Journal of Nonlinear and Convex Analysis
Volume 26, Number 4, 2025, 781–796

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

CONVERGENCE RATE ANALYSIS OF VARIOUS FIXED-POINT ITERATIONS FOR GENERALIZED AVERAGED NONEXPANSIVE OPERATORS

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Abstract. This paper evaluates the rate of convergence of various fixed-point iterative techniques for a class of nonlinear operators inspired by problems in convex optimization, referred to as Generalized Averaged Nonexpansive (GAN) operators. We demonstrate the convergence of iterative techniques, including Mann, Ishikawa, Normal-S, and PV iterations, to fixed points of GAN operators. Additionally, we show that these methods achieve an exponential global convergence rate for GAN operators with a positive exponent less than 1 and establish that the rate depends on the exponents of generalized nonexpansiveness and Hölder regularity. Numerical experiments validate the theory, showing PV and Normal-S outperform Picard.

1. Introduction

From enhancing image quality to powering breakthroughs in machine learning, non-differentiable optimization problems are at the forefront of countless technological advancements. Fixed-point algorithms are essential tools for tackling these problems, with traditional approaches relying on contractive or averaged nonexpansive operators for their convergence analysis [2, 7–9]. However, these traditional methods are limited in their applicability, as not all optimization problems can be readily transformed into fixed-point problems using averaged nonexpansive operators. To address this limitation, Lin and Xu introduced the concept of the Generalized Averaged Nonexpansive operator [10], which, despite being weaker than averaged nonexpansiveness, still guarantees convergence of fixed-point iterations. Moreover, the GAN operator's exponent provides a mechanism for refining local convergence rates, potentially leading to faster convergence than traditional methods. This development raises a crucial and unexplored question: Can iterative methods beyond Picard iteration be effectively applied to GAN operators, potentially leading to faster and more efficient convergence?. This paper delves into this question, providing, for the first time, global convergence rates for the Mann, Ishikawa, Normal-S, and PV iterations applied to GAN operators, significantly advancing the understanding and applicability of GAN-based optimization. Our analysis also reveals that the PV iteration demonstrates superior convergence properties compared to the Picard iteration in the context of GAN operators.

2020 *Mathematics Subject Classification.* 47H09, 47H10.

Key words and phrases. Convex optimization, fixed-point iteration, convergence rate.

*The second author also wishes to thank the University Grants Commission (UGC), India, for providing financial assistance in the form of a Senior Research Fellowship.

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2. Preliminaries

Now, we will give the definition of the μ-GAN operator with exponent λ. Here, $\mathbb{R}^+$ denotes the set of all positive real numbers.

Definition 2.1 ([10])**.** An operator G : $\mathbb{R}^n \to \mathbb{R}^n$ is said to be generalized averaged nonexpansive if there exist $\lambda, \mu \in \mathbb{R}^+$ such that

$$\|Gx - Gy\|^\lambda + \mu\|(I - G)x - (I - G)y\|^\lambda \leq \|x - y\|^\lambda, for\ all\ x, y \in \mathbb{R}^n.$$

Let $Fix(G)$ denote the set of all fixed points of operator G .

One can easily verify that GAN operators are nonexpansive thus set of all fixed points of GAN operators are closed and convex set.

The next theorem, proved in [10], provides the rate of convergence of the fixed-point iterations in GAN with exponent λ for some $\lambda \in (0, 1)$ operators for the Picard case.

Theorem 2.2 ([10])**.** *If G is GAN such that $Fix(G) \neq \phi$ and exponent λ for some $\lambda \in (0, 1)$, then for any initial vector $p_0 \in \mathbb{R}^n$, the Picard sequence p_n of G converges to some $p \in Fix(G)$,and $\|p_n - p\| = o(n^{(\lambda-1)/\lambda})$.*

Now we will define Hölder regularity, which we will denote by HR.

Definition 2.3 ([2])**.** An operator G is said to be **Hölder regular (HR)** if $Fix(G)$ is nonempty, and there exists $\mu \in \mathbb{R}^+$ such that:

$$d(x, Fix(G)) \leq \mu\|x - Gx\|, \quad \text{for all } x \in \mathbb{R}^n,$$

where $d(x, Fix(G))$ denotes the distance from x to the set Fix(G).

The next theorem provides the rate of convergence for exponent $\lambda_1 \geq 1$.

Theorem 2.4 ([10])**.** *If G is GAN such that $Fix(G) \neq \phi$, exponent $\lambda_1 \geq 1$and HR with exponent $\lambda_2 \in \mathbb{R}^+$, then for any initial vector $p_0 \in \mathbb{R}^n$, the Picard algorithm $\{p_n\}$ for G converges to some $p \in Fix(G)$, and there exists $\rho \in (0, 1)$ such that*

$$\|p_n - p\| = \begin{cases} O\left(n^{-\frac{\lambda_2}{\lambda_1(1-\lambda_2)}}\right), & 0 < \lambda_2 < 1 \\ O\left(\rho^n\right), & \lambda_2 \geq 1 \end{cases}$$

A useful theorem that will help in constructing example.

Theorem 2.5 ([10])**.** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous convex function that is differentiable. If there exist $L_1 \geq L_2 > 0$ such that*

$$L_2\|x - y\| \leq \|\nabla f(x) - \nabla f(y)\| \leq L_1\|x - y\|, \quad for\ all\ x, y \in \mathbb{R}^n$$

then, for $\alpha \in \left(0, \frac{2}{L_1}\right)$, we have $S_1 = I - \alpha\nabla f$ is both GAN with exponent 1 and HR with exponent 1.

Definition 2.6. A Banach space X with norm $\|\cdot\|$ is called uniformly convex if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$ with $\|x\| = \|y\| = 1$, the condition

$$\|x - y\| \geq \epsilon \implies \left\|\frac{x + y}{2}\right\| \leq 1 - \delta.$$

Throughout this study, we will assume that our space is uniformly convex.

Definition 2.7. Assume $\mathcal{K}$ is a real Hilbert space and $F \subseteq \mathcal{K}$ is a nonempty, closed, and convex set, and suppose $\{p_k\}_{k=0}^{\infty}$ is a sequence in $\mathcal{K}$. We say that $\{p_k\}_{k=0}^{\infty}$ is Fejér monotone with respect to F if

$$\|p_{k+1} - z\| \leq \|p_k - z\| \tag{2.1}$$

for all $z \in F$ and every integer $k = 0, 1, 2, \ldots$

Now, we present a lemma that we are going to use in later results.

