leq \limsup_{n\to\infty} \gamma_n^i < 1$.

Then the sequence $\{x_n\}$ converges weakly to an element in F.

Proof. Let $t \in F$. Since $\{w_n\}$ dominates t and $\mathcal{T}_i$ is edge-preserving, we get $(\mathcal{T}_i w_n, t) \in E(G)$ for all $i = 1, 2, \ldots, N$. Implying there by $(z_n^i, t) = ((1 - \gamma_n^i)w_n + \gamma_n^i \mathcal{T}_i w_n, t) \in E(G)$ by $E(G)$ is convex. Again, by edge-preserving of $\mathcal{T}_i (i = 1, 2, \ldots, N)$ and $(z_n^i, t) \in E(G)$, we have $(\mathcal{T}_i z_n^i, t) \in E(G)$, then $(y_n^i, t) = ((1 - \beta_n^i)w_n^i + \beta_n^i \mathcal{T}_i z_n^i, t) \in E(G)$, since $E(G)$ is convex. For all $i = 1, 2, \ldots, N$, we get

$$
\begin{aligned}
\left\|z_n^i - t\right\| &= \left\|(1 - \gamma_n^i)(w_n - t) + \gamma_n^i(\mathcal{T}_i w_n - t)\right\| \\
&\leq (1 - \gamma_n^i)\|w_n - t\| + \gamma_n^i \|\mathcal{T}_i w_n - t\| \\
&\leq (1 - \gamma_n^i)\|w_n - t\| + \gamma_n^i \|w_n - t\| \\
&\leq \|w_n - t\|,
\end{aligned}
$$

$$
\begin{aligned}
\left\|y_n^i - t\right\| &= \left\|(1 - \beta_n^i)(w_n - t) + \beta_n^i(\mathcal{T}_i z_n^i - t)\right\| \\
&\leq (1 - \beta_n^i)\|w_n - t\| + \beta_n^i \left\|\mathcal{T}_i z_n^i - t\right\| \\
&\leq (1 - \beta_n^i)\|w_n - t\| + \beta_n^i \left\|z_n^i - t\right\| \\
&\leq (1 - \beta_n^i)\|w_n - t\| + \beta_n^i \|w_n - t\| \\
&\leq \|w_n - t\|,
\end{aligned}
$$

and so

$$
\left\|h_n^i - t\right\| = \left\|(1 - \alpha_n^i)(\mathcal{T}_i z_n^i - t) + \alpha_n^i(\mathcal{T}_i y_n^i - t)\right\|
$$

$$
\begin{aligned}
&\leq (1-\alpha_n^i)\left\|\mathcal{T}_i z_n^i - t\right\| + \alpha_n^i \left\|\mathcal{T}_i y_n^i - t\right\| \\
&\leq (1-\alpha_n^i)\left\|z_n^i - t\right\| + \alpha_n^i \left\|y_n^i - t\right\| \\
&\leq (1-\alpha_n^i)\left\|w_n - t\right\| + \alpha_n^i \left\|w_n - t\right\| \\
&\leq \left\|w_n - t\right\| \\
&\leq \left\|x_n - t\right\| + \theta_n \left\|x_n - x_{n-1}\right\|.
\end{aligned}
$$

This implies that $\|x_{n+1}-t\| \leq \|x_n - t\| + \theta_n \|x_n - x_{n-1}\|$. From Lemma 2.1 and the assumption (i), we obtain $\lim_{n\to\infty}\|x_n - t\|$ exists, in particular, $\{x_n\}$ is bounded and also $\{z_n^i\}$, $\{y_n^i\}$ and $\{h_n^i\}$. By the properties in a real Hilbert space $\mathcal{H}$, we have

