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WEIGHTED COMPOSITION OPERATORS ON VECTOR-VALUED LIPSCHITZ SPACES

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ABSTRACT. In this paper we characterize boundedness, injectivity, surjectivity and compactness of weighted composition operators between $Lip_0(X, E)$ and $Lip_0(Y, F)$.

1. INTRODUCTION

Let E be a Banach space, and let (X, d) be a pointed metric space, that is, a metric space with base point $x_0 \in X$. A function $f: X \to E$ is called Lipschitz if there exists a constant k > 0 such that

$$||f(x) - f(y)|| \le kd(x, y) \text{ for all } x, y \in X.$$

The smallest such k is called the Lipschitz constant L(f) of f. We have

$$L(f) = \sup_{\substack{x,y \in X \\ x \neq y}} \frac{||f(x) - f(y)||}{d(x,y)}.$$

The space of all Lipschitz functions $f : X \to E$ such that $f(x_0) = 0$, denoted by $Lip_0(X, E)$, is a Banach space with the norm L(f). We will also consider the Banach space of all bounded Lipschitz functions $f : X \to E$, denoted by Lip(X, E), with the norm

$$||f||_s = L(f) + ||f||_{\infty}$$
, where $||f||_{\infty} = \sup_{x \in X} ||f(x)||.$

When E is the scalar field, $Lip_0(X, E)$ and Lip(X, E) will be denoted by $Lip_0(X)$ and Lip(X) respectively.

In this paper we shall study an important class of operators from $Lip_0(X, E)$ into $Lip_0(Y, F)$ known as weighted composition operators. Here, F is a Banach space, and (Y, ρ) is a pointed metric space with base point $y_0 \in Y$. The study of such operators on Lipschitz spaces and on other classical Banach spaces had received a lot of attention in past as well as in recent time, see [10, 13, 15] and the references therein. A weighted composition operator $T : Lip_0(X, E) \to Lip_0(Y, F)$ is an operator which can be written as a composition with operator weights, that is, it has the form:

(1.1)
$$Tf(y) = W(y)(f(\phi(y))), \text{ for all } y \in Y \text{ and } f \in Lip_0(X, E);$$

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where ϕ is a map from Y to X, and W(y), denoted by W_y for simplicity, is a linear operator from E to F.

Weighted composition operators are intimately connected with isometries. It is known that for many vector-valued function spaces, the surjective linear isometries are of the form (1.1), see [5,7,11]. The study of isometries between Banach spaces is one of the most important research areas in functional analysis.

Moreover, such operators arise in a very natural way in many situations. Given Banach spaces, E, F, B(E, F) denotes the space of all bounded linear operators from E to F, E^* denotes the dual space of E, and ext(E) denotes the set of extreme points of the unit ball of E. An operator $T \in B(E, F)$ is said to be nice if $T^*(ext(F^*)) \subseteq ext(E^*)$. Nice operators between continuous function spaces are weighted composition operators [3,6,11].

Furthermore, weighted composition operators are also zero product preserving maps. Zero product preserving maps on operator algebras have become the subject of a systematic study for quite some time [2, 18, 19]. On function algebras these maps are usually called disjointness preserving maps or separating maps. A linear map $T : Lip(X, E) \to Lip(Y, F)$ is called disjoint preserving or separating if ||Tf(y)||||Tg(y)|| = 0 for all $y \in Y$, whenever $f, g \in Lip(X, E)$ satisfy ||f(x)||||g(x)|| = 0 for all $x \in X$. The map T is said to be biseparating if it is bijective and both T and T^{-1} are separating. It was shown in [1] that biseparating maps from Lip(X, E) to Lip(Y, F) are weighted composition operators. Similar results were proved in [12, 15] for continuous vector-valued function spaces. For related results on second-order Cesàro function spaces, we refer the reader to the papers [4] and [14].

The boundedness, compactness, weak compactness and spectral properties of composition and weighted composition operators on scaler and vector-valued Lipschitz spaces have been studied extensively by many authors recently. In [16], the authors characterized compact composition operators on Lip(X) and $Lip_0(X)$. In [13], Golbaharan and Mahyar described compact weighted composition operators on Lip(X). They also obtain necessary and sufficient conditions for the injectivity and surjectivity of these operators. In [10], the authors characterized compact weighted composition operators on $Lip_0(X)$. They also give necessary and sufficient conditions for the injectivity and surjectivity of these operators. In this paper we try to complete this circle of ideas by characterizing boundedness, injectivity, surjectivity and compactness of weighted composition operators between $Lip_0(X, E)$ and $Lip_0(Y, F)$. Our proofs depend on the ideas presented in [10].