Lemma 2.8 ([14]). *Let V be a uniformly convex Banach space and $0 < \alpha \leq r_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Let τ_n and σ_n be two sequences in V such that $\limsup_{n\to\infty} \|\tau_n\| \leq c$, $\limsup_{n\to\infty} \|\sigma_n\| \leq c$, and $\limsup_{n\to\infty} \|r_n\tau_n + (1-r_n)\sigma_n\| = c$ hold for some $c \geq 0$. Then,*

$$\lim_{n\to\infty} \|\tau_n - \sigma_n\| = 0.$$

The iterative algorithms described below are referred to as the Mann [11], Ishikawa [5], Normal-S [13], and PV [4] algorithms, respectively, for the self-mapping G defined on V, where V is a closed and convex subset of a Banach space B.

$$\begin{cases} p_0 \in V, \\ p_{n+1} = (1 - r_n)p_n + r_n G p_n, n \in \mathbf{Z}_+ \end{cases} \tag{2.1}$$

$$\begin{cases} p_0 \in V, \\ p_{n+1} = (1 - r_n)p_n + r_n G q_n, \\ q_n = (1 - s_n)p_n + s_n G p_n, \ n \in \mathbf{Z}_+ \end{cases} \tag{2.2}$$

$$\begin{cases} p_0 \in V, \\ p_{n+1} = G((1 - r_n)p_n + r_n G p_n), \ n \in \mathbf{Z}_+ \end{cases} \tag{2.3}$$

$$\begin{cases} p_0 \in V, \\ p_{n+1} = G q_n, \\ q_n = G((1 - r_n)G^2 p_n + r_n G p_n), \quad n \in \mathbf{Z}_+, \end{cases} \tag{2.4}$$

3. Main results

Here, we present key Lemmas essential for the main results.

Lemma 3.1. *Let $\{p_n\}$ be the sequence generated by the PV algorithm for a GAN operator G with $Fix(G) \neq \emptyset$. Then, $\{p_n\}$ is a Fejér monotone (FM) sequence with respect to $Fix(G)$, and $\lim_{n\to\infty} \|p_n - p\|$ exists for all $p \in Fix(G)$.*

Proof. Since G is nonexpansive and $\|\cdot\|$ is convex, we have

$$\begin{aligned}\|p_{n+1}-p\| &= \|Gq_n - Gp\| \leq \|q_n - p\|,\\ \|q_n - p\| &= \|G\left((1-r_n)G^2p_n + r_nGp_n\right) - p\|\\ &\leq (1-r_n)\|G^2p_n - p\| + r_n\|Gp_n - p\|\\ &\leq \|p_n - p\|.\end{aligned}$$

Thus, $\|p_{n+1}-p\| \leq \|p_n - p\|$.

Hence, $\{p_n\}$ is Fejér monotone with respect to $Fix(G)$. Moreover, the sequence $\{\|p_n - p\|\}$, being decreasing and bounded below by 0, converges. □

Similarly, Mann, Normal-S, and Ishikawa iterations are Fejér monotone with respect to $Fix(G)$. Next, we prove a Lemma for the main result.

Theorem 3.2. *Let G be a GAN operator with $Fix(G) \neq \emptyset$, and let $\{p_n\}$ be the sequence generated by the PV iteration. Then,*

$$\lim_{n\to\infty} \|p_n - Gp_n\| = \lim_{n\to\infty} \|p_n - G^2p_n\| = 0$$

.

Proof. As, we have $\lim_{n\to\infty} \|p_n - p\|$ exist from Lemma 3.1 . Let us assume that $\lim_{n\to\infty} \|p_n - p\| = c$. We will use Lemma 2.8 with $\tau_n = G^2p_n - p$ and $\sigma_n = Gp_n - p$. Since G is nonexpansive, we have

$$\|\tau_n\| = \|G^2p_n - p\| \leq \|p_n - p\| \text{ and } \|\sigma_n\| = \|Gp_n - p\| \leq \|p_n - p\|.$$

Hence,

$$\limsup_{n\to\infty} \|\tau_n\| \leq \limsup_{n\to\infty} \|p_n - p\| \text{ and } \limsup_{n\to\infty} \|\sigma_n\| = \limsup_{n\to\infty} \|Gp_n - p\| = c.$$

Now, according to the Lemma 2.8 we need to show that

$$\limsup_{n\to\infty} \|(1-r_n)\tau_n + r_n\sigma_n\| = c.$$

Substituting the value of τ_n and σ_n this is equivalent to show that

$$\limsup_{n\to\infty} \|(1-r_n)G^2p_n + r_nGp_n - p\| = c.$$

Using traingle inequality and G is nonexpansive

$$\begin{aligned}\|(1-r_n)G^2p_n + r_nGp_n - p\| &\leq (1-r_n)\|G^2p_n - p\| + r_n\|Gp_n - p\|\\ &= \|p_n - p\|.\end{aligned}$$

Taking limit superior on both side and using the fact $\lim_{n\to\infty} \|p_n - p\| = c$ we get

$$\limsup_{n\to\infty} \|(1-r_n)G^2p_n + r_nG^2p_n - p\| \leq c. \tag{3.1}$$

Now, consider

$$\|p_{n+1}-p\| = \|Gq_n - Gp\| \leq \|q_n - p\| = \|G\left((1-r_n)G^2p_n + r_nGp_n\right) - Gp\|$$

$$\implies \qquad c \leq \limsup_{n\to\infty} \|(1-r_n)G^2p_n + r_nGp_n - p\|. \tag{3.2}$$

Using (3.1) and (3.2) we get

$$\limsup_{n\to\infty} \|(1-r_n)G^2p_n + r_nGp_n - p\| = c.$$

Therefore, from Lemma 2.8 we have

$$\lim_{n\to\infty} \|G^2p_n - Gp_n\| = 0.$$

Now, using the triangle inequality, one can obtain

$$\|Gq_n - q_n\| \le \|q_n - Gp_n\| + (1-r_n)\|G^2p_n - Gp_n\|. \tag{3.3}$$

Now, consider $\|q_n - Gp_n\|$

$$\begin{aligned}\|q_n - Gp_n\| &= \|G\left((1-r_n)G^2p_n + r_nGp_n\right) - Gp_n\| \\ &\le (1-r_n)\|G^2p_n - Gp_n\| + \|G^2p_n - Gp_n\|.\end{aligned}$$

We have obtained

$$\|q_n - Gp_n\| \le (2-r_n)\|G^2p_n - Gp_n\|. \tag{3.4}$$

Now, since $\{r_n\}$ is a bounded sequence and $\lim_{n\to\infty}\|G^2p_n - Gp_n\| = 0$, it follows from (3.4) that $\lim_{n\to\infty}\|q_n - Gp_n\| = 0$. Similarly, using (3.3) and (3.4), we get $\lim_{n\to\infty}\|q_n - Gq_n\| = 0$, and hence $\lim_{n\to\infty}\|p_{n+1} - Gp_{n+1}\| = 0$.