$$
\begin{aligned}
\left\|h_n^i - t\right\|^2 &= \left\|((1-\alpha_n^i)\mathcal{T}_i z_n^i + \alpha_n^i \mathcal{T}_i y_n^i) - t\right\|^2 \\
&\leq (1-\alpha_n^i)\left\|\mathcal{T}_i z_n^i - t\right\|^2 + \alpha_n^i \left\|\mathcal{T}_i y_n^i - t\right\|^2 \\
&\quad - (1-\alpha_n^i)\alpha_n^i \left\|\mathcal{T}_i z_n^i - \mathcal{T}_i y_n^i\right\|^2 \\
&\leq (1-\alpha_n^i)\left\|\mathcal{T}_i z_n^i - t\right\|^2 + \alpha_n^i \left\|\mathcal{T}_i y_n^i - t\right\|^2 \\
&\leq (1-\alpha_n^i)\left\|z_n^i - t\right\|^2 + \alpha_n^i \left\|y_n^i - t\right\|^2 \\
&\leq (1-\alpha_n^i)\left\|z_n^i - t\right\|^2 + \alpha_n^i \left\|w_n - t\right\|^2 \\
&\leq (1-\alpha_n^i)((1-\gamma_n^i)\left\|w_n - t\right\|^2 + \gamma_n^i \left\|\mathcal{T}_i w_n - t\right\|^2 \\
&\quad - (1-\gamma_n^i)\gamma_n^i \left\|w_n - \mathcal{T}_i w_n\right\|^2) + \alpha_n^i \left\|w_n - t\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\alpha_n^i)((1-\gamma_n^i)\left\|w_n - t\right\|^2 + \gamma_n^i \left\|w_n - t\right\|^2 \\
&\quad - (1-\gamma_n^i)\gamma_n^i \left\|\mathcal{T}_i w_n - w_n\right\|^2) + \alpha_n^i \left\|w_n - t\right\|^2 \\
&= \left\|w_n - t\right\|^2 - \gamma_n^i \left\|w_n - t\right\|^2 - \alpha_n^i \left\|w_n - t\right\|^2 \\
&\quad + \alpha_n^i\gamma_n^i \left\|w_n - t\right\|^2 + \gamma_n^i \left\|w_n - t\right\|^2 - \alpha_n^i\gamma_n^i \left\|w_n - t\right\|^2 \\
&\quad - (1-\alpha_n^i)(1-\gamma_n^i)\gamma_n^i \left\|\mathcal{T}_i w_n - w_n\right\|^2 + \alpha_n^i \left\|w_n - t\right\|^2 \\
&= \left\|w_n - t\right\|^2 - (1-\alpha_n^i)(1-\gamma_n^i)\gamma_n^i \left\|\mathcal{T}_i w_n - w_n\right\|^2 \\
&\leq \left\|x_n - t\right\|^2 + 2\theta_n \langle x_n - x_{n-1}, w_n - t\rangle \\
&\quad - (1-\alpha_n^i)(1-\gamma_n^i)\gamma_n^i \left\|\mathcal{T}_i w_n - w_n\right\|^2. \qquad (3.1)
\end{aligned}
$$

This implies that there exist $i_n \in \{1,2,\dots,N\}$ such that

$$
\begin{aligned}
(1-\alpha_n^{i_n})(1-\gamma_n^{i_n})\gamma_n^{i_n} \left\|\mathcal{T}_{i_n} w_n - w_n\right\|^2 &\leq \left\|x_n - t\right\|^2 - \left\|x_{n+1} - t\right\|^2 \\
&\quad + 2\theta_n \langle x_n - x_{n-1}, w_n - t\rangle. \qquad (3.2)
\end{aligned}
$$

By the assumption (i), (iii) and (iv), from (3.1), (3.2) and $\lim_{n\to\infty}\|x_n - t\|$ exist, we have

$$
\lim_{n\to\infty}\left\|\mathcal{T}_{i_n} w_n - w_n\right\| = 0. \qquad (3.3)
$$

In addition,

$$
\left\|z_n^{i_n} - w_n\right\| \leq \gamma_n^{i_n} \left\|\mathcal{T}_{i_n} w_n - w_n\right\|. \qquad (3.4)
$$

Using (3.3) and (3.4), we have

$$\lim_{n\to\infty} \left\| z_n^{i_n} - w_n \right\| = 0. \tag{3.5}$$

Since (w_n, t), $(t, z_n^{i_n}) \in E(G)$, so $(w_n, z_n^{i_n}) \in E(G)$. From (3.3) and (3.5), we have

$$\begin{aligned} \left\| y_n^{i_n} - \mathcal{T}_{i_n} w_n \right\| &\leq (1 - \beta_n^{i_n}) \left\| w_n - \mathcal{T}_{i_n} w_n \right\| + \beta_n^{i_n} \left\| \mathcal{T}_{i_n} z_n^{i_n} - \mathcal{T}_{i_n} w_n \right\| \\ &\leq (1 - \beta_n^{i_n}) \left\| w_n - \mathcal{T}_{i_n} w_n \right\| + \beta_n^{i_n} \left\| z_n^{i_n} - w_n \right\| \\ &\to 0 \ (as\ n \to \infty). \end{aligned} \tag{3.6}$$

Using (3.3) and (3.6), we have

$$\begin{aligned} \left\| y_n^{i_n} - w_n \right\| &= \left\| y_n^{i_n} - \mathcal{T}_{i_n} w_n + \mathcal{T}_{i_n} w_n - w_n \right\| \\ &\leq \left\| y_n^{i_n} - \mathcal{T}_{i_n} w_n \right\| + \left\| \mathcal{T}_{i_n} w_n - w_n \right\| \\ &\to 0 \ (as\ n \to \infty). \end{aligned} \tag{3.7}$$

Since (w_n, t), $(t, y_n^{i_n}) \in E(G)$, so $(w_n, y_n^{i_n}) \in E(G)$. It follows from (3.5) and (3.7) that

$$\begin{aligned} \left\| x_{n+1} - \mathcal{T}_{i_n} w_n \right\| &\leq (1 - \alpha_n^{i_n}) \left\| \mathcal{T}_{i_n} z_n^{i_n} - \mathcal{T}_{i_n} w_n \right\| + \alpha_n^{i_n} \left\| \mathcal{T}_{i_n} y_n^{i_n} - \mathcal{T}_{i_n} w_n \right\| \\ &\leq (1 - \alpha_n^{i_n}) \left\| z_n^{i_n} - w_n \right\| + \alpha_n^{i_n} \left\| y_n^{i_n} - w_n \right\| \\ &\to 0 \ (as\ n \to \infty). \end{aligned} \tag{3.8}$$