2. Preliminaries and basic results

In this section we recall some basic definitions and results that will be used throughout the paper. The operator $T : Lip_0(X, E) \to Lip_0(Y, F)$ will always denote a weighted composition operator (WCO, for short) of the form

$$Tf(y) = W_y(f(\phi(y)))$$
, for all $y \in Y$ and $f \in Lip_0(X, E)$;

where ϕ is a map from Y to X, and W_y is a linear operator from E to F. We will also refer to such T as the weighted composition operator induced by W and ϕ . The map ϕ is called base point preserving map if $\phi(y_0) = x_0$. We recall that x_0 and y_0 are base points of X and Y respectively. The diameter of a metric space X, denoted by diam(X), is defined as

$$\operatorname{diam}(X) = \sup_{x,y \in X} d(x,y).$$

It is a natural question to ask how the properties of T, W_y and ϕ are related to each other. What conditions the operator W_y and the map ϕ should satisfy so that T is a (bounded) WCO from $Lip_0(X, E)$ to $Lip_0(Y, F)$, and vice-versa.

Consider the following examples.

Example 2.1. Let $X = \{-1, 0\}$ and Y = (-1, 1) with the usual metric d and base point 0. Define the map $\phi : Y \to X$ by $\phi(y) = [y]$, where [y] stands for the greatest integer less than or equal to y. It is clear that ϕ is not Lipschitz. For $y \in Y$, we define $W_y \in B(E)$ by $W_y(e) = ye$. It can be shown that the operator T induced by ϕ and W is a bounded WCO from $Lip_0(X, E)$ into $Lip_0(Y, E)$.

Example 2.2. Let X = (-3, 3), Y = (-2, 2) with the usual metric d and base point -1. Define the map $\phi : Y \to X$ by $\phi(y) = y$. For $y \in Y$, we define $W_y \in B(E)$ as $W_y(e) = [\chi_{(-2,-1]}(y) - \chi_{(-1,2]}(y)]e$, where χ_A is the characteristic function of $A \subseteq \mathbb{R}$.

We observe that W is not Lipschitz. Indeed,

$$\frac{||W_{(-1-\frac{1}{n})} - W_{(-1+\frac{1}{n})}||}{|(-1-\frac{1}{n}) - (-1+\frac{1}{n})|} = \frac{1}{|-1-\frac{1}{n}+1-\frac{1}{n}|} \sup_{||e||=1} ||W_{(-1-\frac{1}{n})}(e) - W_{(-1+\frac{1}{n})}(e)||$$
$$= \frac{n}{2} \sup_{||e||=1} ||e+e|| = n.$$

We can also show that T is bonded linear operator from $Lip_0(X, E)$ to $Lip_0(Y, E)$.

Remark 2.3. In the above two examples we see that $T : Lip_0(X, E) \to Lip_0(Y, E)$ is a bounded WCO, but in one case ϕ is not a base point preserving Lipschitz map, and in the other case W is not Lipschitz.

For a function $f \in Lip_0(X, E)$, under what conditions we have that $Tf \in Lip_0(Y, F)$. Our first proposition gives an answer to this question in the case when diameter of X is finite.

Proposition 2.4. Let diam $(X) < \infty$. Suppose that $W \in Lip(Y, B(E, F))$ and $\phi: Y \to X$ is a base point preserving map such that $\sup \left\{ \|W_x\| \frac{d(\phi(x), \phi(y))}{\rho(x, y)} : x, y \in Y, x \neq y \right\} < \infty$. Then T is a WCO from $Lip_0(X, E)$ into $Lip_0(Y, F)$.

Proof. Take $C = \sup \left\{ \|W_x\| \frac{d(\phi(x),\phi(y))}{\rho(x,y)} : x, y \in Y, x \neq y \right\}$. Let $f \in Lip_0(X, E)$. Then for each $x, y \in Y$ with $\phi(x) \neq \phi(y)$, we have

$$\frac{\|Tf(x) - Tf(y)\|}{\rho(x, y)} = \frac{\|W_x(f(\phi(x))) - W_y(f(\phi(y)))\|}{\rho(x, y)}$$
$$\leq \frac{\|W_x(f(\phi(x))) - W_x(f(\phi(y)))\|}{\rho(x, y)}$$

$$+ \frac{\|W_x(f(\phi(y))) - W_y(f(\phi(y)))\|}{\rho(x,y)}$$

= $\frac{\|W_x\|\|f(\phi(x)) - f(\phi(y))\|}{d(\phi(x),\phi(y))} \frac{d(\phi(x),\phi(y))}{\rho(x,y)}$
+ $\frac{\|W_x - W_y\|\|f(\phi(y))\|}{\rho(x,y)}$
 $\leq CL(f) + L(W)L(f) \operatorname{diam}(X).$

Moreover, for each $x, y \in Y$ with $x \neq y$ and $\phi(x) = \phi(y)$, we have

$$\frac{\|Tf(x) - TF(y)\|}{\rho(x, y)} \le L(W)L(f)\operatorname{diam}(X).$$

An important property which is related to the compactness of weighted composition operators is supercontractivity. A map $\phi: Y \to X$ is called supercontractive if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\frac{d(\phi(x),\phi(y))}{\rho(x,y)} < \epsilon$ whenever $0 < \rho(x,y) < \delta$. Constant maps are Lipschitz and supercontractive. A supercontractive Lipschitz function is often called a little Lipschitz function.

