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ON THE F-ITERATIVE METHOD FOR THE CLASS OF MAPS SATISFYING ENRICHED CONDITION (C_γ) IN BANACH SPACES

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Abstract. This paper introduces the class of maps satisfying the enriched condition (C_γ) and approximates fixed points of maps satisfying the enriched condition (C_γ) using a modified three-step F-iterative method in a Banach space. First, we establish a weak convergence theorem, followed by deriving several strong convergence theorems for our iterative method under certain mild conditions. Second, we provide some numerical examples to support our main results. Using examples, numerical computations, and graphs derived from various iterative methods, we demonstrate that the studied scheme exhibits a better rate of convergence compared to other methods.

1. Introduction

Let W be a nonempty subset of a normed space $\mathcal{Z}$. A map $D : W \to W$ has a fixed point if there exists a point u in W such that $Du = u$. The set of all fixed points of D is denoted by F_D. The fixed point theory has been used in many areas (see, [15, 22, 23, 32]). The map D is said to be nonexpansive if

$$\|Du - Dv\| \leq \|u - v\|$$

for all $u, v \in W$. There is extensive literature concerning the fixed point theory of nonexpansive maps and their generalizations (see, [4, 17, 18, 28]).

In 2008, Suzuki [27] defined a class of generalized nonexpansive maps as follows.

Definition 1.1 ([27]). A map $D : W \to W$ is described as satisfying condition (C), also known as being Suzuki nonexpansive, if

$$\frac{1}{2}\|u - Du\| \leq \|u - v\| \quad \text{implies} \quad \|Du - Dv\| \leq \|u - v\|$$

for all $u, v \in W$.

Subsequently, this definition was widened in [11].

Definition 1.2 ([11]). For $\mu \geqslant 1$, we say that a map $D : W \to W$ satisfies condition (E_μ) on W if there exists $\mu \geqslant 1$ such that

$$\|u - Dv\| \leqslant \mu\|u - Du\| + \|u - v\|$$

for all $u, v \in W$. We say that D satisfies condition (E) on W whenever D satisfies (E_μ) for $\mu = 1$.

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Remark 1.3. (i) If $D : W \to W$ is nonexpansive then it satisfies condition (E). The converse is not true, as in Example 1 in [11].
(ii) From Lemma 7 in [27], if $D : W \to W$ satisfies condition (C) on W then it satisfies condition (E_3).

There are continuous maps satisfying condition (E) but not condition (C), as in Example 1 in [11].

Definition 1.4 ([11])**.** Let $\gamma \in (0, 1)$. A map $D : W \to W$ is said to satisfy condition (C_γ) if

$$\gamma\|u - Du\| \leq \|u - v\| \text{ implies } \|Du - Dv\| \leq \|u - v\|$$

for all $u, v \in W$.

Remark 1.5. (i) If $\gamma = \frac{1}{2}$, we get the map satisfying condition (C).
(ii) If $0 < \gamma_1 < \gamma_2 < 1$ then the condition (C_{γ_1}) implies the condition (C_{γ_2}).

We do not know if condition (C_γ) for $\gamma \neq \frac{1}{2}$ implies condition (E_μ). Nevertheless, this fact holds for Lipschitzian maps.

Proposition 1.6 ([11])**.** *Let W be a closed subset of a Banach space $\mathcal{Z}$.*

(i) *If $D : W \to W$ is a Lipschitzian map with Lipschitz constant L satisfying condition (C_γ) for some $\gamma \in (0, 1)$ then the map D satisfies condition* (E_μ) *for $\mu = \max\{1, 1 + \gamma\,(L - 1)\}$.*
(ii) *If D satisfies condition (C_γ) for some $\gamma \in (0, 1)$ then F_D is closed.*

In 2019, Berinde [8] was the first to present a novel generalization of nonexpansive maps, which he named the new class of maps as a class of enriched nonexpansive maps, defined as follows.

Definition 1.7 ([8])**.** A map $D : W \to W$ is said to be enriched nonexpansive if there exists $b \in [0, \infty)$ such that

$$\|b(u - v) + Du - Dv\| \leq (b + 1)\|u - v\|,$$

for all u, v in W.

The class of enriched nonexpansive maps and the class of maps satisfying condition (C) are important classes of nonlinear maps. In 2021, Ullah et al. [34] were the first to present a novel Suzuki generalized nonexpansive maps. They introduced the class of enriched Suzuki nonexpansive maps in Hilbert spaces. Later, in 2022, Abdeljawad et al. [1] studied this class of maps in Banach spaces.

Definition 1.8 ([34])**.** Let W be a nonempty subset of a Hilbert space $\mathcal{Z}$. A map $D : W \to W$ is said to be enriched Suzuki nonexpansive if there exists $b \in [0, \infty)$ such that

$$\frac{1}{2}\|u - Du\| \leq (b + 1)\|u - v\| \Rightarrow \|b(u - v) + Du - Dv\| \leq (b + 1)\|u - v\|,$$

for all $u, v \in W$.