In addtion,

$$\left\| x_{n+1} - w_n \right\| \leq \left\| x_{n+1} - \mathcal{T}_{i_n} w_n \right\| + \left\| \mathcal{T}_{i_n} w_n - w_n \right\|.$$

From (3.3) and (3.8), we have

$$\lim_{n\to\infty} \left\| x_{n+1} - w_n \right\| = 0. \tag{3.9}$$

It follows from (3.9) that

$$\left\| h_n^i - w_n \right\| \leq \left\| x_{n+1} - w_n \right\| \to 0 \tag{3.10}$$

as $n \to \infty$ for all $i = 1, 2, \ldots, N$. From (3.1), we have

$$(1 - \alpha_n^i)(1 - \gamma_n^i)\gamma_n^i \left\| \mathcal{T}_i w_n - w_n \right\|^2 \leq \left\| w_n - t \right\|^2 - \left\| h_n^i - t \right\|^2. \tag{3.11}$$

By our assumption (iii) and (iv), it follows from (3.10) and (3.11) that

$$\lim_{n\to\infty} \left\| \mathcal{T}_i w_n - w_n \right\| = 0 \tag{3.12}$$

for all $i = 1, 2, \ldots, N$. Since $\{w_n\}$ is bounded and $\mathcal{H}$ is reflexive, $\omega_w(w_n) = \{x \in \mathcal{H} : w_{n_k} \rightharpoonup p, \{w_{n_k}\} \subset \{w_n\}\}$ is nonempty. Let $p \in \omega_w(w_n)$ be an arbitrary element. Then there exists a subsequence $\{w_{n_k}\} \subset \{w_n\}$ converging weakly to p. Let $q \in \omega_w(w_n)$ and $\{w_{n_m}\} \subset \{w_n\}$ be such that $w_{n_m} \rightharpoonup q$. From Lemma 2.3 and (3.12), we have $p, q \in F$. Applying Lemma 2.2, we obtain $p = q$. □

Note that if $\mathcal{T}$ is nonexpansive, then $\mathcal{T}$ is G-nonexpansive. As a direct convergence of Theorem 3.1, we can get the following result.

Corollary 3.2. *Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{T}_i : \mathcal{H} \to \mathcal{H}$ a family of nonexpansive mappings for all $i = 1, 2, \ldots, N$ such that $F = \cap_{i=1}^N F(\mathcal{T}_i) \neq \emptyset$. Let $\{x_n\}, \{w_n\}$ generated by $x_0, x_1 \in \mathcal{H}$ and*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n^i = (1-\gamma_n^i)w_n + \gamma_n^i \mathcal{T}_i w_n, \\ y_n^i = (1-\beta_n^i)w_n^i + \beta_n^i \mathcal{T}_i z_n^i, \\ h_n^i = (1-\alpha_n^i)\mathcal{T}_i z_n^i + \alpha_n^i \mathcal{T}_i y_n^i, \\ x_{n+1} = argmax\{\|h_n^i - w_n\|, i = 1, 2, \ldots, N\}, \end{cases} \tag{3.13}$$

where $\{\theta_n\} \subset [0, \theta]$ for each $\theta \in (0, 1]$ and $\{\alpha_n^i\}$, $\{\beta_n^i\}$, $\{\gamma_n^i\} \subset [0, 1]$. Assume that the following additional conditions hold:

(i) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$;
(ii) $\limsup_{n\to\infty} \alpha_n^i < 1$;
(iii) $0 < \liminf_{n\to\infty} \gamma_n^i \leq \limsup_{n\to\infty} \gamma_n^i < 1$.

Then the sequence $\{x_n\}$ converges weakly to an element in F.

4. Signal recovery problems

In this section, we apply the MPIMHT to solve signal recovery under situations without knowing the type of noises. In signal processing, compressed sensing can be modeled as the following under determinated linear equation system $\mathbf{y} = A\mathbf{x} + n$, where $A \in \mathbb{R}^{m\times n}$ is a degraded matrix, $\mathbf{x} \in \mathbb{R}^n$ is an original signal with n components to be recovered and $n, \mathbf{y} \in \mathbb{R}^m$ are noise and the observed signal with noisy for m components respectively. Finding the solutions of previous determinated linear equation system can be seen as solving the LASSO problem

$$\min_{\mathbf{x}\in\mathbb{R}^N} \frac{1}{2}\|\mathbf{y} - A\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_1, \tag{4.1}$$

where $\lambda > 0$. As a result various techniques and iterative schemes have been developed to solve the Lasso problem. We can apply the minimization problem of the sum of two functions for solving the LASSO problem (4.1) by setting $\mathcal{T}(\mathbf{x}) = prox_{\psi h}(\mathbf{x} - \psi\nabla f(\mathbf{x}))$, where $f(\mathbf{x}) = \|\mathbf{y} - A\mathbf{x}\|_2^2/2$, $h(x) = \lambda\|\mathbf{x}\|_1$, $\nabla f(\mathbf{x}) = A^T(A\mathbf{x} - \mathbf{y})$.