□

Similar results can be established for the Ishikawa iteration in a similar manner. The following lemma holds true for all iterations discussed in this study.

Lemma 3.3. *Let $\{p_n\}$ be any Fejér monotone sequence with respect to $Fix(G)$. Then, $\lim_{n\to\infty} d(p_n, Fix(G)) = 0$ if and only if $\{p_n\}$ converges to a point in $Fix(G)$, where $d(p_n, Fix(G)) = \inf\{\|p_n - a\| : a \in Fix(G)\}$.*

Proof. One side of the implication is easy to see, we will prove the converse. Now to show converse we have $\lim_{n\to\infty} d(p_n, Fix(G)) = 0$ thus for any $\frac{\epsilon}{2} > 0$ we have $d(p_n, Fix(G)) < \frac{\epsilon}{2}\ \forall\ n \ge n_1$for some $n_1 \in \mathbb{N}$ which implies that

$$\inf\{\|p_n - p\|\, p \in Fix(G)\} < \frac{\epsilon}{2} \implies \inf\{\|p_{n_1} - p\|\, p \in Fix(G)\} < \epsilon/2.$$

Therefore, there exists $p_1 \in F(G)$ such that

$$\|p_{n_1} - p_1\| < \frac{\epsilon}{2}. \tag{3.5}$$

Using the fact that $\{p_n\}$ is Fejér monotone sequence we have $\forall\ n, m \ge n_1$

$$\begin{aligned}\|p_{n+m} - p_n\| &\le \|p_{n+m} - p_1\| + \|p_n - p_1\| \\ &\le \|p_{n_1} - p_1\| + \|p_{n_1} - p_1\| = 2\,\|p_{n_1} - p_1\| < \epsilon.\end{aligned}$$

Thus, $\{p_n\}$ is a Cauchy in Y. Since Y is closed, $\lim_{n\to\infty} p_n = q$ for some $q \in Y$.

Now, as

$$\lim_{n\to\infty} d\left(p_n, Fix(G)\right) = 0 \implies \lim_{n\to\infty} \inf\{\|p_n - p\| : p \in Fix(G)\} = 0.$$

Using the definition of infimum we get for every $\epsilon > 0\ \exists\ p_\epsilon \in Fix(G)$ such that

$$\|p_n - p_\epsilon\| < \frac{\epsilon}{2}\ \forall\ n \ge M. \tag{3.6}$$

As we have

$$\lim_{n\to\infty} p_n = q \implies \|p_n - q\| < \frac{\epsilon}{2} \quad \forall n \geq M_1, \quad \text{where } M_1 \in \mathbb{N}. \tag{3.7}$$

Using (3.6) and (3.7)

$$\|q - p_\epsilon\| \leq \|q - p_n\| + \|p_n - p_\epsilon\| < \epsilon \;\forall\; n \geq max\{M, M_1\}.$$

Therefore, we have

$$\inf\{\|q-p\| : p \in Fix(G)\} = 0 \implies d(q, Fix(G)) = 0 \implies q \in Fix(G).$$

□

For real sequences $\{p_n\}$ with $p_n \geq 0$ and $\{q_n\}$ with $q_n > 0$, both tending to zero, we write $p_n = o(q_n)$ if $\lim_{n\to\infty} \frac{p_n}{q_n} = 0$. If there exist constants $c > 0$ and $K \in \mathbb{N}_0$ such that $p_n \leq cq_n$ for all $n \geq K$, we write $p_n = O(q_n)$.

Next, we present the main result, beginning with the results for Mann, Ishikawa, and Normal-S.

Theorem 3.4. *If G is a GAN operator with $Fix(G) \neq \phi$, and HR with exponent $\lambda_1 \geq 1$ and $\lambda_2 \in \mathbb{R}^+$ respectively. Then, for any initial vector $p_0 \in \mathbb{R}^n$, the sequence $\{p_n\}$ generated by the Mann and Ishikawa algorithms (with $s_n < \frac{1}{2}$ and $0 < M < r_n < 1$) and the Normal-S algorithm (with $1 - 2r_n > M_3 > 0$) converges to some $p \in Fix(G)$. Moreover, there exist ρ_1, ρ_2, and $\rho_3 \in (0,1)$ such that the rate of convergence for each iteration is as follows:*

*for **Mann***

$$\|p_n - p\| = \begin{cases} O\left(n^{-\frac{\lambda_2}{\lambda_1(1-\lambda_2)}}\right), & 0 < \lambda_2 < 1 \\ O\left(\rho_1^n\right), & \lambda_2 \geq 1 \end{cases} \tag{3.8}$$

*for **Ishikawa***

$$\|p_n - p\| = \begin{cases} O\left(n^{-\frac{\lambda_2}{\lambda_1(1-\lambda_2)}}\right), & 0 < \lambda_2 < 1 \\ O\left(\rho_2^n\right), & \lambda_2 \geq 1 \end{cases} \tag{3.9}$$

*for **Normal-S***

$$\|p_n - p\| = \begin{cases} O\left(n^{-\frac{\lambda_2}{\lambda_1(1-\lambda_2)}}\right), & 0 < \lambda_2 < 1 \\ O\left(\rho_3^n\right), & \lambda_2 \geq 1 \end{cases} \tag{3.10}$$

Proof. For the **Mann** Iteration from the definition of GAN, we have

$$\|Gx - Gy\|^{\lambda_1} + \mu_1\|(I-G)x - (I-G)y\|^{\lambda_1} \leq \|x - y\|^{\lambda_1}, for\ all\ x, y \in R^n.$$

Substituting $x = p_n$ and $y = p \in \mathrm{Fix}(G)$, we obtain

$$\|Gp_n - p\|^{\lambda_1} \leq -\mu_1\|p_n - Gp_n\|^{\lambda_1} + \|p_n - p\|^{\lambda_1}.$$

As $r_n > 0$, multiplying the entire equation by r_n yields:

$$r_n\|Gp_n - p\|^{\lambda_1} \leq -r_n\mu_1\|p_n - Gp_n\|^{\lambda_1} + r_n\|p_n - p\|^{\lambda_1}. \tag{3.11}$$

From the Mann iteration (2.1) and using that $\|\cdot\|^{\lambda_1}$ is a convex function, we have

$$\|p_{n+1}-p\|^{\lambda_1} \leq (1-r_n)\|p_n-p\|^{\lambda_1} + r_n\|Gp_n-p\|^{\lambda_1}. \tag{3.12}$$

From equation (3.11)and (3.12),

$$\|p_{n+1}-p\|^{\lambda_1} \leq -r_n\mu_1\|p_n-Gp_n\|^{\lambda_1} + \|p_n-p\|^{\lambda_1}. \tag{3.13}$$

From, the Hölder inequality we have $d(p_n, \mathrm{Fix}(G)) \leq \mu_2\|p_n-Gp_n\|^{\lambda_2}$.