It can be seen that every Suzuki nonexpansive map is an enriched Suzuki nonexpansive map with $b = 0$.

In scenarios where maps ensure a fixed point under defined conditions, crafting an iterative method to closely approximate this fixed point becomes crucial. The initial iterative method by Picard [24] was designed for locating fixed points of nonlinear maps. Later, Banach [7] demonstrated the effectiveness of Picard's method in identifying fixed points of contraction maps. Subsequently, several authors have confirmed the convergence of Picard's iterative method for the classes of various maps. However, implementing the Picard iterative method for nonexpansive maps poses challenges, so several authors have proposed general iterative methods (e.g., Mann [19], Ishikawa [13], Noor [20], Ali and Ali [3], Alshehri et al. [6], Şahin et al. [29]).

In 2020, Ali and Ali [5] introduced the F-iterative method and compared the rates of convergences of different iterative methods with this iterative method. They proved that the F-iterative method converges faster than the iterative methods proposed by Agarwal et al. [2], Gürsoy and Karakaya [12], Karakaya et al. [16], Thakur et al. [31], Ullah and Arshad (M^*) [35] and Ullah and Arshad (M) [36] in Banach spaces. In 2021, Jubair et al. [14] studied this iterative scheme to estimate fixed points of nonexpansive maps in the framework of Banach spaces. In 2022, Uddin et al. [33] studied this iteration process in the framework of Banach spaces and obtained convergence results for Suzuki nonexpansive maps.

Let's consider a map D on a nonempty convex subset W of a normed space $\mathcal{Z}$. Let D_r denote an averaged map of D, defined as $D_r u = (1 - r)\, u + rDu$, where $r = 1/(b+1)$. It is well-known that the set of all fixed points D_r coincides with the set of F_D. Using the map D_r, the Agarwal iterative, Thakur iterative, M-iterative and F-iterative methods can be described as follows, respectively,

$$\left\{\begin{array}{l} u_1 \in W, \\ z_l = (1-\beta_l)\, u_l + \beta_l D_r u_l, \\ u_{l+1} = (1-\alpha_l)\, D_r u_l + \alpha_l D_r z_l, \quad l \in \mathbb{N}, \end{array}\right. \tag{1.1}$$

$$\left\{\begin{array}{l} u_1 \in W, \\ v_l = (1-\beta_l)\, u_l + \beta_l D_r u_l, \\ z_l = D_r((1-\alpha_l)\, u_l + \alpha_l v_l), \\ u_{l+1} = D_r z_l, \quad l \in \mathbb{N}, \end{array}\right. \tag{1.2}$$

$$\left\{\begin{array}{l} u_1 \in W, \\ v_l = (1-\alpha_l)\, u_l + \alpha_l D_r u_l, \\ z_l = D_r v_l, \\ u_{l+1} = D_r z_l, \quad l \in \mathbb{N}, \end{array}\right. \tag{1.3}$$

and

$$\left\{\begin{array}{l} u_1 \in W, \\ v_l = D_r((1-\alpha_l)\, u_l + \alpha_l D_r u_l), \\ z_l = D_r v_l, \\ u_{l+1} = D_r z_l, \quad l \in \mathbb{N}, \end{array}\right. \tag{1.4}$$

where $\alpha_l, \beta_l \in (0,1)$ for all $l \in \mathbb{N}$ and $r = \frac{1}{b+1}$.

In this paper, we aim to study the convergence of the F-iterative method defined by (1.4) for maps satisfying an enriched version of condition (C_γ) in a Banach space. We employ this iterative method to establish both weak and strong convergence results for maps satisfying the enriched condition (C_γ) in a Banach space under diverse conditions. Also, we present some numerical examples to compare the rates of convergences of the iterative methods (1.1)-(1.4). Our work is more general and unifies the comparable results in the existing literature, for instance, the results given in [14, 33].

2. Preliminaries

This section presents some well-known definitions and results that will be used to establish the paper's primary outcomes.

Definition 2.1 ([10, 30])**.** Let W be a nonempty subset of a normed space $\mathcal{Z}$, and $\{u_l\}$ be a bounded sequence in $\mathcal{Z}$. In this case, the number

$$R\left(W, \{u_l\}\right) = \inf\left\{\limsup_{l\to\infty}\|u_l - u\| : u \in W\right\}$$

is called the asymptotic radius of $\{u_l\}$ relative to W, and the set

$$A\left(W, \{u_l\}\right) = \left\{u \in W : \limsup_{l\to\infty}\|u_l - u\| = R\left(W, \{u_l\}\right)\right\}$$

is called the asymptotic center of $\{u_l\}$ relative to W. Moreover, if $\{u_l\}$ is a bounded sequence in W and W is a closed, convex subset of a uniformly convex Banach space ($UCBS$ for short) $\mathcal{Z}$ then the set $A\left(W, \{u_l\}\right)$ contains one and only one element.