Now, we present the parallel iterative method in recovering the original signal $\mathbf{x}$ when the observed signals $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_M$ can be recovered by using the degraded matrices $A_1, A_2, \ldots, A_M$, repectively in which

$$\mathbf{y}_i = A_i\mathbf{x} + n_i, i = 1, 2, \ldots, M. \tag{4.2}$$

That is, the original signal $\mathbf{x}$ is a common solution of the M-determinated system of linear equations (4.2). Let us consider the following M-LASSO problems which is called as the LASSO system introduced by Suantai et al. [30]:

$$
\begin{aligned}
&\min_{\mathbf{x}\in\mathbb{R}^N} \frac{1}{2}\|A_1\mathbf{x}-\mathbf{y}_1\|_2^2+\lambda_1\|\mathbf{x}\|_1,\\
&\min_{\mathbf{x}\in\mathbb{R}^N} \frac{1}{2}\|A_2\mathbf{x}-\mathbf{y}_2\|_2^2+\lambda_2\|\mathbf{x}\|_1,\\
&\qquad\vdots\\
&\min_{\mathbf{x}\in\mathbb{R}^N} \frac{1}{2}\|A_M\mathbf{x}-\mathbf{y}_M\|_2^2+\lambda_M\|\mathbf{x}\|_1,
\end{aligned} \tag{4.3}
$$

where the original signal $\mathbf{x}$ is common solution of LASSO system (4.3). We will find the true signal $\mathbf{x}$ through the common solution of LASSO system. Let $\mathcal{T}_i(\mathbf{x}) = prox_{\psi_i g_i}\big(\mathbf{x}+\psi_i A_i^t(A_i\mathbf{x}-\mathbf{y}_i)\big)$. We apply the MPIMHT in finding the common solution $\mathbf{x}$ for the LASSO system:

$$
\begin{aligned}
\mathbf{w}_n &= \mathbf{x}_n+\theta_n(\mathbf{x}_n-\mathbf{x}_{n-1})\\
\mathbf{z}_n^i &= (1-\gamma_n^i)\mathbf{w}^n+\gamma_n^i\mathcal{T}_i(\mathbf{w}^n),\\
\mathbf{y}_n^i &= (1-\beta_n^i)\mathbf{w}_n^i+\beta_n^i\mathcal{T}_i(\mathbf{z}_n^i),\\
\mathbf{h}_n^i &= (1-\alpha_n^i)\mathcal{T}_i(\mathbf{z}_n^i)+\alpha_n^i\mathcal{T}_i(\mathbf{y}_n^i),\\
\mathbf{x}_{n+1} &= \operatorname{argmax}\left\{||\mathbf{h}_n^i-\mathbf{w}_n||, i=1,2,\dots,M\right\},
\end{aligned} \tag{4.4}
$$

where $g_i(\mathbf{x}) = \lambda_i\,\|\mathbf{x}\|_1$, $\psi_i = 2/\|A_i^T A_i\|_2$. The following stopping criterion is used $\|\mathbf{x}_{n+1}-\mathbf{x}_n\|_2 < \epsilon_l$, and after that set $\mathbf{x}_{n-1} = \mathbf{x}_n$ and $\mathbf{x}_n = \mathbf{x}_{n+1}$. The default parameters θ_n and $\{\alpha_n^i\}, \{\beta_n^i\}, \{\gamma_n^i\}$ are set as follows:

$$
\alpha_n^i = \frac{n}{n+1}, \quad \beta_n^i = \alpha_n^i, \quad \gamma_n^i = \alpha_n^i,
$$

$$
\theta_n = \begin{cases} \min\left\{\dfrac{1}{n^2\|\mathbf{x}_n-\mathbf{x}_{n-1}\|_2^2}, 0.1\right\} & \text{if } (\mathbf{x}_n \neq \mathbf{x}_{n-1})\ \&\ (1\le n<\widetilde{N}),\\ 0.15 & \text{otherwise,} \end{cases}
$$

when $\widetilde{N}$ is a number of iterations that we want to stop with $\epsilon_l = 10^{-7}$. And, we called the algorithm (4.4) as the MPIMHT with degraded matrices $A_i, i = 1\dots M$.

Next, some experiments are provided to illustrate the convergence and the effectiveness of the MPIMHT (4.4) and compare with the FISTA algorithm [14], Suantai et al. [30], Cholamjiak et al. [15] and Jun-on et al. [20]. The original signal x with $n = 1024$ generated by the uniform distribution in the interval $[-2, 2]$ with 70 nonzero elements is used to create the observation signal with $m = 512$ and $\mathbf{y}_i = A_i\mathbf{x}+n_i$ where $i \le 3$.

