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DYNAMICAL ANALYSIS OF A NEW 3D CHAOTIC SYSTEM WITH APPLICATION IN SYNCHRONIZATION

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Abstract. In this paper, we have studied the dynamical analysis of a new 3D chaotic system by investigating its Lyapunov exponents, time series and Hamilton energy. An optimization problem for exploring the precise ultimate bound set of the system has been solved analytically by using Lagrange multiplier method. Also, numerical simulation is applied to test the obtained results. Furthermore, an application of the obtained bound set in the synchronization of two identical chaotic systems has been discussed.

1. Introduction

Researchers have explored the fascinating subject of chaos in various nonlinear systems, focusing on their dynamics and stability. Hamilton energy is one of the beautiful ideas to understand the dynamics and stability of a chaotic system more deeply. Studies show that researchers are successful in regulating the stability of the chaotic system, if the energy flow is completely under control. Furthermore, studies have shown that varying the Hamilton energy can achieve this objective, as the stability of chaotic systems relies on parameters and equilibrium points. Recently, research on the study of chaotic systems has taken attention to the Hamilton energy functions [9, 32] and its importance for the emergence of nonlinear oscillations [15, 22, 26]. In [10], Ma et al. developed the Hamilton energy function for each of the three types of attractors and explored energy modulation to control chaos in the system. From a physical standpoint, the dynamic systems are expressed as ordinary differential equations (ODEs), which are built as nonlinear circuits and may be defined as vector field. Using the appropriate Hamilton energy function, we estimate the inner field energy. So, we can use the evolution of the general Lyapunov function to predict the transition mode, make sure that the dynamics of the attractor are stable, and figure out how the variables in the dynamical system are connected. For stability, chaotic systems rely on equilibrium points and parameters. According to Zhou et al. [35], the stability of dynamical systems is strongly impacted by the growth and decay of the Hamilton energy function. The push continuously emits energy, which regulates the energy flow and stabilizes the chaos. Therefore, it is important to study the Hamilton energy function for chaotic dynamical systems.

In order to explore the dynamical behavior of systems more deeply, one interesting field in chaos theory has emerged related to estimation of the boundaries for the solution of chaotic systems. The boundary of the solution of chaotic systems and

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their ultimate bound sets (UBSs) play a significant role in chaos control [23, 34]. Numerous researchers have been fascinated by the concept of ultimate bound set. It can be used to calculate attractors numerically, which is significant in the studies related to hidden attractors and points of equilibrium. It is well known that chaotic systems can be expressed explicitly as differential equations and it is a crucial task to study the chaotic systems from viewpoint of the boundary. Therefore, investigation of a bound set of chaotic systems is a beautiful idea for the deep understanding of the subject. Chaos can be controlled and synchronized for different systems such as fractional order chaotic systems, integer order chaotic and hyperchaotic systems [20,30] via ultimate bound estimation technique. As compared to the search of attractors of chaotic systems, the search of ultimate bound sets and Lyapunov functions corresponding to attractors of hyperchaotic systems are more challenging. One possible approach to solve the problem of ultimate bound estimate for the hyper-chaotic systems is through optimization. Li et al. [16] have analyzed the ultimate bound set for the hyper-chaotic Lorenz-Haken system. Using dimension reduction method, Wang et al. [28] calculated explicit bound sets for the hyper-chaotic Lorenz-Stenflo system. Through the extremum principle of function and generalized Lyapunov function theory, Nik et al. [18] obtained UBSs for a system. Recently, Lei et. [13] brought to light the complete inaccuracy regarding the results of estimating the ultimate bound set [33]. The bound sets of different chaotic and hyperchaotic systems are analytically estimated using the standard methodology of an optimization approach [29]. Numerous authors are interested in estimating the bounded sets of various hyper-chaotic and chaotic systems by using an optimization technique [1, 6, 7, 21]. These types of investigations in case of bound sets have been carried out for few chaotic systems which seems to be a key component of present scenario.