3. Bounded weighted composition operators

In this section we characterize boundedness of weighted composition operators between vector-valued Lipschitz spaces.

We start by stating the following elementary lemma.

Lemma 3.1. Let $f_n, f \in Lip_0(X, E)$ such that $f_n \to f$. Then $f_n \to f$ pointwise on X.

Proposition 3.2. Let $W : Y \to B(E, F)$ be a map, and let $\phi : Y \to X$ be a base point preserving map. Let $T : Lip_0(X, E) \to Lip_0(Y, F)$ be a WCO. Then

- (1) T is bounded.
- (1) I us occurrent. (2) $\sup \left\{ \|W_x\| \frac{d(\phi(x),\phi(y))}{\rho(x,y)} : x, y \in Y, x \neq y \right\} \le 2\|T\|.$

Proof. Let $f_n \in Lip_0(X, E)$ such that $f_n \to 0$ and $Tf_n \to g$ in $Lip_0(Y, F)$. Then $f_n(\phi(y)) \to 0$ and $Tf_n(y) \to g(y)$ for every $y \in Y$. Applying Closed Graph Theorem and using the fact that each W_y is bounded, we conclude that T is bounded. This proves the first assertion.

Let $x, y \in Y$ with $x_0 \neq \phi(x) \neq \phi(y)$. We can assume without loss of generality that $d(\phi(y), x_0) \leq d(\phi(x), x_0)$. This implies that $d(\phi(x), \phi(y)) \leq 2d(\phi(x), x_0)$. Take $\delta = \min\{d(\phi(x), x_0), d(\phi(x), \phi(y))\} > 0$. For fixed $e \in E$ we define the function $f: X \to E$ by

$$f(z) = d(\phi(x), \phi(y)) \max\left\{0, 1 - \frac{d(\phi(x), z)}{\delta}\right\}e_{z}$$

One can show that $f \in Lip_0(X, E)$ and $L(f) \leq 2||e||$. We also note that $f(\phi(x)) = d(\phi(x), \phi(y))e$ and $f(\phi(y)) = 0$. Now,

$$\|W_x(e)\|\frac{d(\phi(x),\phi(y))}{\rho(x,y)} = \frac{\|W_x(f(\phi(x))) - W_y(f(\phi(y)))\|}{\rho(x,y)} = \frac{\|Tf(x) - Tf(y)\|}{\rho(x,y)}$$

$$\leq L(Tf) \leq 2||T|| ||e||.$$

For the case $x_0 = \phi(x) \neq \phi(y)$, we define the function $f: X \to E$ by $f(z) = d(z, x_0)e$ and the proceed in the same way to get the required conclusion. Hence, the second assertion is also proved.

Corollary 3.3. Let $W: Y \to B(E, F)$ be a bounded map which is continuous on $Y \setminus N$, where $N = \{y \in Y : W_y = 0\}$, and let $\phi : Y \to X$ be a base point preserving map. Let $T: Lip_0(X, E) \to Lip_0(Y, F)$ be a WCO. Then ϕ is Lipschitz on every nonempty compact subset of $Y \setminus N$.

Proof. Let U be a nonempty compact subset of $Y \setminus N$. Take $B = \inf_{y \in U} ||W_y||$. Suppose that $x, y \in U$ with $x \neq y$. By Proposition 3.2 we have $||W_x|| \frac{d(\phi(x), \phi(y))}{\rho(x, y)} \leq$ 2||T||. This implies that $\frac{d(\phi(x),\phi(y))}{\rho(x,y)} \leq \frac{2||T||}{B}$. Therefore, ϕ is a Lipschitz.

4. Injective and surjective weighted composition operators

In this section we characterize injectivity and surjectivity of weighted composition operators between vector-valued Lipschitz spaces.