Definition 2.2 ([21])**.** A normed space $\mathcal{Z}$ is said to have Opial property if every weakly convergent sequence $\{u_l\}$ in $\mathcal{Z}$ which admits weak limit, namely, $u \in \mathcal{Z}$ satisfies

$$\liminf_{l\to\infty}\|u_l - u\| < \liminf_{l\to\infty}\|u_l - v\|, \quad \forall v \in \mathcal{Z} - \{u\}.$$

Proposition 2.3. *Let W be a closed subset of a Banach space $\mathcal{Z}$ with the Opial property. Assume that $D : W \to W$ is a Lipschitzian map with the Lipschitz constant L satisfying condition (C_γ) for some $\gamma \in (0, 1)$. If $\{u_l\}$ converges weakly to a^* and $\lim_{l\to\infty}\|Du_l - u_l\| = 0$ then $Da^* = a^*$.*

Proof. By Proposition 1.6 (i), we have

$$\|u_l - Da^*\| \leqslant \mu\|u_l - Du_l\| + \|u_l - a^*\|$$

for $\mu = \max\{1, 1 + \gamma\,(L - 1)\}$. Then, we get

$$\liminf_{l\to\infty}\|u_l - Da^*\| \leqslant \liminf_{l\to\infty}\|u_l - a^*\|.$$

Hence, from the Opial property, we obtain $Da^* = a^*$. □

Lemma 2.4 ([25])**.** *Consider a $UCBS$ $\mathcal{Z}$ such that $0 < i \leq \alpha_l \leq j < 1$. In this case, if a real number $z \geq 0$ exists such that for any sequences $\{u_l\}$ and $\{v_l\}$ in $\mathcal{Z}$ satisfying $\limsup_{l\to\infty}\|u_l\| \leq z$, $\limsup_{l\to\infty}\|v_l\| \leq z$ and $\lim_{l\to\infty}\|\alpha_l u_l + (1 - \alpha_l)\,v_l\| = z$ then $\lim_{l\to\infty}\|u_l - v_l\| = 0$ holds.*

3. Main results

We begin with the definition of maps satisfying an enriched condition (C_γ).

Definition 3.1. Let W be a nonempty subset of a normed space $\mathcal{Z}$. We say that a map $D : W \to W$ satisfies the enriched condition (C_γ) if for all $u, v \in W$ and $\gamma \in (0,1)$, there exists $b \in [0,\infty)$ such that
$\gamma\|u - Du\| \le (b+1)\|u - v\| \Rightarrow \|b(u-v) + Du - Dv\| \le (b+1)\|u - v\|.$

Clearly, every map satisfying condition (C_γ) satisfies the enriched condition (C_γ) for $b = 0$.

Now, we establish an important lemma for this class of maps as follows.

Lemma 3.2. *Let W be a nonempty convex subset of a normed space $\mathcal{Z}$. If $D : W \to W$ satisfies the enriched condition (C_γ) for some $\gamma \in (0,1)$ and $b \in [0,\infty)$ then for every $r \in (0,1]$, the map $D_r : W \to W$ defined by $D_r u = (1-r)u + rDu$ satisfies the condition (C_γ).*

Proof. Since D satisfies enriched condition (C_γ), there exist constants $\gamma \in (0,1)$ and $b \in [0,\infty)$ such that
$\gamma\|u - Du\| \le (b+1)\|u - v\| \Rightarrow \|b(u-v) + Du - Dv\| \le (b+1)\|u - v\|$, for all $u, v \in W$. Now, we may put $b = \frac{1-r}{r} = \frac{1}{r} - 1$. Then $b+1 = \frac{1}{r}$. It is easy to see that $r \in (0,1]$. The above condition becomes

$$\gamma\|u - Du\| \le \frac{1}{r}\|u - v\| \Rightarrow \left\|\left(\frac{1-r}{r}\right)(u-v) + Du - Dv\right\| \le \frac{1}{r}\|u - v\|.$$

It follows that

$$\gamma\|ru - rDu\| \le \|u - v\| \Rightarrow \|(1-r)(u-v) + rDu - rDv\| \le \|u - v\|$$

or

$$\begin{aligned} &\gamma\|u - [(1-r)u + rDu]\| \le \|u - v\| \\ &\Rightarrow \|[(1-r)u + rDu] - [(1-r)v + rDv]\| \le \|u - v\|. \end{aligned}$$

Since $(1-r)u + rDu = D_r u$, then we have

$$\gamma\|u - D_r u\| \le \|u - v\| \Rightarrow \|D_r u - D_r v\| \le \|u - v\|, \text{ for all } u, v \in W.$$

This shows that the averaged operator D_r satisfies condition (C_γ). □

Next, we prove new findings concerning the F-iterative scheme (1.4) applied to the category of maps satisfying the enriched (C_γ) condition. Our discussion begins with an elementary lemma.