An important technique for controlling chaos is predictive control and synchronization [19]. Researchers have utilized a diverse range of synchronization schemes and control techniques to achieve synchronization in various integer chaotic or hyperchaotic systems. Recently, [2, 4, 8, 11], and many other researchers have studied synchronization in fractional order chaotic and hyperchaotic system [24,25]. Among the different methods of chaos synchronization, linear feedback control is the most common, simple, and straightforward approach to realize in practice. Several authors have used the linear feedback control approach to synchronize other nonlinear systems. These authors have also used Lyapunov stability theory to make sure that nonlinear error systems are stable on a global level. They use this method to achieve globally exponential synchronization through linear feedback control. The main contribution of our studies in this manuscript are highlighted as follows:

–This paper focuses on two aspects: first is dynamic analysis of new 3D chaotic system from perceptive of Hamilton energy and bound sets, and second is the application of bound set in chaos synchronization.

–To the best of our knowledge, no one constructed this type of 3D Lorenz-like system.

–To study more deeply about the dynamics of the new system, Hamilton energy function is also investigated.

–Available literature does not yet address the bound sets for the 3D chaotic system

neither analytically nor numerically.
–With the help of bound set, global exponential synchronization has been achieved between two identical 3D chaotic systems.

Complete description of the paper is as follows: Section 2 describes a new 3D chaotic system with Lyapunov exponent and phase portraits. Section 3 investigates the Hamilton energy function. The explicit ultimate bound set for the new 3D chaotic system is determined in Section 4. Section 5 describes the application of the bound set in synchronization. Conclusion is drawn in Section 6.

2. The 3D chaotic system and its dynamics

Consider the general system [17], which includes a variety of significant systems, including the classical Lorenz system, the Chen system, and other Lorenz-like systems, which have been addressed for various parameter values. The following is the mathematical equations that describes the general system:

$$\begin{aligned} \dot{w}_1 &= \mu(w_2 - w_1), \\ \dot{w}_2 &= \kappa w_1 + \beta w_2 - w_1 w_3, \\ \dot{w}_3 &= w_1 w_2 - \delta w_3, \end{aligned} \tag{2.1}$$

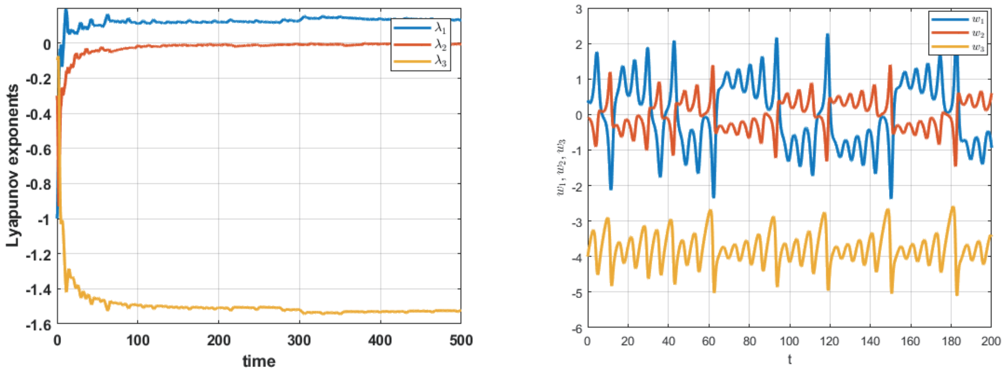
where μ, β, κ, and δ are unknown parameters. Throughout the fields of chaos theory, dynamical systems, chaos control, synchronization and many other area of research [14, 27, 31] have exclusively concentrated on Lorenz-like systems. As a result of these studies and ideas, we decided to develop and analyze a new chaotic system designed in the Lorenz-like fashion. The mathematical equations of a new 3D Lorenz-like system are described as follows:

$$\begin{aligned} \dot{w}_1 &= -w_1 - \mu w_2, \\ \dot{w}_2 &= -\beta w_2 - w_1 w_3 - \kappa w_1, \\ \dot{w}_3 &= w_1 w_2 - \delta w_3, \end{aligned} \tag{2.2}$$