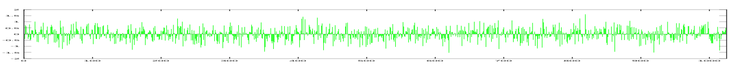


FIGURE 1. Original Signal (x) with $m = 70$.

The observation signal $\mathbf{y}_i, i = 1, 2, 3$ show on Figure 2.

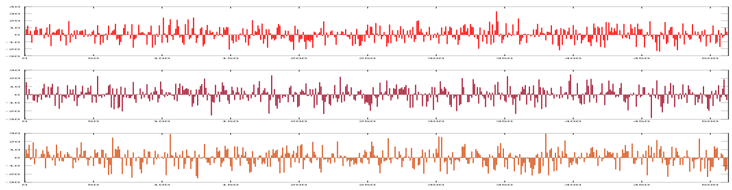


FIGURE 2. Degraded Signals $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$, respectively.

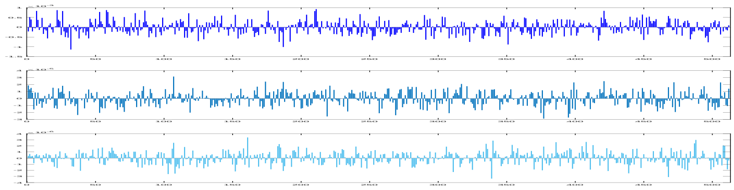


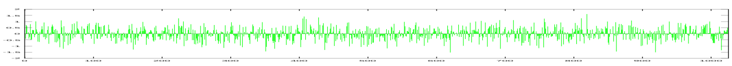
FIGURE 3. Noise Signals $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ respectively.

The matrices A_i generated by the normal distribution with mean zero and variance one and the white Gaussian noise $\mathbf{n_i}$ (See on Figure 3).

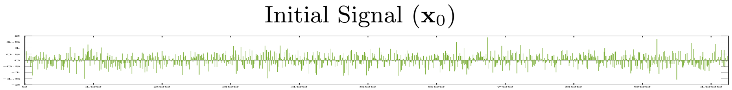
Both theoretical and experimental results for the convergence properties of the MPIMHT with the permutation of the blurring matrices A_1, A_2 and A_3 are demonstrated and discussed on the following cases:

Case I: The MPIMHT with $\mathcal{T}_1$. Case II: The MPIMHT with $\mathcal{T}_2$. Case III: The MPIMHT with $\mathcal{T}_3$. Case IV: The MPIMHT with $\mathcal{T}_1$–$\mathcal{T}_2$. Case V: The MPIMHT with $\mathcal{T}_1$–$\mathcal{T}_3$. Case VI: The MPIMHT with $\mathcal{T}_2$–$\mathcal{T}_3$. Case VII: The MPIMHT with $\mathcal{T}_1$–$\mathcal{T}_2$–$\mathcal{T}_3$.

The process is started when the signal initial data $\mathbf{x}_0$ and $\mathbf{x}_1$ with $n = 1024$ is picked randomly.



Initial Signal ($\mathbf{x}_0$)



Initial Signal ($\mathbf{x}_1$)

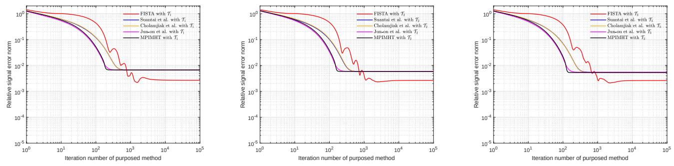
FIGURE 4. Initial Signals $\mathbf{x}_0$ and $\mathbf{x}_1$.

The relative signal error is measured by the following formula $\|\mathbf{x}_n - \mathbf{x}\|_2/\|\mathbf{x}\|_2$ in order to check the convergence of all comparative algorithms. The performance of the tested methods at n^{th} iteration is measured quantitatively by the means of the the signal-to-noise ratio (SNR), which is defined by

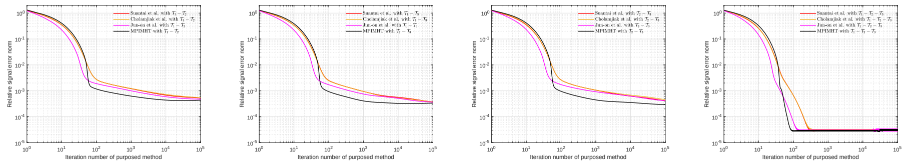
$$\mathrm{SNR}(\mathbf{x}_n) = 20\log_{10}\left(\frac{\|\mathbf{x}_n\|_2}{\|\mathbf{x}_n - \mathbf{x}\|_2}\right),$$

where $\mathbf{x}_n$ is the recovered signal at n^{th} iteration by using the considered method.

The signal relative error and SNR quality of all comparative methods for recovering the degraded signal are shown on Figures 5 and Figure 6.