Lemma 3.3. *Let W be a nonempty closed and convex subset of a normed space $\mathcal{Z}$. If $D : W \to W$ is a map satisfying enriched condition (C_γ) with $F_D \neq \emptyset$ and $\{u_l\}$ is a sequence obtained from (1.4) then $\lim_{l\to\infty}\|u_l - a^*\|$ exists for each $a^* \in F_D$.*

Proof. Consider any point $a^* \in F_D$. It follows that $a^* \in F_{D_r}$. Hence using Lemma 3.2, we have that D_r satisfies the condition (C_γ). In particular, $\gamma\|a^* - D_r a^*\| \le \|a^* - u\|$ implies $\|D_r a^* - D_r u\| \le \|a^* - u\|$ for all $u \in W$. Using this, we get

$$\begin{aligned} \|v_l - a^*\| &= \|D_r[(1-\alpha_l)\,u_l + \alpha_l D_r u_l] - a^*\| \\ &\le \|(1-\alpha_l)\,u_l + \alpha_l D_r u_l - a^*\| \end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_l)\|u_l-a^*\|+\alpha_l\|D_ru_l-a^*\| \\
&\leq (1-\alpha_l)\|u_l-a^*\|+\alpha_l\|u_l-a^*\| \\
&= \|u_l-a^*\|. \qquad (3.1)
\end{aligned}
$$

From (3.1), we have

$$
\begin{aligned}
\|z_l-a^*\| &= \|D_rv_l-a^*\| \\
&\leq \|v_l-a^*\| \leq \|u_l-a^*\|.
\end{aligned}
$$

While using this inequality, we have

$$
\begin{aligned}
\|u_{l+1}-a^*\| &= \|D_rz_l-a^*\| \\
&\leq \|z_l-a^*\| \leq \|u_l-a^*\|.
\end{aligned}
$$

Now, we can see that $\|u_{l+1}-a^*\| \leq \|u_l-a^*\|$. So the sequence $\{\|u_l-a^*\|\}$ is non-increasing and bounded below. It follows that $\lim_{l\to\infty}\|u_l-a^*\|$ exists, where $a^* \in F_{D_r} = F_D$ is any point. □

We can prove the following theorem from Lemma 3.3, which plays a vital role in this paper.

Theorem 3.4. *Let W be a nonempty closed and convex subset of a UCBS $\mathcal{Z}$, $D: W \to W$ be a Lipschitzian map satisfying the enriched condition (C_γ), and $\{u_l\}$ be a sequence obtained from (1.4). Then $F_D \neq \emptyset \iff \{u_l\}$ is bounded in W and $\lim_{l\to\infty}\|u_l-D_ru_l\|=0$, where $r=\frac{1}{b+1}$.*

Proof. First, we note from Lemma 3.3 that for any specific point $a^* \in F_D$, $\lim_{l\to\infty}\|u_l-a^*\|$ eventually exists and $\{u_l\}$ is bounded in W. Thus, we put

$$\lim_{l\to\infty}\|u_l-a^*\| = z. \qquad (3.2)$$

We have proved in Lemma 3.3 that

$$\|v_l-a^*\| \leq \|u_l-a^*\|.$$

It follows that

$$\limsup_{l\to\infty}\|v_l-a^*\| \leq \limsup_{l\to\infty}\|u_l-a^*\| = z. \qquad (3.3)$$

Now, by Lemma 3.2, D_r satisfies the condition (C_γ). Then, we have

$$\gamma\|a^*-D_ra^*\| \leq \|u_l-a^*\| \Rightarrow \|D_ru_l-a^*\| \leq \|u_l-a^*\|.$$

Hence

$$\limsup_{l\to\infty}\|D_ru_l-a^*\| \leq \limsup_{l\to\infty}\|u_l-a^*\| = z.$$

Again, from the proof of Lemma 3.3,

$$\|u_{l+1}-a^*\| \leq \|v_l-a^*\|.$$

It follows that

$$z \leq \liminf_{l\to\infty}\|v_l-a^*\|. \qquad (3.4)$$

From (3.3) and (3.4), we get

$$z = \lim_{l\to\infty}\|v_l-a^*\|. \qquad (3.5)$$

From (3.1), (3.2), and (3.5), we have that

$$z = \lim_{l\to\infty} \|(1-\alpha_l)(u_l - a^*) + \alpha_l (D_r u_l - a^*)\|.$$

Applying Lemma 2.4, we obtain

$$\lim_{l\to\infty} \|u_l - D_r u_l\| = 0.$$

Conversely, we suppose that $\{u_l\} \subseteq W$ is bounded and $\lim_{l\to\infty} \|u_l - D_r u_l\| = 0$, and we need to prove that $F_D \neq \emptyset$. For this purpose, it is sufficient to prove that for every choice of $a^* \in A(W, \{u_l\})$, the element $D_r a^*$ is contained in the set $A(W, \{u_l\})$. Using Lemma 3.2 and Proposition 1.6 (i), one has

$$\begin{aligned} R(D_r a^*, \{u_l\}) &= \limsup_{l\to\infty} \|u_l - D_r a^*\| \\ &\leq \limsup_{l\to\infty} (\mu \|u_l - D_r u_l\| + \|u_l - a^*\|) \\ &= \limsup_{l\to\infty} \|u_l - a^*\| = R(a^*, \{u_l\}), \end{aligned}$$

where $\mu = \max\{1, 1+\gamma(L-1)\}$. We have seen that $D_r a^* \in A(W, \{u_l\})$. But the set $A(W, \{u_l\})$ consists of one point. It follows that $a^* = D_r a^*$. But $F_{D_r} = F_D$, hence we proved that the set $F_D \neq \emptyset$. □

We establish a weak convergence result under the Opial property as follows.