where μ, β, κ, δ are the unknown parameters and w_i for $i = 1, 2, 3$ are variables of state for the system (2.2). By taking the parameter values as $\mu = 2.5$, $\beta = 0.3$, $\kappa = 4$, $\delta = 0.1$ with initial conditions $(-0.4, 0.1, -4)$, Lyapunov exponents of the system (2.2) at $t = 1000$ are $\lambda_1 = 0.125875$, $\lambda_2 = -0.001774$, and $\lambda_3 = -1.524101$. In Fig. 1(A), we observe that one of the three Lyapunov exponents value is positive, which is a necessary requirement for chaotic behavior of the system. Moreover, the times series and phase portraits of the proposed system with initial conditions $(-0.4, 0.1, -4)$ can be see in the Fig. 1(B) and Fig. 2, respectively. For $\mu+\beta+\delta > 0$, the system (2.2) is dissipative with the divergence

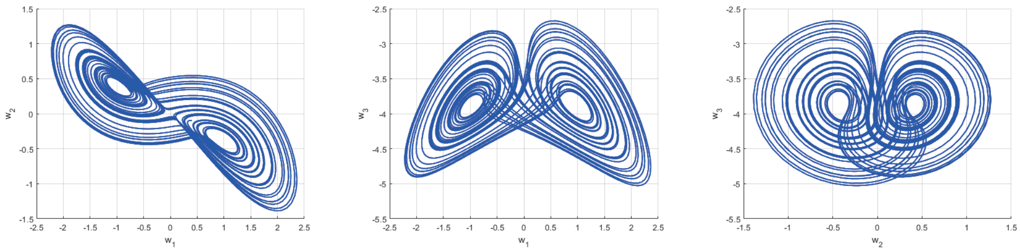
$$\frac{\partial \dot{w}_1}{\partial w_1} + \frac{\partial \dot{w}_2}{\partial w_2} + \frac{\partial \dot{w}_3}{\partial w_3} = -(\mu + \beta + \delta) < 0.$$

The equilibrium points of system (2.2) and their corresponding eigenvalues for the parameters $\mu = 2.5$, $\beta = 0.3$, $\kappa = 4$ and $\delta = 0.1$ are given in the following table: From the perspective of the Routh-Hurwitz criteria [5], it may be said that all equilibrium points are unstable. Taking into consideration the suggested system (2.2), it is worth noting that it is not comparable to the original Lorenz system



(A) (B)

FIGURE 1. (A) Lyapunov exponents; (B) Time Series of the system (2.2) with $\mu = 2.5$, $\beta = 0.3$, $\kappa = 4$ and $\delta = 0.1$



(A) $w_1 - w_2$ plane (B) $w_1 - w_3$ plane (C) $w_2 - w_3$ plane

FIGURE 2. (A), (B) and (C) denote the phase portraits of the 3D chaotic system (2.2) with $\mu = 2.5$, $\beta = 0.3$, $\kappa = 4$ and $\delta = 0.1$

Equilibrium points	Eigenvalues	Stability
$E_1(0, 0, 0)$	$(-3.8316, 2.5316, 0.1000)$	Saddle point
$E_2(-0.9849, 0.3940, -3.8800)$	$(-1.5175, 0.0588 + 1.1290i, 0.0588 - 1.1290i)$	Saddle point
$E_3(0.9849, -0.3940, -3.8800)$	$(-1.5175, 0.0588 + 1.1290i, 0.0588 - 1.1290i)$	Saddle point

or any other existing Lorenz-like systems. In addition, the Chen system and the Lorenz system, both of which were originally developed, are linked in the sense that is provided by [3], which also demonstrates a transition from one system to the other. In light of this, it would be fascinating to get knowledge about the unusual dynamics that this system demonstrates.