Case 1. If $\lambda_2 \geq 1$.

As we have $\|p_n-Gp_n\| \to 0$, for sufficiently large n, we have

$$\|p_n-Gp_n\| < 1 \implies \|p_n-Gp_n\|^{\lambda_2} \leq \|p_n-Gp_n\|.$$

Hence, using Hölder's inequality, we obtain

$$d(p_n, \mathrm{Fix}(G)) = \|p_n - P_{\mathrm{Fix}(G)}\| \leq \mu_2\|p_n-Gp_n\|.$$

Taking $d_n = d\,(p_n, \mathrm{Fix}(G))$ and assuming $\lambda_1 \geq 1$, we obtain $\frac{d_n^{\lambda_1}}{\mu_2^{\lambda_1}} \leq \|p_n-Gp_n\|^{\lambda_1}$. Multiplying both sides by $-r_n\mu_1$ and putting in(3.13) ,we obtain

$$\|p_{n+1}-p\|^{\lambda_1} \leq -r_n\mu_1 d_n^{\lambda_1}\mu_2^{-\lambda_1} + \|p_n-p\|^{\lambda_1}.$$

Taking $p = P_{Fix(G)}(p_n)$ in above equation we obtain

$$\|p_{n+1}-P_{Fix(G)}(p_n)\|^{\lambda_1} \leq -r_n\mu_1\mu_2^{-\lambda_1}d_n^{\lambda_1} + r_n d_n^{\lambda_1}.$$

Using the definition of projection, d_n, and the above inequality, we obtain that

$$d_{n+1}^{\lambda_1} \leq -r_n\mu_1\mu_2^{-\lambda_1}d_n^{\lambda_1} + d_n^{\lambda_1}.$$

Since $0 < m < r_n$, it follows that $-r_n < -m$ for all $n \in \mathbb{N}$.

$$d_{n+1}^{\lambda_1} \leq -m\mu_1\mu_2^{-\lambda_1}d_n^{\lambda_1} + d_n^{\lambda_1}. \tag{3.14}$$

Case 2. If $0 < \lambda_2 < 1$ we get $d_n \leq \mu_2\|p_n-Gp_n\|^{\lambda_2} \implies \left(\frac{d_n}{\mu_2}\right)^{\frac{\lambda_1}{\lambda_2}} \leq \|p_n-Gp_n\|^{\lambda_1}$.

Now, proceeding in the same way as in **Case 1** ,we obtain

$$d_{n+1}^{\lambda_1} \leq -m\mu_1\mu_2^{\frac{-\lambda_1}{\lambda_2}}d_n^{\frac{\lambda_1}{\lambda_2}} + d_n^{\lambda_1}. \tag{3.15}$$

We now derive a similar inequality for the **Ishikawa**(2.2) iteration.

Using Definition 2, we have

$$\|Gq_n-p\|^{\lambda_1} \leq -\mu_1\|q_n-G(q_n)\|^{\lambda_1} + \|q_n-p\|^{\lambda_1}. \tag{3.16}$$

From the Ishikawa iteration, we can easily show that

$$\|p_{n+1}-p\|^{\lambda_1} \leq (1-r_n)\|p_n-p\|^{\lambda_1} + \alpha_n\|Gq_n-p\|^{\lambda_1}. \tag{3.17}$$

From equation (3.16)and (3.17), we obtain

$$\|p_{n+1}-p\|^{\lambda_1} \leq (1-r_n)\|p_n-p\|^{\lambda_1} - r_n\mu_1\|q_n-G(q_n)\|^{\lambda_1} + r_n\|q_n-p\|^{\lambda_1}. \tag{3.18}$$

Now, as

$$\|q_n-p\|^{\lambda_1} \leq (1-s_n)\|p_n-p\|^{\lambda_1} + s_n\|Gp_n-p\|^{\lambda_1}.$$

Multiplying above equation by α_n and using (3.18),we get

$$\|p_{n+1}-p\|^{\lambda_1}-(1-r_n)\|p_n-p\|^{\lambda_1}\leq -r_n\mu_1\|q_n-G(q_n)\|^{\lambda_1}+r_n(1-s_n)\|p_n-p\|^{\lambda_1} + r_ns_n\|p_n-p\|^{\lambda_1}$$

$$\|p_{n+1}-p\|^{\lambda_1}\leq -r_n\mu_1\|q_n-G(q_n)\|^{\lambda_1}+\|p_n-p\|^{\lambda_1}. \tag{3.19}$$

Using nonexpansivity of G and putting value of q_n, we have

$$\begin{aligned}\|p_n-Gp_n\| &\leq \|p_n-q_n\|+\|Gq_n-q_n\|+\|Gp_n-Gq_n\|\\ \|p_n-Gp_n\| &\leq 2\|p_n-((1-s_n)p_n+s_nGp_n)\|+\|Gq_n-q_n\|\end{aligned} \tag{3.20}$$

$$-\mu_1r_n\|Gq_n-q_n\|^{\lambda_1}\leq -\mu_1r_n(1-2(s_n))^{\lambda_1}\|p_n-Gp_n\|^{\lambda_1}.$$

Using (3.19) and above equation, we get

$$\|p_{n+1}-p\|^{\lambda_1}\leq -\mu_1r_n(1-2(s_n))^{\lambda_1}\|p_n-Gp_n\|^{\lambda_1}+\|p_n-p\|^{\lambda_1}$$

as, $0<M<r_n<1$, $1-2(s_n)>M_1$ and let $A={M_1}^{\lambda_1}$ thus, we get

$$\|p_{n+1}-p\|^{\lambda_1}\leq -\mu_1MA\|p_n-Gp_n\|^{\lambda_1}+\|p_n-p\|^{\lambda_1}.$$

Using Hölder's inequality and following the steps of Mann case, we obtain that
Case1: $\lambda_2\geq 1$

$$d_{n+1}^{\lambda_1}\leq -MA\mu_1\mu_2^{-\lambda_1}d_n^{\lambda_1}+d_n^{\lambda_1}$$

Case2: $0<\lambda_2<1$

$$d_{n+1}^{\lambda_1}\leq -MA\mu_1\mu_2^{\frac{-\lambda_1}{\lambda_2}}d_n^{\lambda_1}+d_n^{\lambda_1}$$

We now derive a similar inequality for the **Normal-S**(2.3) iteration.
In the definition (2) of GAN, put $x=q_n$ and $y=p$ for some $p\in \mathrm{Fix}(G)$.