Theorem 3.5. *Let W be a nonempty closed and convex subset of a UCBS $\mathcal{Z}$, $D : W \to W$ be a Lipschitzian map which satisfies the enriched condition (C_γ) for some $\gamma \in (0,1)$ with $F_D \neq \emptyset$, and $\{u_l\}$ be a sequence obtained from (1.4). Then $\{u_l\}$ converges weakly to a fixed point of D if $\mathcal{Z}$ satisfies the Opial property.*

Proof. The space $\mathcal{Z}$, being uniformly convex, inherently exhibits reflexivity. Furthermore, $\{u_l\}$ is a bounded sequence in W due to Theorem 3.4. It follows that we can select a subsequence $\{u_{l_j}\}$ of sequence $\{u_l\}$ such that $\{u_{l_j}\}$ converges weakly to some $z \in W$. If z denotes a weak limit of $\{u_{l_j}\}$ then we want to prove that z is the weak limit of $\{u_l\}$ and a fixed point of D. For this, in the view of Theorem 3.4, it follows that $\lim_{j\to\infty} \|D_r u_{l_j} - u_{l_j}\| = 0$. By Proposition 2.3, we have $z \in F_{D_r}$; that is, z is a fixed point of D_r and hence a fixed point of D.

It remains to be proven that z is the weak limit of $\{u_l\}$. For this, arguing by contradiction, assume that z is not a weak limit of $\{u_l\}$, so there exists another subsequence $\{u_{l_k}\}$ of $\{u_l\}$ which admits a weak limit w such that $w \neq z$. Employing identical computations as before reveals that w belongs to $F_{D_r} = F_D$. In light of Lemma 3.3 and considering the Opial property of the space $\mathcal{Z}$, we derive

$$\begin{aligned} \lim_{l\to\infty} \|u_l - z\| &= \lim_{j\to\infty} \|u_{l_j} - z\| < \lim_{j\to\infty} \|u_{l_j} - w\| \\ &= \lim_{l\to\infty} \|u_l - w\| = \lim_{k\to\infty} \|u_{l_k} - w\| \\ &< \lim_{k\to\infty} \|u_{l_k} - z\| = \lim_{l\to\infty} \|u_l - z\|. \end{aligned}$$

Accordingly, we proved that $\lim_{l\to\infty} \|u_l - z\| < \lim_{l\to\infty} \|u_l - z\|$, which is a contradiction. Hence, we obtain the desired result. □

Next, we prove the following strong convergence theorems under some conditions.

Theorem 3.6. *Let W be a nonempty convex subset of a UCBS $\mathcal{Z}$, $D : W \to W$ be a Lipschitzian map that satisfies the enriched condition (C_γ) with $F_D \neq \emptyset$ and $\{u_l\}$ be a sequence obtained from (1.4). If W is compact then the sequence $\{u_l\}$ converges strongly to a fixed point of D.*

Proof. Due to the compactness of the set W, there exists a subsequence $\{u_{l_i}\}$ of $\{u_l\}$ that satisfies $\lim_{i\to\infty} \|u_{l_i} - u_0\| = 0$, for some $u_0 \in W$. Also, in the view of Theorem 3.4, $\lim_{i\to\infty} \|u_{l_i} - D_r u_{l_i}\| = 0$. According to Lemma 3.2, D_r satisfies the condition (C_γ). Thus, using Proposition 1.6 (i), we get

$$\|u_{l_i} - D_r u_0\| \leq \mu \|u_{l_i} - D_r u_{l_i}\| + \|u_{l_i} - u_0\| .$$

Let's take $\lim_{i\to\infty}$ on both sides of the above inequality, we get $\lim_{i\to\infty} \|u_{l_i} - D_r u_0\| = 0$, that is, $u_{l_i} \to D_r u_0$. It follows that $D_r u_0 = u_0$. This proves that u_0 is a fixed point for D_r and hence for D. But, by Lemma 3.3, $\lim_{l\to\infty} \|u_l - u_0\|$ exists. Eventually, u_0 is the strong limit for $\{u_l\}$. □

Theorem 3.7. *Let W be a nonempty closed and convex subset of a UCBS $\mathcal{Z}$, $D : W \to W$ be a Lipschitzian map satisfies the enriched condition (C_γ) with $F_D \neq \emptyset$ and $\{u_l\}$ be a sequence obtained from (1.4). Then the sequence $\{u_l\}$ converges strongly to a fixed point of D if and only if* $\liminf_{l\to\infty} \operatorname{dist}(u_l, F_D) = 0$ *or* $\limsup_{l\to\infty} \operatorname{dist}(u_l, F_D) = 0$, *where* $\operatorname{dist}(u_l, F_D) = \inf\{\|u_l - a^*\| : a^* \in F_D\}$.