Our first theorem gives a necessary and sufficient condition for a WCO to be injective. We recall that $N = \{y \in Y : W_y = 0\}.$

Theorem 4.1. Let $W: Y \to B(E, F)$ be a map, $\phi: Y \to X$ a base point preserving map, and $T: Lip_0(X, E) \to Lip_0(Y, F)$ a WCO. Suppose that $y_0 \in Y \setminus N$, and for every $y \in Y \setminus N$, W_y is injective. Then T is injective if and only if $\phi(Y \setminus N)$ is dense in X.

Proof. Suppose that T is injective, and $\phi(Y \setminus N)$ is not dense in X. Then there exists $x' \in X$ such that $\operatorname{dist}(x', \phi(Y \setminus N) > 0$. Let $\varepsilon = \operatorname{dist}(x', \phi(Y \setminus N))$. Define $f: X \to E$ by

$$f(x) = \max\left\{0, 1 - \frac{d(x', x)}{\varepsilon}\right\}e.$$

It is clear that $f(x_0) = 0$ and $f \in Lip_0(X, E)$. Moreover, Tf = 0 and f(x') = e. This implies that T is not injective, a contradiction.

Conversely, assume that $\phi(Y \setminus N)$ is dense in X. Let $g \in Lip_0(X, E)$ with T(g) = 0. Let $x \in \phi(Y \setminus N)$ and $y \in Y \setminus N$ such that $\phi(y) = x$. Now, Tg(y) = x $W_y(g(\phi(y))) = W_y(g(x)) = 0$. As W_y is injective g(x) = 0. This implies that g = 0. Therefore, T is injective.

Theorem 4.2. Let $W \in Lip(Y, B(E, F))$ and let $\phi : Y \to X$ be a Lipschitz homeomorphism. Let $T : Lip_0(X, E) \to Lip_0(Y, F)$ be a WCO. Suppose that $y_0 \in Y \setminus N$. If T is surjective, then the following conditions hold.

- (1) $Y \setminus N = Y$,
- (2) $\inf \left\{ \frac{d(\phi(x),\phi(y))}{\rho(x,y)} : x, y \in Y, x \neq y \right\} > 0,$ (3) $\inf \left\{ ||W_x|| \frac{d(\phi(x),\phi(y))}{\rho(x,y)} : x \in K, y \in Y, x \neq y \right\} > 0,$ where K is nonempty compact subset of Y.

Proof. Let $y \in Y \setminus \{y_0\}$. Take $\varepsilon < \rho(y, y_0)$. For fix $s \in F$, we define $g: Y \to F$ by

$$g(x) = \max\left\{0, 1 - \frac{\rho(y, x)}{\varepsilon}\right\}s, \ x \in Y.$$

We have $g(y_0) = 0$, g(y) = s and $g \in Lip_0(Y, F)$. As T is surjective, there exists $f \in Lip_0(X, E)$ such that Tf = g. Now, $Tf(y) = W_y(f(\phi(y))) = g(y) = s$. This implies that $W_y \neq 0$ for all $y \in Y$. This proves the first assertion.

As ϕ is Lipschitz homeomorphism, there exists M > 0 such that $\rho(x, y) \leq Md(\phi(x), \phi(y))$ for all $x, y \in Y$. This implies that, $\frac{d(\phi(x), \phi(y))}{\rho(x, y)} \geq \frac{1}{M}$. Thus,

$$\inf\left\{\frac{d(\phi(x),\phi(y))}{\rho(x,y)}: x,y\in Y, x\neq y\right\}\geq \frac{1}{M}>0,$$

which proves the second assertion.

Suppose that K is a nonempty compact subset of Y. Then $\inf_{x \in K} ||W_x|| = ||W_k||$ for some $k \in K$. This implies that

$$||W_x||\frac{d(\phi(x),\phi(y))}{\rho(x,y)} \ge ||W_k||\frac{d(\phi(x),\phi(y))}{\rho(x,y)}$$

for all $x \in K$, $y \in Y$ with $x \neq y$. Hence, we have

$$\inf\left\{ ||W_x|| \frac{d(\phi(x), (\phi(y)))}{\rho(x, y)} : x \in K, y \in Y, x \neq y \right\}$$

$$\geq ||W_k|| \inf\left\{ \frac{d(\phi(x), \phi(y))}{\rho(x, y)} : x, y \in Y, x \neq y \right\}$$

$$\geq ||W_k|| \frac{1}{M} > 0.$$

This completes the proof.

5. Compact weighted composition operators

In this section we characterize compactness of weighted composition operators between vector-valued Lipschitz spaces. For this we need to generalize a result of Antonio and Moisés [16, Proposition 2.3] to vector-valued Lipschitz spaces. Our generalization uses the same idea presented in their paper. This result uses the fact that $Lip_0(X)$ is a dual space. The predual of $Lip_0(X)$ is $\mathcal{E}(X)$, the Arens-Eells space of X. For more details see [20]. We also refer the reader to [8, Theorem 4.3]. The vector-valued version of this result is true if the range space is a dual space as it is evident from a result due to Johnson.