$$\|p_{n+1}-p\|^{\lambda_1}=\|Gq_n-p\|^{\lambda_1}\leq -\mu\|q_n-G(q_n)\|^{\lambda_1}+\|q_n-p\|^{\lambda_1}. \tag{3.21}$$

$$\|q_n-p\|^{\lambda_1}\leq (1-r_n)\|p_n-p\|^{\lambda_1}+r_n\|Gp_n-p\|^{\lambda_1}.$$

Using the definition of GAN, we obtain:

$$\|q_n-p\|^{\lambda_1}\leq \|p_n-p\|^{\lambda_1}-r_n\mu_1\|p_n-Gp_n\|^{\lambda_1}. \tag{3.22}$$

Further, we have:

$$\|p_n-Gp_n\|\leq \|q_n-p_n\|+\|q_n-Gp_n\|\leq \|q_n-p_n\|+\|q_n-Gq_n\|+\|Gq_n-Gp_n\|.$$

Using the nonexpansivity of the GAN operator, we get:

$$\begin{aligned}\|p_n-Gp_n\| &\leq 2\|q_n-p_n\|+\|q_n-Gq_n\|\\ (1-2r_n)\|p_n-Gp_n\| &\leq \|q_n-Gq_n\|.\end{aligned}$$

As we have r_n such that $1-2r_n>M_3>0$.

$$-\mu_1\|q_n-Gq_n\|^{\lambda_1}\leq -\mu_1(1-2r_n)^{\lambda_1}\|p_n-Gp_n\|^{\lambda_1}. \tag{3.23}$$

As, $1-2r_n>M_3>0$ and $-(1-2r_n)^{\lambda_1}<-M_3^{\lambda_1}=-A_1$ (say).
Using (3.21), (3.22) and (3.23), we obtain

$$\|p_{n+1} - p\|^{\lambda_1} \leq -\mu_1(A_1 + m)\|p_n - Gp_n\|^{\lambda_1} + \|p_n - p\|^{\lambda_1}.$$

Using Hölder's inequality and following the same steps as in the Mann case, we get:

Case 1: $\lambda_2 \geq 1$

$$d_{n+1}^{\lambda_1} \leq -\mu_1(A_1 + m)\mu_2^{-\lambda_1} d_n^{\lambda_1} + d_n^{\lambda_1}.$$

Case 2: $0 < \lambda_2 < 1$

$$d_{n+1}^{\lambda_1} \leq -\mu_1(A_1 + m)\mu_2^{\frac{-\lambda_1}{\lambda_2}} d_n^{\lambda_1} + d_n^{\lambda_1}.$$

Now, consider first **case 1** for all the iterations. Let $a_n = {d_n}^{\lambda_1}$, then for each of the cases above we have $a_{n+1} \leq (1 - \alpha)a_n$ where α varies for each iteration as,

(1) **Mann** $\alpha = m\mu_1\mu_2^{-\lambda_1}$
(2) **Ishikawa** $\alpha = MA\mu_1\mu_2^{-\lambda_1}$
(3) **Normal-S** $\alpha = (m + A)\mu_1\mu_2^{-\lambda_1}$.

Further, as $a_{n+1} > 0 \implies 1 - \alpha > 0 \implies 0 < 1 - \alpha < 1$, and

$$a_{n+1} \leq (1 - \alpha)a_n \implies a_{n+1} \leq (1 - \alpha)^{n-m+1} a_m,$$

we obtain

$$d_{n+1} \leq (1 - \alpha)^{\frac{n-m+1}{\lambda_1}} d_m.$$

Therefore, as $n \to \infty$, we get $d_n \to 0$ for each iteration.

Now, as $d_n = d(p_n, \mathrm{Fix}(G))$, using Lemma 3.3, we can conclude that each iteration converges to the fixed point of G. Since all the sequences are Fejér monotone (FM), for any $m > n$,

$$\begin{aligned} \|p_m - P_{Fix(G)}(p_n)\| &\leq \|p_{m-1} - P_{Fix(G)}(p_n)\| \leq \|p_{m-2} - P_{Fix(G)}(p_n)\| \dots \\ &\leq \|p_n - P_{Fix(G)}(p_n)\| = d(p_n, Fix(G)) = d_n. \end{aligned} \tag{3.24}$$

Now, as $m \to \infty$ we have $p_m \to p$ a fixed point then

$$\|p_m - P_{Fix(G)}(p_n)\| \to \|p - P_{Fix(G)}(p_n)\|.$$

Now, in (3.24) keeping n fixed and taking $m \to \infty$,we get

$$\|p - P_{Fix(G)}(p_n)\| \leq d_n. \tag{3.25}$$

Therefore, for sufficiently large n, we have using (3.25)

$$\|p_n - p\| \leq \|p_n - P_{Fix(G)}(p_n)\| + \|p - P_{Fix(G)}(p_n)\| \leq 2d_n \leq 2d_m(1 - \alpha)^{\frac{n-m}{\lambda_1}}. \tag{3.26}$$

Hence, for some C, we have

$$\|p_n - p\| \leq C_1(1 - \alpha)^{\frac{n-m}{\lambda_1}} = C\theta^n \tag{3.27}$$

where $\theta = (1 - \alpha)^{\frac{1}{\lambda_1}}$. From the values of α for different iterations, one can obtain ρ_1, ρ_2, and ρ_3 from the equations (3.8), (3.9), and (3.10).

Now, consider **Case2** for all the iterations

Mann

$$d_{n+1}^{\lambda_1} \leq -m\mu_1\mu_2^{\frac{-\lambda_1}{\lambda_2}} d_n^{\lambda_1} + d_n^{\lambda_1}.$$

Ishikawa

$$d_{n+1}^{\lambda_1} \leq -MA\mu_1\mu_2^{\frac{-\lambda_1}{\lambda_2}} d_n^{\lambda_1} + d_n^{\lambda_1}.$$

Normal-S

$$d_{n+1}^{\lambda_1} \leq -\mu_1(A+m)\mu_2^{\frac{-\lambda_1}{\lambda_2}} d_n^{\lambda_1} + d_n^{\lambda_1}.$$

We can make a general case as

$$d_{n+1}^{\lambda_1} \leq -\theta d_n^{\lambda_1 \cdot \nu} + d_n^{\lambda_1}$$

where ν varies according to the iteration and $\nu = \frac{1}{\lambda_2} > 1$. Now, we can proceed similarly as in Proposition 4.12 of [10]to get

$$\|p_n - p\| = O\left(k^{-\frac{\lambda_2}{\lambda_1(1-\lambda_2)}}\right), 0 < \lambda_2 < 1.$$

□

Now, we will prove the result for the PV iteration.