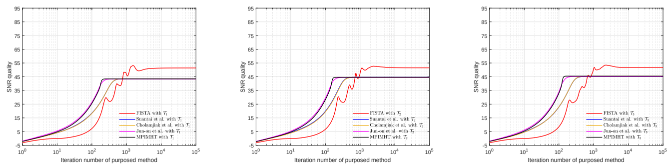


Non-Parallell situation (MPIMHT with case I-III).

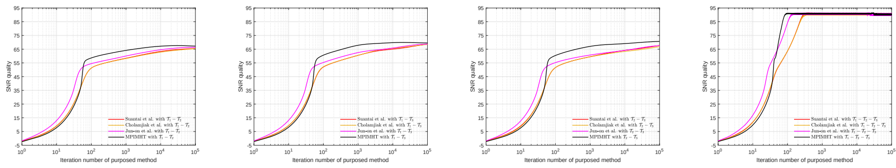


Parallell situation (MPIMHT with case IV-VII).

FIGURE 5. The relative error norm of all comparative methods.



Non-Parallell situation (MPIMHT with case I-III).



Parallell situation.

FIGURE 6. The SNR plots of all comparative methods (MPIMHT with case IV-VII).

Figure 5 shows that the relative error plots of all algorithms are decreased as the iteration number increases and after that they converge to some constants. The relative error plot demonstrates the validity of all comparative algorithms and confirms their convergence. The first three figures in Figure 5 show that when the number of iterations is large enough, the FISTA method provides us the least relative error. It should also be highlighted that within the first 500 iterations, the MPIMHT and Jun-on et al. methods approaches converge similarly and faster than the other comparable methods. With the exception of FISTA techniques, the remaining figures of Figure 5 depict the convergence behavior of all comparison approaches (all methods that can be parallel computing). The other parallel approaches converge significantly better than the proposed method, as can be seen. However, after 100 iterations the proposed method converges better.

Figure 6 shows that the SNR quality of the restored signal using all comparative methods increases until it converges to some constant value. The FISTA method outperforms the other approaches when the quality of the recovered signal is attained using only one of the dregraded matrices, as seen in the first three figures of Figure 6. The remaining figures of Figure 6 show the SNR quality of all comparative methods excepted FISTA methods. It can be seen that the SNR quality of the restored image for all parallel algorithms are improved and better than FISTA method. And, when all degrading matrices are used in finding the common solutions of the signal recovery problem, we get the best quality of the recovering signal. With the exception of FISTA techniques, the remaining figures of Figure 6 illustrate the SNR quality of all comparing methods. All parallel methods improve and outperform the FISTA method in terms of SNR quality of the restored image. We acquire the best quality of the recovered signal when all degrading matrices are applied in discovering the common solutions of the signal recovering challenge.

Figure 7 displays the SNR plots for the best case of FISTA technique and all comparative parallel methods that use all degradation matrices within 100^{th} iterations. We discovered that all parallel approaches outperform the FISTA method in terms of quality. And, after 50 iterations, the MPIMHT approach provides us the best quality. Moreover, the highest SNR quality can be achieved within the first 100 iterations.

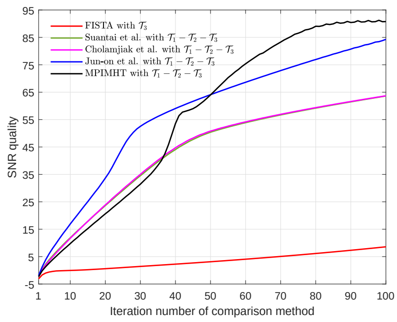


FIGURE 7. The SNR plots of FISTA method and all comparative parallel methods in which all degrading matrices within 100^{th} iterations.

Figure 8 shows that the proposed technique, which uses all degradation matrices, takes the longest average time on each iteration step and also consumes the greatest CPU time during the procedure. That's the one of the disadvantage of the proposed method.

The last figure shows the best quality of the restored signals at 90^{th} step of iterations for all comparative methods.

5. Conclusions

In this manuscript, we present a modified version of the parallel inertial monotone hybrid three-step iteration algorithm (referred to as MPIMHT) to solve the common fixed point problem for a finite family of G-nonexpansive mappings in a Hilbert