Theorem 5.1 ([17, Theorem 3.1]). For any metric space X and any Banach space E, Lip(X, E) is a dual space whenever E is.

Throughout this section we will assume that E and F are dual Banach spaces. For $x \in X$ and r > 0, B(x, r) denotes the closed ball in X of radius r centered at x.

Lemma 5.2. Let E be a dual Banach space, and let (f_n) be a bounded sequence in $Lip_0(X, E)$. Then (f_n) has a subsequence which converges pointwise on X to a function $f \in Lip_0(X, E)$. Moreover, this convergence is uniform on totally bounded subset of X.

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Proof. Let (f_n) be bounded sequence in $Lip_0(X, E)$. Since $Lip_0(X, E)$ is a dual space whenever E is, [8, Theorem 4.3], its unit ball is w^* -compact (Banach-Alaoglu Theorem). Hence, there exist a subsequence (f_{n_k}) and a function $f \in Lip_0(X, E)$ such that $f_{n_k} \xrightarrow{w^*} f$. Now, since on bounded subsets of $Lip_0(X, E)$ the w^* -topology agrees with the topology of pointwise convergence [9, Corollary 3.8], we conclude that $f_{n_k} \to f$ pointwise on X.

For the second part, suppose that C is totally bounded subsets of X. Let $\varepsilon > 0$ and $N = \sup \{L(f), L(f_{n_k}) (k \in \mathbb{N})\}$. As C is totally bounded, there exists a finite set $\{x_1, x_2, \ldots, x_n\}$ in C such that $C \subset \bigcup_{i=1}^n B(x_i, \frac{\varepsilon}{3N})$. We choose k' large enough so that $||f_{n_k}(x_i) - f(x_i)|| < \frac{\varepsilon}{3}$ whenever $k \ge k'$ and $1 \le i \le n$. Given $x \in C$, we choose i such that $x \in B(x_i, \frac{\varepsilon}{3N})$. Now, for $k \ge k'$ we have

$$\begin{split} ||f_{n_k}(x) - f(x)|| &\leq ||f_{n_k}(x) - f_{n_k}(x_i)|| + ||f_{n_k}(x) - f(x_i)|| + ||f(x_i) - f(x)|| \\ &< L(f_{n_k})d(x, x_i) + \frac{\varepsilon}{3} + L(f)d(x, x_i) \\ &\leq 2N\frac{\varepsilon}{3N} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{split}$$

Therefore, $f_{n_k} \to f$ uniformly on C.

In order to prove our main theorem for compactness of WCO, we need the following proposition.

Proposition 5.3. Let E and F be dual Banach spaces. Let $W \in Lip(Y, B(E, F))$, $\phi : Y \to X$ be a base point preserving map, and $T : Lip_0(X, E) \to Lip_0(Y, F)$ be a WCO. Then T is compact if and only if for each bounded sequence $f_n \in Lip_0(X, E)$ which converge to 0 uniformly on totally bounded subsets of X, there exists a subsequence (f_{n_k}) such that $Tf_{n_k} \to 0$ as $k \to \infty$.

Proof. Let T be compact and let $f_n \in Lip_0(X, E)$ be a bounded sequence which converges uniformly to 0 on totally bounded subsets of X. The compactness of T implies that there exist a subsequence (f_{n_k}) and $g \in Lip_0(Y, F)$ such that $Tf_{n_k} \to g$ as $k \to \infty$. By Lemma 3.1, $Tf_{n_k} \to g$ pointwise on Y. On the other hand, $f_{n_k} \to 0$ uniformly on totally bounded subsets of X implies that $f_{n_k} \to 0$ pointwise on X (as $\{x\}$ is a totally bounded subsets of X). Thus,

$$||Tf_{n_k}(y)|| = ||W_y(f_{n_k}(\phi(y)))|| \le ||W_y||||f_{n_k}(\phi(y))|| \to 0, \text{ as } n \to \infty.$$

Therefore, $Tf_{n_k} \to 0$ pointwise on Y. Hence, g = 0.

For the converse part, let $f_n \in Lip_0(X, E)$ be a bounded sequence. By Lemma 5.2, there exist a subsequence (f_{n_k}) and $f \in Lip_0(X, E)$ such that $f_{n_k} \to f$ uniformly on totally bounded subset of X. This implies that $(f_{n_k} - f)$ has a subsequence $(f_{n_{k_l}} - f)$, such that $T(f_{n_{k_l}} - f) \to 0$ as $l \to \infty$. Hence, T is compact.