Theorem 3.5. *Let G be a GAN operator with $Fix(G) \neq \phi$ and HR with exponent $\lambda_1 \geq 1$, and $\lambda_2 \in \mathbb{R}^+$ respectively. Then, for any initial vector $p_0 \in \mathbb{R}^n$, the PV algorithm generates the sequence $\{p_n\}$, which converges to some $p \in Fix(G)$. Moreover, there exists $\rho_4 \in (0,1)$ such that*

$$\|p_n - p\| = \begin{cases} O\left(n^{-\frac{\lambda_2}{\lambda_1(1-\lambda_2)}}\right), & 0 < \lambda_2 < 1, \\ O\left(\rho_4^n\right), & \lambda_2 \geq 1. \end{cases}$$

Proof. For the sake of conveinence, we rewrite the PV (2.4) iteration as

$$\begin{cases} p_{n+1} = Gq_n, \\ q_n = Ga_n \\ a_n = (1-r_n)G^2p_n + r_nGp_n, \quad n \in Z_+. \end{cases} \tag{3.28}$$

Takeing $x = q_n$and $y = p$ for some $p \in Fix(G)$ we get

$$\|p_{n+1} - p\|^{\lambda_1} = \|Gq_n - p\|^{\lambda_1} \leq -\mu_1\|q_n - G(q_n)\|^{\lambda_1} + \|q_n - p\|^{\lambda_1}. \tag{3.29}$$

As, $\|q_n - p\|^{\lambda_1} = \|Ga_n - p\|^{\lambda_1}$ using the definition of the GAN ,we have

$$\begin{aligned} \|q_n - p\|^{\lambda_1} &\leq \|a_n - p\|^{\lambda_1} - \mu_1\|a_n - Ga_n\|^{\lambda_1} \\ \|a_n - p\|^{\lambda_1} &= \|(1-r_n)G^2p_n + r_nGp_n - p\|^{\lambda_1}. \end{aligned} \tag{3.30}$$

Using the convexity of $\|.\|^{\lambda_1}$,we obtain

$$\|a_n - p\|^{\lambda_1} \leq (1-r_n)\|G^2p_n - p\|^{\lambda_1} + r_n\|Gp_n - p\|^{\lambda_1}. \tag{3.31}$$

Using the definition of GAN , we have

$$\|a_n - p\|^{\lambda_1} \leq -(1-r_n)\mu_1\|G^2p_n - Gp_n\|^{\lambda_1} + \|Gp_n - p\|^{\lambda_1}. \tag{3.32}$$

Using (3.30) and (3.32)

$$\|q_n - p\|^{\lambda_1} \leq -(1-r_n)\mu_1\|G^2p_n - Gp_n\|^{\lambda_1} + \|Gp_n - p\|^{\lambda_1} - \mu_1\|a_n - Ga_n\|^{\lambda_1}. \tag{3.33}$$

In the definition of GAN putting $x = p_n$ and $y = p$ for some $p \in Fix(G)$, we get

$$\|Gp_n - p\|^{\lambda_1} \leq -\mu_1\|p_n - Gp_n\|^{\lambda_1} + \|p_n - p\|^{\lambda_1}. \tag{3.34}$$

Using (3.34) and (3.33) ,we get

$$\begin{aligned}\|q_n - p\|^{\lambda_1} \leq &-(1-r_n)\mu_1\|G^2p_n - Gp_n\|^{\lambda_1} - \mu_1\|p_n - Gp_n\|^{\lambda_1} \\ &+ \|p_n - p\|^{\lambda_1} - \mu_1\|a_n - Ga_n\|^{\lambda_1}.\end{aligned} \tag{3.35}$$

Using (3.29) and (3.35),we have

$$\begin{aligned}\|p_{n+1} - p\|^{\lambda_1} \leq &-\mu_1\|q_n - G(q_n)\|^{\lambda_1} - (1-r_n)\mu_1\|G^2p_n - Gp_n\|^{\lambda_1} \\ &- \mu_1\|p_n - Gp_n\|^{\lambda_1} + \|p_n - p\|^{\lambda_1} - \mu_1\|a_n - Ga_n\|^{\lambda_1}.\end{aligned} \tag{3.36}$$

Case 1. In the Hölder inequality taking $x = Gp_n$,we get,

$$d(Gp_n, FixG) \leq \mu_2\|Gp_n - G^2p_n\|^{\lambda_2}.$$

For the case $\lambda_2 \geq 1$, since $\|Gp_n - G^2p_n\| \to 0$ as $n \to \infty$, it follows that there exists $n_0 \in \mathbb{N}$ such that $\|Gp_n - G^2p_n\| \leq 1$ for all $n \geq n_0$.

Hence, we have

$$\begin{aligned}d(Gp_n, FixG) &\leq \mu_2\|Gp_n - G^2p_n\| - (1-r_n)\mu_1\|Gp_n - G^2p_n\| \\ &\leq -(1-r_n)\mu_1\frac{d(Gp_n, FixG)}{\mu_2}.\end{aligned} \tag{3.37}$$

Using nonexpansive property of G, we have

$$\begin{aligned}d_{n+1} = d(p_{n+1}, Fix(G)) &= \|p_{n+1} - P_{\mathrm{Fix}(G)}(p_{n+1})\| \\ &\leq \|a_n - P_{\mathrm{Fix}}(Gp_n)\| \\ &\leq (1-r_n)\|Gp_n - P_{\mathrm{Fix(G)}}(Gp_n)\| + r_n\|Gp_n - P_{\mathrm{Fix(G)}}(Gp_n)\| \\ \|Gp_n - P_{\mathrm{Fix(G)}}(Gp_n)\| &= d(Gp_n, \mathrm{Fix(G)}) - (1-r_n)\mu_1{\mu_2}^{-\lambda_1}d_{n+1} \\ &\geq -(1-r_n)\mu_1{\mu_2}^{-\lambda_1}d(Gp_n, \mathrm{Fix(G)}) - \mu_1\|p_n - Gp_n\|^{\lambda_1} \\ &\leq -\mu_1{\mu_2}^{-\lambda_1}d(Gp_n, \mathrm{Fix(G)}) \\ &\leq -\mu_1{\mu_2}^{-\lambda_1}(d_{n+1})^{\lambda_1}.\end{aligned}$$

Also, we have that

$$-\mu_1\|p_n - Gp_n\|^{\lambda_1} \leq -\mu_1{\mu_2}^{-\lambda_1}{d_n}^{\lambda_1},$$

Thus, from (3.35), we obtain that

$$\|p_{n+1} - p\|^{\lambda_1} \leq \|p_n - p\|^{\lambda_1} - (1-r_n)\mu_1{\mu_2}^{-\lambda_1}d_{n+1}^{\lambda_1} - \mu_1{\mu_2}^{-\lambda_1}(d_n)^{\lambda_1}.$$

Therefore, we have

$$(d_{n+1})^{\lambda_1} \leq \frac{(1-\mu_1{\mu_2}^{-\lambda_1}){d_n}^{\lambda_1}}{(1+(1-r_n)\mu_1{\mu_2}^{-\lambda_1})}.$$

Let

$${d_{n+1}}^{\lambda_1} \leq \frac{1-\alpha}{a}(d_n)^{\lambda_1} \tag{3.38}$$

where, $a = \left(1+(1-r_n)\mu_1{\mu_2}^{-\lambda_1}\right)$ and $\alpha = (1-\mu_1{\mu_2}^{-\lambda_1})$.