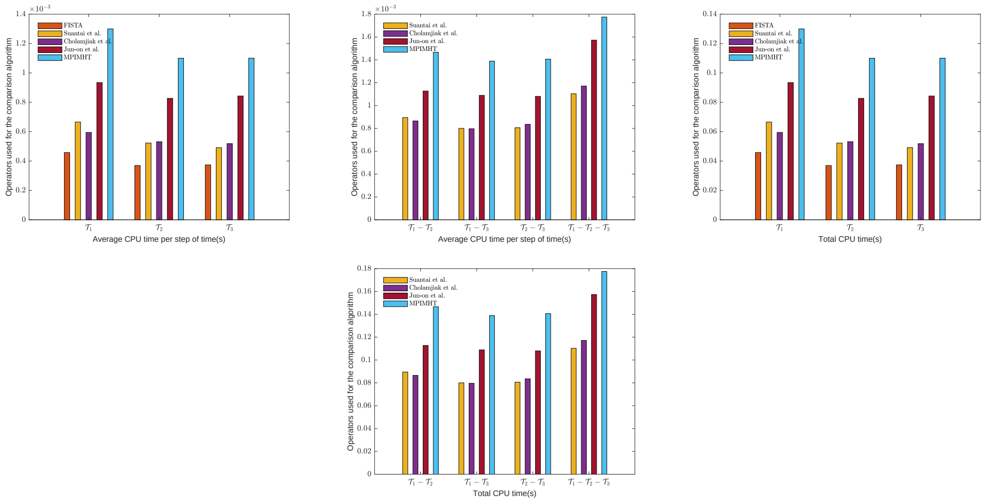
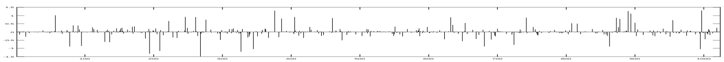
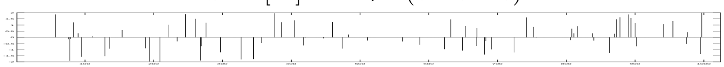


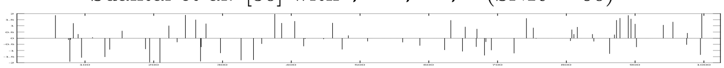
FIGURE 8. The CPU time consumption through out the process for FISTA and all parallel methods within 100^{th} iterations.



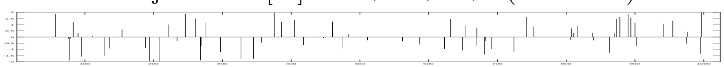
FISTA [14] with $\mathcal{T}^3$ (SNR = 6)



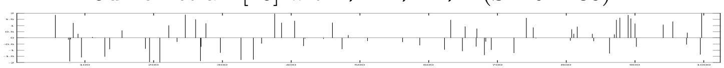
Suantai et al. [30] with $\mathcal{T}^1 - \mathcal{T}^2 - \mathcal{T}^3$ (SNR = 60)



Cholamjiak et al. [15] with $\mathcal{T}^1 - \mathcal{T}^2 - \mathcal{T}^3$ (SNR = 60)



Jun-on et al. [20] with $\mathcal{T}^1 - \mathcal{T}^2 - \mathcal{T}^3$ (SNR = 80)



MPIMHT with $\mathcal{T}^1 - \mathcal{T}^2 - \mathcal{T}^3$ (SNR = 91)

FIGURE 9. Recovering signals being used the FISTA method and all comparative parallel methods in which all degrading matrices at 90^{th} iterations.

space with a directed graph. We have proved weak convergence of the sequence generated by MPIMHT to an element of the problem's solution set under certain conditions. As an application, the algorithm is then used to solve the signal recovery problem involving several filters. We discovered that the numerical experiment on a signal recovery problem outcome of MPIMHT is better than that of some previous algorithms.

References

[1] S. M. A. Aleomraninejad, S. Rezapour and N. Shahzad, *Some fixed point result on a metric space with a graph*, Topol. Appl. **159** (2012), 659–663.

[2] M. R. Alfuraidan, *Fixed points of monotone nonexpansive mappings with a graph*, Fixed Point Theory Appl. **49** (2015): Article number 49.

[3] M. R. Alfuraidan and M. A. Khamsi, *Fixed points of monotone nonexpansive mappings on a hyperbolic metric space with a graph*, Fixed Point Theory Appl. **44** (2015): Article number 44.

[4] F. Alvarez and H. Attouch, *An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping*, Set-Valued Anal. **9** (2001), 3–11.

[5] P. K. Anh and D. V. Hieu, *Parallel and sequential hybrid methods for a finite family of asymptotically quasi* ϕ*-nonexpansive mappings*, J. Appl. Math. Comput. **48** (2015), 241–263.

[6] P. K. Anh and D. V. Hieu, *Parallel hybrid iterative methods for variational inequalities, equilibrium problems, and common fixed point problems*, Vietnam J. Math. **44** (2016), 351–374.

[7] J. H. Asl, B. Mohammadi, S. Rezapour and S. M. Vaezpour, *Some fixed point results for generalized quasi-contractive multifunctions on graphs*, Filomat **27** (2013), 311–315.

[8] H. Attouch, J. Peypouquet and P. Redont, *A dynamical approach to an inertial forward-backward algorithm for convex minimization*, SIAM J. Optimiz. **24** (2014), 232–256.

[9] S. Banach, *Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales*, Fund. Math. **3** (1922), 133–181.

[10] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, vol. 243 2nd ed, Springer, New York, NY, USA, 2017.

[11] F. Bojor, *Fixed point of* ϕ*-contraction in metric spaces endowed with a graph*, Ann. Univ. Craiova Math. Ser.Mat. Inform. **37** (2010), 85–92.

[12] F. Bojor, *Fixed point theorems for Reich type contractions on metric spaces with a graph*, Nonlinear Anal. **75** (2012), 3895–3901.