The next theorem gives a necessary and sufficient condition for the compactness of weighted composition operators.

Theorem 5.4. Let E and F be dual Banach spaces. Suppose that $W \in Lip(Y, B(E, F))$, $\phi : Y \to X$ is base point preserving map, and $\phi(Y \setminus N)$ be totally bounded in X.

Let $T: Lip_0(X, E) \to Lip_0(Y, F)$ be a WCO. Then T is compact if and only if

$$\lim_{d(\phi(x),\phi(y))\to 0} \|W_x\| \frac{d(\phi(x),\phi(y))}{\rho(x,y)} = 0.$$

Proof. Suppose that T is compact and that there exist $\varepsilon > 0$ and two sequence (x_n) and (y_n) in Y such that $x_n \neq y_n$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} d(\phi(x_n), \phi(y_n)) = 0$, but $\|W_{x_n}\| \frac{d(\phi(x_n), \phi(y_n))}{\rho(x_n, y_n)} \ge \varepsilon$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, we define the sequence $f_n : X \to \mathbb{K}$ by

$$f_n(t) = \begin{cases} d(t, \phi(y_n)), & \text{if } d(t, \phi(y_n)) \le d(\phi(x_n), \phi(y_n)), \\ d(\phi(x_n), \phi(y_n)), & \text{if } d(t, \phi(y_n)) \ge d(\phi(x_n), \phi(y_n)), \end{cases}$$

for all $t \in X$. We observe that $f_n \in Lip(X)$, $||f_n||_{\infty} \leq d(\phi(x_n), \phi(y_n))$ and $L(f_n) \leq 1$. The sequence $f_n \to 0$ uniformly, and hence, $f_n(x) \to 0$ for every $x \in X$. For each $n \in \mathbb{N}$ and fixed $e \in E$ with ||e|| = 1, we define the sequence $g_n : X \to E$ by

$$g_n(t) = [f_n(t) - f_n(x_0)]e.$$

The sequence $(g_n) \in Lip_0(X, E)$ and $g_n \to 0$ uniformly on X. Moreover, $L(g_n) = L(f_n) \leq 1$ for all $n \in \mathbb{N}$. By Proposition 5.3, there exists a subsequence (g_{n_k}) such that $Tg_{n_k} \to 0$ as $k \to \infty$, that is, $L(Tg_{n_k}) \to 0$ as $k \to \infty$. Moreover, the sequence $L(Tg_{n_k}) + |f_{n_k}(x_0)|L(W) \to 0$ as $k \to \infty$. This implies that there exists a k_0 such that $L(Tg_{n_k}) + |f_{n_k}(x_0)|L(W) < \frac{\varepsilon}{2}$ for all $k \geq k_0$. Now,

$$\begin{split} L(Tg_{n_k}) &\geq \frac{\|Tg_{n_k}(x_{n_k}) - Tg_{n_k}(y_{n_k})\|}{\rho(x_{n_k}, y_{n_k})} \\ &= \frac{\|W_{x_{n_k}}([f_{n_k}(\phi(x_{n_k})) - f_{n_k}(x_0)]e) - W_{y_{n_k}}([f_{n_k}(\phi(y_{n_k})) - f_{n_k}(x_0)]e)\|}{\rho(x_{n_k}, y_{n_k})} \\ &= \frac{\|W_{x_{n_k}}(d(\phi(x_{n_k}), \phi(y_{n_k}))e) + f_{n_k}(x_0)(W_{y_{n_k}} - W_{x_{n_k}})e\|}{\rho(x_{n_k}, y_{n_k})}. \end{split}$$

Thus, for all $k \ge k_0$,

$$\begin{split} \|W_{x_{n_{k}}}(e)\|\frac{d(\phi(x_{n_{k}}),\phi(y_{n_{k}}))}{\rho(x_{n_{k}},y_{n_{k}})} \\ &= \frac{\|W_{x_{n_{k}}}(d(\phi(x_{n_{k}}),\phi(y_{n_{k}}))e) + f_{n_{k}}(x_{0})(W_{y_{n_{k}}} - W_{x_{n_{k}}})e - f_{n_{k}}(x_{0})(W_{y_{n_{k}}} - W_{x_{n_{k}}})e\|}{\rho(x_{n_{k}},y_{n_{k}})} \\ &\leq L(Tg_{n_{k}}) + |f_{n_{k}}(x_{0})|\frac{\|W_{y_{n_{k}}} - W_{x_{n_{k}}}\|}{\rho(x_{n_{k}},y_{n_{k}})} \\ &\leq L(Tg_{n_{k}}) + |f_{n_{k}}(x_{0})|L(W) < \frac{\varepsilon}{2}. \end{split}$$

This implies that

$$\|W_{x_{n_k}}\|\frac{d(\phi(x_{n_k}),\phi(y_{n_k}))}{\rho(x_{n_k},y_{n_k})} < \frac{\varepsilon}{2}, \ \forall \ k \ge k_0,$$

a contradiction.