Proof. The first part is trivial. So, we prove the converse part. Suppose that

$$\liminf_{l\to\infty} \operatorname{dist}(u_l, F_D) = 0.$$

It follows from Lemma 3.3 that $\lim_{l\to\infty} \operatorname{dist}(u_l, F_D)$ exists and hence $\lim_{l\to\infty} \operatorname{dist}(u_l, F_D) = 0$. Therefore, for a given $\delta > 0$, there exists $l_0 \in \mathbb{N}$ such that for all $l \geq l_0$, we have

$$\operatorname{dist}(u_l, F_D) = \inf\{\|u_l - a^*\| : a^* \in F_D\} < \frac{\delta}{2}.$$

In particular, $\inf\{\|u_{l_0} - a^*\| : a^* \in F_D\} < \frac{\delta}{2}$. Hence, there exists $u \in F_D$ such that $\|u_{l_0} - a^*\| < \frac{\delta}{2}$. Now for $l_1, l \geq l_0$, we have

$$\begin{aligned} \|u_{l+l_1} - u_l\| &\leq \|u_{l+l_1} - a^*\| + \|a^* - u_l\| \\ &\leq \|u_{l_0} - a^*\| + \|u_{l_0} - a^*\| = 2\|u_{l_0} - a^*\| < \delta. \end{aligned}$$

Thus, $\{u_l\}$ is a Cauchy sequence in W. Since W is a closed subset of the Banach space $\mathcal{Z}$, there exists a point $u \in W$ such that $\lim_{l\to\infty} u_l = u$. Now $\lim_{l\to\infty} \operatorname{dist}(u_l, F_D) = 0$ implies $\operatorname{dist}(u, F_D) = 0$. Hence, by Proposition 1.6 (ii), we get $u \in F_D$. □

Now, we prove a strong convergence theorem under condition (I). Before this, we give the definition of condition (I).

Definition 3.8 ([26]). Let W be a nonempty subset of a normed space $\mathcal{Z}$. A map $D : W \to W$ is said to satisfy condition (I), provided that one can find a function f such that $f(a) = 0$ for whenever $a = 0, f(a) > 0$ for every choice of $a > 0$ and $\|u - Du\| \geq f(\operatorname{dist}(u, F_D))$ for all $u \in W$.

We close the section with the proven result under condition (I).

Theorem 3.9. *Let W be a nonempty closed and convex subset of a UCBS $\mathcal{Z}$, $D : W \to W$ be a Lipschitzian map that satisfies the enriched condition (C_γ) with $F_D \neq \emptyset$ and $\{u_l\}$ be a sequence obtained from (1.4). If D_r satisfies condition (I) then $\{u_l\}$ converges strongly to a fixed point of D.*

Proof. From Lemma 3.2, we get that D_r satisfies the condition (C_γ). We may write from Theorem 1 that $\lim_{l\to\infty} \|u_l - D_r u_l\| = 0$. From the condition (I) associated with D_r, combined with the preceding equation, we conclude that $\lim_{l\to\infty} \operatorname{dist}(u_l, F_{D_r}) = 0$. Utilizing Theorem 3.7, it follows that the sequence $\{u_l\}$ exhibits strong convergence to a fixed point of D. □

Remark 3.10.
(i) If we take $b = 0$ in Theorems 3.4-3.9 then we obtain some convergence results for the map satisfying condition (C_γ), which are new in the literature.
(ii) If we take $b = 0$ and $\gamma = \frac{1}{2}$ in Definition 3.1, we obtain Suzuki nonexpansive map. Then, Theorems 3.4-3.9 generalized the convergence results of [33].
(iii) The map satisfying enriched condition (C_γ) is a generalization of the nonexpansive map. Then, Theorems 3.4-3.9 generalized the correspondence results of [14].

4. The rates of convergences with numerical examples

In this section, using two numerical examples, we shall demonstrate that the F-iterative method converges to a fixed point better than other methods.

First, we provide a numerical example of a map satisfying the enriched condition (C_γ) with $b = 0$.

Example 4.1. For a given $\gamma \in (0,1)$, let $D : [0,1] \to [0,1]$ be a map defined by

$$Du = \begin{cases} \frac{u}{2}, & u \neq 1, \\ \frac{1+\gamma}{2+\gamma}, & u = 1. \end{cases}$$

When $\gamma = \frac{3}{8}$, we have

$$Du = \begin{cases} \frac{u}{2}, & u \neq 1, \\ \frac{11}{19}, & u = 1. \end{cases}$$

As seen from Example 5 of [11], we observe that the map D satisfies the condition $\left(C_{\frac{3}{8}}\right)$ and so the enriched condition $\left(C_{\frac{3}{8}}\right)$ with $b = 0$. Hence, we obtain $r = 1$ and so $D_r u = D_1 u = Du$. The point $u = 0$ is the unique fixed point of D. In the table and graphic below, it can be seen that the F-iterative process is faster than other iterative processes with the control sequences $\alpha_l = 0.35$ and $\beta_l = 0.55$ for all $l \in \mathbb{N}$.