3. The Hamilton energy

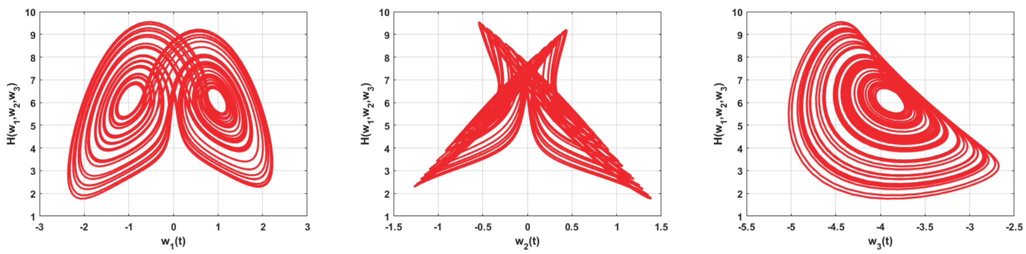
In this section, the calculation of the Hamilton energy for 3D chaotic system (2.2) has been included. The investigation focuses on the link between the Hamilton energy and many attractors of the system (2.2), as well as the energy reliance on the attractors. Rewrite the system (2.2) as follows:

$$\begin{bmatrix}\dot{w}_1\\ \dot{w}_2\\ \dot{w}_3\end{bmatrix} = F_c + F_d = \begin{bmatrix}-\mu w_2\\ -w_1w_3-\kappa w_1\\ w_1w_2\end{bmatrix} + \begin{bmatrix}-w_1\\ -\beta w_2\\ -\delta w_3\end{bmatrix}, \tag{3.1}$$

where F_c is the conservative field containing the full rotation and F_d denotes the dissipative field including the divergence. Then system (2.2) can be rewritten as follows:

$$\begin{aligned}\begin{bmatrix}\dot{w}_1\\ \dot{w}_2\\ \dot{w}_3\end{bmatrix} &= \begin{bmatrix}0 & -\mu & 0\\ \mu & 0 & -w_1\\ 0 & w_1 & 0\end{bmatrix}\begin{bmatrix}\frac{-\kappa}{\mu}w_1\\ w_2\\ w_3\end{bmatrix} + \begin{bmatrix}\frac{\mu}{\kappa} & 0 & 0\\ 0 & -\beta & 0\\ 0 & 0 & -\delta\end{bmatrix}\begin{bmatrix}\frac{-\kappa}{\mu}w_1\\ w_2\\ w_3\end{bmatrix},\\ &= J(W)\nabla H + R(W)\nabla H,\end{aligned} \tag{3.2}$$

where the gradient vector of a smooth energy function is denoted by ∇H and $J(W)$ denotes a skew-symmetric matrix, while $R(W)$ denotes a symmetric matrix. The



(A) (B) (C)

FIGURE 3. (A) H versus w_1; (B) H versus w_2; and (C) H versus w_3 for the system (2.2) with $\mu = 2.5$, $\beta = 0.3$, $\kappa = 4$ and $\delta = 0.1$.

Hamilton energy function is obtained by the following approach:

$$\frac{dH}{dt} = \nabla H^T R(W)\nabla H, \tag{3.3}$$

$$\nabla H^T J(W)\nabla H = 0. \tag{3.4}$$

The Hamilton energy function can be determined by using Helmhotz's theorem [12, 35] as follows:

$$\nabla H^T F_c(W) = 0, \tag{3.5}$$

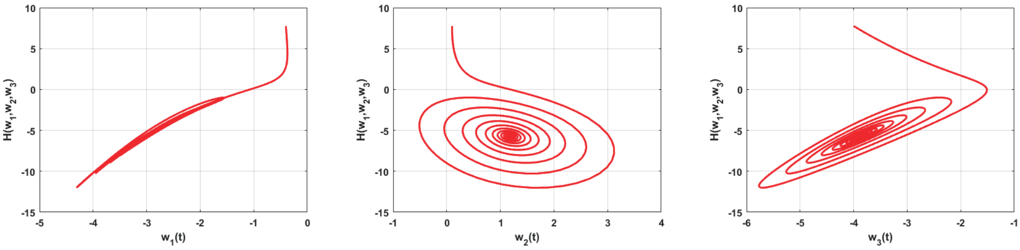
$$\nabla H^T F_d(W) = \frac{dH}{dt}. \tag{3.6}$$

Thus, the Hamilton energy function is,

$$H = -\frac{\kappa}{\mu}w_1^2 + \frac{1}{2}w_2^2 + \frac{1}{2}w_3^2. \tag{3.7}$$