For the converse part, suppose that

$$\lim_{d(\phi(x),\phi(y))\to 0} \|W_x\| \frac{d(\phi(x),\phi(y))}{\rho(x,y)} = 0.$$

Let $f_n \in Lip_0(X, E)$ be a bounded sequence such that $f_n \to 0$ uniformly on totally bounded subsets of X. Let A > 0 with $L(f_n) < A$ for all $n \in \mathbb{N}$ and

$$B = \sup\left\{ \|W_x\| \frac{d(\phi(x), \phi(y))}{\rho(x, y)} : x, y \in Y, x \neq y \right\}.$$

We know that $B \leq 2||T||$ by Theorem 3.2. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$x, y \in Y, \ 0 < d(\phi(x), \phi(y)) < \delta \Rightarrow \|W_x\| \frac{d(\phi(x), \phi(y))}{\rho(x, y)} < \frac{\varepsilon}{2A}$$

Now, $f_n \to 0$ uniformly on $\phi(Y \setminus N)$, as $\phi(Y \setminus N)$ is totally bounded. Hence, there exists $k \in \mathbb{N}$ such that for all $y \in Y \setminus N$ and $n \ge k$, $||f_n(\phi(y))|| < \frac{\varepsilon}{C}$, where $C = 3(\frac{2B}{\delta} + \frac{1}{2} + L(W)).$

To prove that T is compact, we show that

$$\frac{\|Tf_n(x) - Tf_n(y)\|}{\rho(x, y)} < \varepsilon$$

for all $x, y \in Y$, with $x \neq y$, and $n \geq k$. This will imply that $L(Tf_n) \to 0$ as $n \to \infty$, see Proposition 5.3.

Let $x, y \in Y$, with $x \neq y$, and $n \geq k$. We consider the following cases. Case 1: $x, y \in Y \setminus N$ with $\phi(x) \neq \phi(y)$.

Then we have

$$\frac{\|Tf_n(x) - Tf_n(y)\|}{\rho(x, y)} = \frac{\|W_x(f_n(\phi(x))) - W_y(f_n(\phi(y)))\|}{\rho(x, y)} \\
\leq \frac{\|W_x\|\|f_n(\phi(x)) - f_n(\phi(y))\|}{d(\phi(x), \phi(y))} \frac{d(\phi(x), \phi(y))}{\rho(x, y)} + \|f_n(\phi(y))\|L(W).$$

If $0 < d(\phi(x), \phi(y)) < \delta$, then

$$\frac{\|Tf_n(x) - Tf_n(y)\|}{\rho(x, y)} < L(f_n)\frac{\varepsilon}{2A} + \frac{\varepsilon}{C}L(W) < \frac{\varepsilon}{2} + \frac{\varepsilon}{3} < \frac{5\varepsilon}{6}.$$

If
$$d(\phi(x), \phi(y)) \ge \delta$$
, then
 $||Tf(x) - Tf(y)||$

$$\frac{\|If_n(x) - If_n(y)\|}{\rho(x, y)} \leq \frac{\|W_x\| \|f_n(\phi(x)) - f_n(\phi(y))\|}{d(\phi(x), \phi(y))} \frac{d(\phi(x), \phi(y))}{\rho(x, y)} + \|f_n(\phi(y))\| L(W)$$

$$\leq \frac{\|f_n(\phi(x))\| + \|f_n(\phi(y))\|}{\delta} B + \frac{\varepsilon}{C} L(W)$$

$$\leq \frac{2\varepsilon B}{C\delta} + \frac{\varepsilon}{C} L(W) < \frac{\varepsilon}{2}.$$

Case 2: $x, y \in Y \setminus N$ with $x \neq y$ and $\phi(x) = \phi(y)$. The proof is obvious.

Case 3: $x \in Y \setminus N$ and $W_y = 0$. Then

$$\frac{\|Tf_n(x) - Tf_n(y)\|}{\rho(x, y)} \le \frac{\|W_x(f_n(\phi(x)))\|}{\rho(x, y)} = \frac{\|W_x - W_y\|}{\rho(x, y)} \|f_n(\phi(x))\| < \frac{\varepsilon}{2}.$$

Case 4: $y \in Y \setminus N$ and $W_x = 0$. The proof is similar to **Case 3**.

Case 5: $x, y \in Y$ with $x \neq y$ and $W_x = W_y = 0$. The proof is obvious. Combining all the cases we get $L(Tf_n) < \varepsilon$ for all $n \geq k$. Therefore, T is compact.