[13] F. Bojor, *Fixed points of Kannan mappings in metric spaces endowed with a graph*, An. St. Univ. Ovidius Constanta Ser. Mat. **20** (2012), 31–40.

[14] A. Beck and M. Teboulle, *A fast iterative shrinkage-thresholding algorithm for linear inverse problems*, SIAM J. Imaging Sci. **2** (2009), 183–202.

[15] W. Cholamjiak, S.A. Khan, D. Yambangwai and K.R. Kazmi, *Strong convergence analysis of common variational inclusion problems involving an inertial parallel monotone hybrid method for a novel application to image restoration*, RACSAM Rev. R. Acad. A. **114** (2020), 1–20.

[16] R. Glowinski and P. L. Tallec, *Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanic*, SIAM, Philadelphia, 1989.

[17] S. Haubruge, V. H. Nguyen and J. J. Strodiot, *Convergence analysis and applications of the Glowinski Le Tallec splitting method for finding a zero of the sum of two maximal monotone operators*, J. Optim. Theory Appl. **97** (1998), 645–673.

[18] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc. **136** (2008), 1359–1373.

[19] R. Johnsonbaugh, *Discrete Mathematics*, New Jersey, 1997.

[20] N. Jun-on, R. Suparatulatorn, M. Gamal and W. Cholamjiak, *An inertial parallel algorithm for a finite family of G-nonexpansive mappings applied to signal recovery*, AIMS Mathematics **7** (2021), 1775–1790.

[21] R. P. Kelisky and T. J. Rivlin, *Iterates of Bernstein polynomials*, Pacific J. Math. **21** (1967), 511–520.

[22] P. E. Maingé, *Regularized and inertial algorithms for common fixed points of nonlinear operators*, J. Math. Anal. Appl. **344** (2008), 876–887.

[23] J. J. Moreau, *Proximité et dualité dans un espace hilbertien*, Bull. Soc. Math. Fr. **93** (1965), 273–299.

[24] A. Nicolae, D. O. Regan and A. Petrusel, *Fixed point theorems for single-valued and multivalued generalized contractions in metric spaces endowed with a graph*, Georgian Math. J. **18** (2011), 307–327.

[25] B. T. Polyak, *Some methods of speeding up the convergence of iterative methods*, USSR Comput. Math. Math. Phys. **4** (1964), 1–17.

[26] M. Samreen and T. Kamran, *Fixed point theorems for integral G-contractions*, Fixed Point Theory Appl. **149** (2013): Article number 149.
[27] P. Sridarat, R. Suparaturatorn, S. Suantai and Y. J. Cho, *Covergence analysis of SP-iteration for G-nonexpansive mappings with directed graphs*, Bull. Malays. Math. Sci. Soc. **42** (2019), 2361–2380.
[28] S. Suantai, *Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **311** (2005), 506–517.
[29] S. Suantai, M. Donganont and W. Cholamjiak, *Hybrid methods for a countable family of G-nonexpansive mappings in Hilbert spaces endowed with graphs*, Mathematics **7** (2019): 936.
[30] S. Suantai, K. Kankam, P. Cholamjiak and W. Cholamjiak, *A parallel monotone hybrid algorithm for a finite family of G-nonexpansive mappings in Hilbert spaces endowed with a graph applicable in signal recovery*, Comp. Appl. Math. **40** (2021): 145.
[31] R. Suparatulatorn, S. Suantai and W. Cholamjiak, *Hybrid methods for a finite family of G-nonexpansive mappings in Hilbert spaces endowed with graphs*, AKCE Int. J. Graphs Co. **14** (2017), 101–111.
[32] D. V. Thong and D. V. Hieu, *Inertial extragradient algorithms for strongly pseudomonotone variational inequalities*, J. Comput. Appl. Math. **341** (2018), 80–98.
[33] D. V. Thong and D. V. Hieu, *Modified subgradient extragradient method for variational inequality problems*, Numer. Algor. **79** (2018), 597–610.
[34] J. Tiammee, A. Kaewkhao and S. Suantai, *On Browder's convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graphs*, Fixed Point Theory Appl. **187** (2015): Article number 187.
[35] J. Tiammee and S. Suantai, *Coincidence point theorems for graph-preserving multivalued mappings*, Fixed Point Theory Appl. **70** (2014): Article number 70.
[36] O. Tripak, *Common fixed points of G-nonexpansive mappings on Banach spaces with a graph*, Fixed Point Theory Appl. **87** (2016): Article number 87.
[37] L. Y. Zhang, H. Zhao, and Y. B. Lv, *A modified inertial projection and contraction algorithms for quasivariational inequalities*, Appl. Set-Valued Anal. Optim. **1** (2019), 63–76.

Manuscript received March 25, 2024
revised August 5, 2024

D. Yambangwai
Department of Mathematics, School of Science, University of Phayao, Phayao, 56000, Thailand
E-mail address: `damrongsak.ya@up.ac.th`

T. Thianwan
Department of Mathematics, School of Science, University of Phayao, Phayao, 56000, Thailand
E-mail address: `tanakit.th@up.ac.th`