From (3.38) and proceeding similarly as in Theorem 3.4, we get $\|p_n - p\| \leq C\rho_4^n$.

Case 2. For $0 < \lambda_2 < 1$ from Hölder inequality and d_n definition ,we have

$$\mu_2^{\frac{-\lambda_1}{\lambda_2}} d_n^{\frac{\lambda_1}{\lambda_2}} \leq \|p_n - Gp_n\|^{\lambda_1} \implies d_{n+1}^{\lambda_1} \leq d_n^{\lambda_1} - \mu_1 \mu_2^{\frac{-\lambda_1}{\lambda_2}} d_n^{\frac{\lambda_1}{\lambda_2}}$$

and then proceed in the same way as in Theorem 3.4. □

Next proposition was given by [10] for the case when the exponent $\lambda_1 \in (0, 1)$.

Proposition 3.6 ([10]). *If G is a GAN operator with $Fix(G) \neq \phi$ and exponent λ_1 for some $\lambda_1 \in (0, 1)$, then it is FP-δ-contractive for some $\delta \in (0, 1)$.*

Table 1. **Convergence Rates of Various Methods**

Iterations	$\lambda_1 \in (0,1)$	$\lambda_1 \in [1,+\infty)$ & $\lambda_2 \in (0,1)$	$\lambda_1 \in [1,+\infty)$ & $\lambda_2 \in [1,+\infty)$
Picard	$O(\delta^{n+1})$	$O\left(n^{-\frac{\lambda_2}{\lambda_1(1-\lambda_2)}}\right)$	$O(\rho^n)$
Mann	$O(\delta^{n+1})$	$O\left(n^{-\frac{\lambda_2}{\lambda_1(1-\lambda_2)}}\right)$	$O(\rho_1^n)$
ISHIKAWA	$O(\delta^{n+1})$	$O\left(n^{-\frac{\lambda_2}{\lambda_1(1-\lambda_2)}}\right)$	$O(\rho_2^n)$
NORMAL-S	$O(\delta^{n+1})$	$O\left(n^{-\frac{\lambda_2}{\lambda_1(1-\lambda_2)}}\right)$	$O(\rho_3^n)$
PV	$O(\delta^{3(n+1)})$	$O\left(n^{-\frac{\lambda_2}{\lambda_1(1-\lambda_2)}}\right)$	$O(\rho_4^n)$

In the following theorem, we provide the convergence rates when $\lambda_1 \in (0, 1)$.

Theorem 3.7. *If G is a GAN operator with $Fix(G) \neq \phi$ and exponent $\lambda_1 \in (0, 1)$. Then, the PV iteration converges to the fixed point of G. The global rate of convergence is given by $O(\delta^{3(n+1)})$.*

Proof. We are given that G is a GAN operator with $Fix(G) \neq \phi$ and exponent $\lambda_1 \in (0, 1)$ therefore from Propostion 3.6 we have G is a FP-δ contraction map therefore there exist a $\delta \in (0, 1)$ such that

$$\|G(x) - p\| \leq \delta\|x - p\| \; \forall x \in R^n/Fix(G).$$

We will prove results for the PV iteration, similar results hold for other iterations

$$\begin{aligned} \|p_{n+1} - p\| &= \|Gq_n - p\| \leq \delta.\|q_n - p\| \\ &= \delta\|G((1 - r_n)G^2 p_n + r_n.Gp_n) - p\| \\ &\leq \delta^2[(1 - r_n)\delta^2 \|p_n - p\| + r_n\delta \|p_n - p\|] \\ &\leq \delta^3\|p_n - p\|. \end{aligned}$$

Inductively, we get

$$\|p_{n+1} - p\| \leq \delta^{3(n+1)}\|p_0 - p\|.$$

□

4. Numerical experiment

Example 4.1. Let $f(x) = \frac{1}{2}\|Ax - b\|_2^2$ where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then A is a full-rank matrix. Now, from the proof of Corollary 5.6 in [10], and by Theorem 2.5, we have $S_1 = I - \alpha\nabla f$ is GAN where $\alpha \approx (0, 0.763)$.Thus, the Mann, Ishikawa, Normal-S, and PV algorithms all converge to the fixed point of S_1 with an exponential rate of convergence by Theorem 3.5 and Theorem 3.4.Here, $S_1(x) = x - \alpha A^\top Ax - A^\top b$ .Putting the value $x = (x_1, x_2)$, we get $S_1(x_1, x_2) = \left(\frac{1}{2}(x_1 - x_2) - 1, -\frac{1}{2}x_1 - 1\right)$.One can easily find its fixed point as $(-2, 0)$. The convergence behavior is shown in the tables and graphs. Using two randomly chosen r_n values, we observe that PV converges faster than Picard.

Table 2. Residual table for Case 1 ($r_n = 0.14302691083025781$)

iteration	PV	Picard	Normal-S	Ishikawa	Mann
1	3.162278	3.162278	3.162278	3.162278	3.162278
2	3.162278	3.162278	3.162278	3.162278	3.162278
3	0.897345	1.802776	1.707093	1.788660	2.788947
4	0.397366	1.346291	1.264030	1.334034	2.499469
5	0.175972	1.075291	0.988109	1.062430	2.275145
6	0.077928	0.868278	0.777032	0.854741	2.100522
7	0.034510	0.702256	0.611418	0.688638	1.963146
8	0.015283	0.568114	0.481132	0.554925	1.853253
9	0.006768	0.459611	0.378611	0.447188	1.763409
10	0.002997	0.371833	0.297936	0.360369	1.688091