Second, we provide a numerical example of a map satisfying the enriched condition (C_γ) but not the ordinary condition (C_γ).

Example 4.2. Set a map D on a closed, convex and bounded subset $W = [-2, \frac{-1}{2}] \cup [\frac{1}{2}, 2]$ by $Du = u^{-1}$ for each $u \in W$. We will show that D satisfies the enriched condition (C_γ) and does not satisfy condition (C_γ).

l	F	M	Thakur	Agarwal
1	1	1	1	1
2	0.1065789	0.2131579	0.2297368	0.5107895
3	0.0109910	0.0439638	0.0519062	0.2308130
4	0.0011334	0.0090675	0.0117275	0.1042986
5	0.0001169	0.0018702	0.0026497	0.0471299
6	0.0000121	0.0003857	0.0005987	0.0212968
7	0.0000012	0.0000796	0.0001353	0.0096235
8	**0**	0.0000164	0.0000306	0.0043486
9	0	0.0000034	0.0000069	0.0019650
10	0	0.0000007	0.0000016	0.0008879
11	0	0.0000001	0.0000004	0.0004012
12	0	**0**	0.0000001	0.0001813
13	0	0	**0**	0.0000819
14	0	0	0	0.0000370
15	0	0	0	0.0000167
16	0	0	0	0.0000076
17	0	0	0	0.0000034
18	0	0	0	0.0000015
19	0	0	0	0.0000007
20	0	0	0	0.0000003
21	0	0	0	0.0000001
22	0	0	0	**0**

TABLE 1. Comparison of the F and other iterations

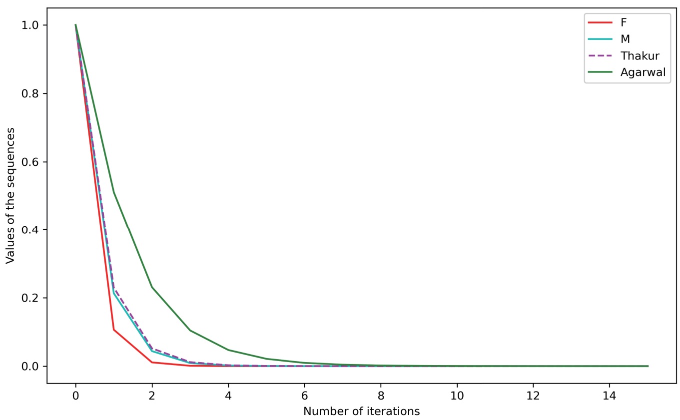

FIGURE 1. The convergence tendencies of the F, M, Thakur, and Agarwal iterations towards the fixed point 0 of the map D

For $\gamma \in (0,1)$, $u = 1$ and $v = \frac{1}{2}$, we have

$$\gamma\|u - Du\| = \gamma \left\|1 - (1)^{-1}\right\| = 0 < \frac{1}{2} = \|u - v\|$$

and

$$\|Du - Dv\| = \left\|(1)^{-1} - (\frac{1}{2})^{-1}\right\| = 1 > \frac{1}{2} = \|u - v\|.$$

Hence, the map D can not satisfy the condition (C_γ).

On the other hand, D satisfies the enriched condition (C_γ). Then for any $u, v \in W$ with $\gamma\|u - Du\| \leq (b+1)\|u - v\|$ or $\gamma\|v - Dv\| \leq (b+1)\|u - v\|$, we have

$$\|b(u - v) + Du - Dv\| \leq (b+1)\|u - v\|.$$

It follows that

$$\|u-v\| \times \left\|b-(uv)^{-1}\right\| \leq (b+1)\|u-v\|.$$

Consequently,

$$\left\|b-(uv)^{-1}\right\| \leq (b+1).$$

This equation now holds for $b=1.5$. Therefore, D satisfies the enriched condition (C_γ) with $b=1.5$. Hence, we obtain $r=\frac{2}{5}$ and so $D_r u = D_{\frac{2}{5}} u = \frac{3u^2+2}{5u}$.