Theorem 5.5. Let E and F be dual Banach spaces, and let $W \in Lip(Y, B(E, F))$. Let $\phi : Y \to X$ be a base point preserving map such that $\phi(Y \setminus N)$ is totally bounded in X, and let $T : Lip_0(X, E) \to Lip_0(Y, F)$ be a WCO. If ϕ is supercontractive on $Y \setminus N$, then T is compact.

Proof. Suppose that ϕ is supercontractive on $Y \setminus N$. For given $\varepsilon > 0$, there exists $0 < \delta_0 < 1$ such that

$$x, y \in Y \setminus N, \ 0 < \rho(x, y) < \delta_0 \Rightarrow \frac{d(\phi(x), \phi(y))}{\rho(x, y)} \le \frac{\varepsilon}{1 + ||W||_s},$$

where $||W||_s = L(W) + ||W||_{\infty}$. Let $\delta = \frac{\varepsilon \delta_0}{1+||W||_s}$ and $x, y \in Y$ such that $0 < d(\phi(x), \phi(y)) < \delta$. To prove that T is compact, we show that $||W_x|| \frac{d(\phi(x), \phi(y))}{\rho(x,y)} < \varepsilon$, see Theorem 5.4.

We consider the following cases.

Case 1: $x, y \in Y \setminus N$ with $0 < \rho(x, y) < \delta_0$. Then

$$||W_x||\frac{d(\phi(x),\phi(y))}{\rho(x,y)} \le ||W||_{\infty} \frac{d(\phi(x),\phi(y))}{\rho(x,y)} < ||W||_{\infty} \frac{\varepsilon}{1+||W||_s} < \varepsilon.$$

Case 2: $x, y \in Y \setminus N$ with $\rho(x, y) \ge \delta_0$. Then

$$||W_x||\frac{d(\phi(x),\phi(y))}{\rho(x,y)} \le ||W||_{\infty} \frac{d(\phi(x),\phi(y))}{\delta_0} < ||W||_{\infty} \frac{\varepsilon \delta_0}{(1+||W||_s)\delta_0} < \varepsilon.$$

Case 3: $x \in Y \setminus N$ and $W_y = 0$. Then

$$||W_x||\frac{d(\phi(x),\phi(y))}{\rho(x,y)} = \frac{||W_x - W_y||}{\rho(x,y)}d(\phi(x),\phi(y)) < L(W)\delta = \frac{L(W)\varepsilon\delta_0}{1+||W||_s} < \varepsilon.$$

Case 4: Same as Case 3. Therefore,

$$\lim_{d(\phi(x),\phi(y))\to 0} \|W_x\| \frac{d(\phi(x),\phi(y))}{\rho(x,y)} = 0.$$

Theorem 5.6. Let E and F be dual Banach spaces, and let $W \in Lip(Y, B(E, F))$. Let $\phi : Y \to X$ be a base point preserving map such that $\phi(Y \setminus N)$ is totally bounded in X, and let $T : Lip_0(X, E) \to Lip_0(Y, F)$ be a WCO. If T is compact, then ϕ is supercontactive on compact subsets of $Y \setminus N$.

Proof. Assume that T is a compact operator. By Theorem 5.4, we have

$$\lim_{t(\phi(x),\phi(y))\to 0} \|W_x\| \frac{d(\phi(x),\phi(y))}{\rho(x,y)} = 0.$$

Let U be a non empty compact subset of $Y \setminus N$. Take $B = \inf_{y \in U} ||W_y||$, and let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that

$$x, y \in Y, \ 0 < d(\phi(x), \phi(y)) < \delta \Rightarrow ||W_x|| \frac{d(\phi(x), \phi(y))}{\rho(x, y)} < B\varepsilon$$

The map ϕ is uniformly continuous as it is Lipschitz on U, see corollary 3.3. Thus, there exits $\delta_1 > 0$ such that $d(\phi(x), \phi(y)) < \delta$ whenever $\rho(x, y) < \delta_1$.

Suppose that $x, y \in U$ with $0 < \rho(x, y) < \delta_1$. Then $d(\phi(x), \phi(y)) < \delta$. If $\phi(x) = \phi(y)$, then $\frac{d(\phi(x), \phi(y))}{\rho(x, y)} = 0 < \varepsilon$.

If $\phi(x) \neq \phi(y)$, then we have

$$\frac{d(\phi(x),\phi(y))}{\rho(x,y)} \le ||W_x|| \frac{d(\phi(x),\phi(y))}{B\rho(x,y)} < \frac{B\varepsilon}{B} = \varepsilon$$

Therefore, ϕ is supercontractive.

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