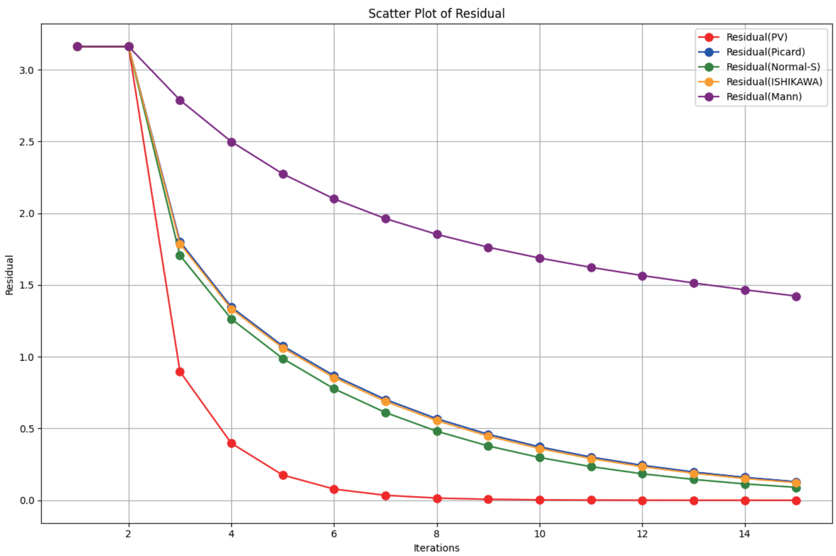

Figure 1. Scattered plot for $r_n = 0.14$

TABLE 3. Residual table for Case 2 ($r_n = 0.4$)

iteration	PV	Picard	Normal-S	Ishikawa	Mann
1	3.162278	3.162278	3.162278	3.162278	3.162278
2	3.162278	3.162278	3.162278	3.162278	3.162278
3	0.950082	1.802776	1.555635	1.696231	2.200000
4	0.445371	1.346291	1.132519	1.254763	1.814166
5	0.208804	1.075291	0.845356	0.978162	1.617860
6	0.097894	0.868278	0.631636	0.766715	1.479761
7	0.045896	0.702256	0.471966	0.601296	1.363131
8	0.021517	0.568114	0.352659	0.471591	1.258114
9	0.010088	0.459611	0.263512	0.369867	1.161786
10	0.004730	0.371833	0.196900	0.290085	1.072980

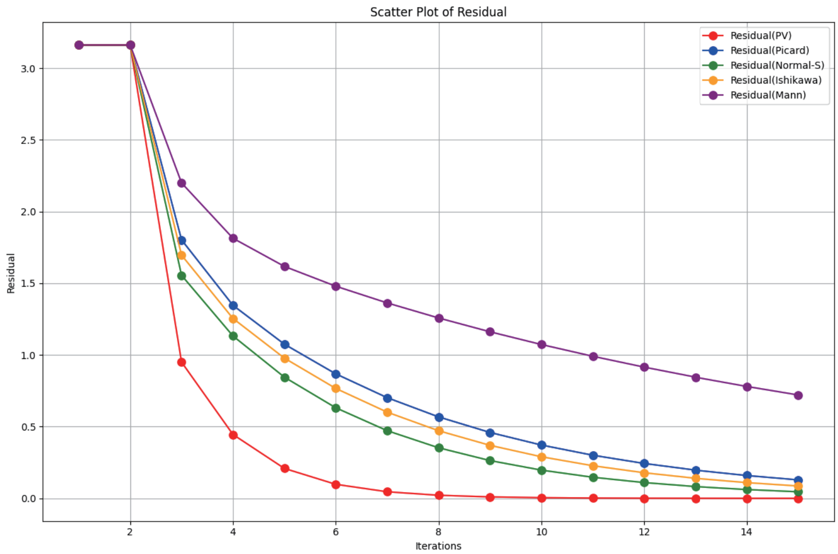

FIGURE 2. Scattered plot for $r_n = 0.4$

5. CONCLUSION

In this paper, we have established the convergence of various iterative algorithms, namely Mann, Ishikawa, Normal-S, and PV, for the GAN operator and obtained their convergence rates. Our results are further verified through numerical experiments, where we observe that PV and Normal-S perform better than Picard. For future work, one could aim to generalize these results to more abstract spaces and explore faster iterations for GAN operators.

Acknowledgment

The authors are grateful to the anonymous referees for their valuable comments and suggestions that improve the paper. The authors are grateful to the Deanship of Scientific Research, Islamic University of Madinah for post publication support.

References

[1] R. Agarwal, O'Regan and D. R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer, 2009.

[2] J. M. Borwein, G. Li and M. K. Tam, *Convergence rate analysis for averaged fixed point iterations in common fixed point problems*, SIAM Journal on Optimization **27** (2017), 1–33.

[3] A. Cegielski, *Iterative Methods for Fixed Point Problems in Hilbert Spaces*, Lecture Notes in Mathematics, vol. 2057. Springer, Heidelberg, 2012.

[4] P. Gautam and Vineet, *Convergence and stability of a novel iterative algorithm for weak contraction in Banach spaces*, Rendiconti del Circolo Matematico di Palermo Series 2 **73** (2024), 1537–1554.

[5] S. Ishikawa, *Fixed points by a new iteration method*, Proceedings of the American Mathematical Society **44** (1974), 147–150.

[6] S. H. Khan, *A Picard-Mann hybrid iterative process*, Fixed Point Theory and Applications **2013** (2013): article 69.

[7] C. Kaur and P. Gautam, *A novel iterative scheme to approximate the fixed points of Zamfirescu operator and generalized non-expansive map with an application*, Lobachevskii Journal of Mathematics **44** (2023), 1316–1331.

[8] S. Khatoon, W. Cholamjiak and I. Uddin, *A modified proximal point algorithm involving nearly asymptotically quasi-nonexpansive mappings*, Journal of Inequalities and Applications **2021** (2021): article 83.

[9] S. Khatoon, I. Uddin and M. Basarir, *A modified proximal point algorithm for a nearly asymptotically quasi-nonexpansive mapping with an application*, Computational and Applied Mathematics **40** (2021), article 250.

[10] Y. Lin and Y. Xu, *Convergence rate analysis for fixed-point iterations of generalized averaged nonexpansive operators*, Journal of Fixed Point Theory and Applications **24** (2022): article 61.

[11] W. R. Mann, *Mean value methods in iteration*, Proceedings of the American Mathematical Society, vol. **4** (1953), 506–510.

[12] É. Picard, *Mémoire sur la théorie des équations aux dérivées partielles et la méthode des approximations successives*, Journal de Math. (1890).

[13] D. R. Sahu, *Applications of the S-iteration process to constrained minimization problems and split feasibility problems*, Fixed Point Theory, vol. **12** (2011), 187–204.

[14] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bulletin of the Australian Mathematical Society **43** (1991), 407–413.

Manuscript received May 20, 2024
revised November 17, 2024

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