The fixed points of D are $u=-1$ and $u=1$. For initial points u_1 close to -1, the sequence $\{u_l\}$ generated by the F-iterative method will converge to -1. Similarly, for initial points u_1 close to 1, the sequence $\{u_l\}$ will converge to 1. In the tables and graphics, it can be easily seen that the F-iterative method is faster than other iterative processes with the control sequences $\alpha_l = 0.35$ and $\beta_l = 0.55$ for all $l \in \mathbb{N}$ for two different initial points.

l	F	M	Thakur	Agarwal
1	-2.0000000000	-2.0000000000	-2.0000000000	-2.0000000000
2	-1.0201256834	-1.0867720042	-1.1036247297	-1.3445323353
3	-1.0001205648	-1.0029202115	-1.0042768299	-1.0878163518
4	-1.000006946	-1.0000845744	-1.0001460409	-1.0171538720
5	-1.000000040	-1.000024361	-1.000049436	-1.0029950718
6	-1.000000000	-1.000000702	-1.000001673	-1.0005096233
7	-1	-1.000000011	-1.000000057	-1.0000863112
8	-1	-1.000000000	-1.000000002	-1.0000146062
9	-1	-1	-1.000000000	-1.0000024714
10	-1	-1	-1	-1.0000004182
11	-1	-1	-1	-1.0000000708
12	-1	-1	-1	-1.0000000120
13	-1	-1	-1	-1.0000000020
14	-1	-1	-1	-1.0000000003
15	-1	-1	-1	-1.0000000001
16	-1	-1	-1	-1

TABLE 2. Comparative analysis of the F-iterative and alternative methods for the initial point $u_1 = -2$

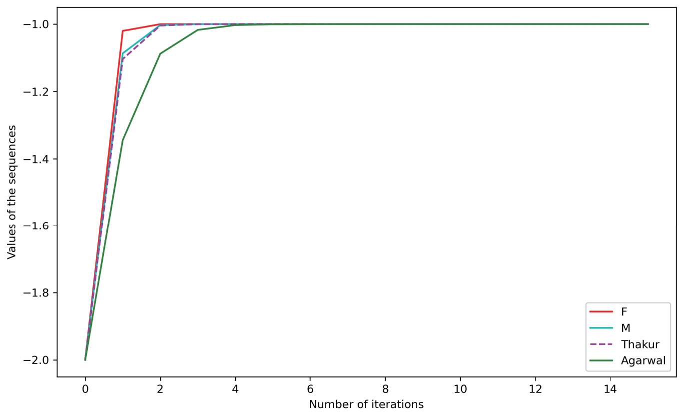

FIGURE 2. Patterns of convergence for the F, M, Thakur, and Agarwal iterative methods toward the fixed point -1 of the map D

l	F	M	Thakur	Agarwal
1	2	2	2	2
2	1.0201256834	1.0867720042	1.1036247297	1.3445323353
3	1.0001205648	1.0029202115	1.0042768299	1.0878163518
4	1.0000006946	1.0000845744	1.0001460409	1.0171538720
5	1.0000000040	1.0000024361	1.0000049436	1.0029950718
6	1.0000000000	1.0000000702	1.0000001673	1.0005096233
7	**1**	1.0000000011	1.0000000057	1.0000863112
8	1	1.0000000000	1.0000000002	1.0000146062
9	1	**1**	1.0000000000	1.0000024714
10	1	1	**1**	1.0000004182
11	1	1	1	1.0000000708
12	1	1	1	1.0000000120
13	1	1	1	1.0000000020
14	1	1	1	1.0000000003
15	1	1	1	1.0000000001
16	1	1	1	**1**

TABLE 3. Comparative analysis of the F-iterative and alternative methods for the initial point $u_1 = 2$

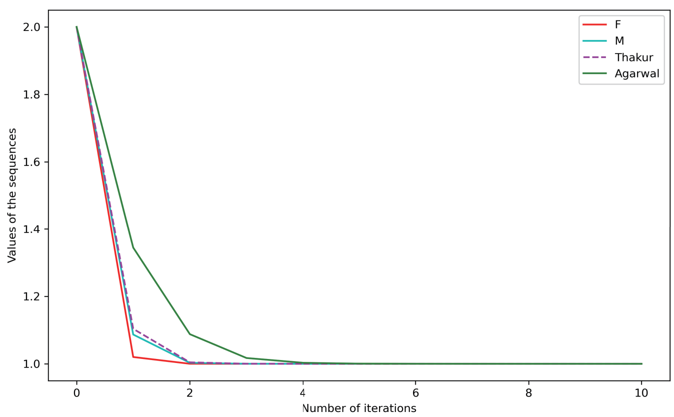

FIGURE 3. Patterns of convergence for the F, M, Thakur, and Agarwal iterative methods toward the fixed point 1 of the map D

5. CONCLUSIONS

In this article, we introduced the class of maps satisfying the enriched condition (C_γ). We provided that if the map D satisfies the enriched condition (C_γ) then the average map D_r satisfies the condition (C_γ). Furthermore, we established that both strong and weak convergence results can be achieved under diverse conditions within a Banach space through the F-iterative process, as specified in (1.4). We compared the rates of convergences of the Agarwal iterative (1.1), Thakur iterative (1.2), M-iterative (1.3) and F-iterative (1.4) methods and found that the F-iterative method provides the faster convergence than other methods.

We suggest as an open problem whether all results of this paper can be improved for the map D satisfying the enriched condition (E_μ) in a different space, such as CAT(0) space, as given in [9].

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