The flow of energy both inside and outside of the system is calculated as:

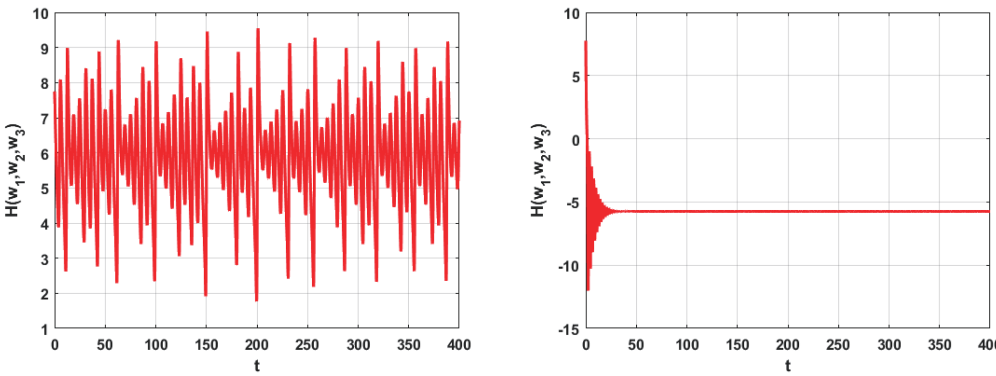
$$\frac{dH}{dt} = \frac{\kappa}{\mu}w_1^2 - \beta w_2^2 - \delta w_3^2. \tag{3.8}$$



(A) (B) (C)

FIGURE 4. (a) H versus w_1; (b) H versus w_2; and (c) H versus w_3 for system (2.2) with $\mu = 2$, $\beta = 0.1$, $\kappa = 3$ and $\delta = 1.2$

Based on the equation (3.8), we made the observation that the time derivative of the Hamilton energy of the system (2.2) is depend upon the parameters and variables of the system. The Hamilton energy is affected by the changes that occur in the behavior of the system, and the behavior of the system is affected by the changes that occur in the value of H. As a consequence of this, the regulation of the Hamilton energy will guarantee the stability of the system (2.2). If we choose the values $\mu = 2.5$, $\beta = 0.3$, $\kappa = 4$, and $\delta = 0.1$, we are able to see the initiation of chaotic states in the system, as shown in Fig. 3. Moreover, the Hamilton energy function is subject to change over time as a consequence of the continuous energy pumping and release that occurs inside the system, as seen in Fig. 5(A). Again, if we choose the parameters as $\mu = 2$, $\beta = 0.1$, $\kappa = 3$, and $\delta = 1.2$, the system becomes stable which can be seen in Fig. 4, and the Hamilton energy is finally stabilized to a constant as seen in Fig. 5(B). Furthermore, the influence of attractors stability on the energy function is seen in Fig. 5.



(A) $\mu = 2.5$, $\beta = 0.3$, $\kappa = 4$ and $\delta = 0.1$. (B) $\mu = 2$, $\beta = 0.1$, $\kappa = 3$ and $\delta = 1.2$.

FIGURE 5. Comparing in terms of energy quantity and consumption of the system (2.2) in chaotic stale (A) and stable state (B).

Remark 3.1. The results of the Hamilton energy investigations guarantee that system (2.2) shown the existence of chaos for the considered parameters $\mu = 2.5$, $\beta = 0.3$, $\kappa = 4$ and $\delta = 0.1$.

4. The bound set of the 3D chaotic system

Within the following subsections, we first provide an explanation of the fundamental terminology and the technique that is used to compute the ultimate bound set. After that, we use the Lagrange multiplier approach to aid in the optimization of bound set, and the outcomes are shown via the use of numerical simulations.

4.1. **Method description.** Assuming that $W = (w_1, w_2, ..., w_n)^T$ is the solution of the following autonomous system:

$$\frac{dW}{dt} = f(W), \tag{4.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinar function. Let $W_0 = W(t_0, t_0, W_0)$ be the initial value of $W(t, t_0, W_0)$, and $\Upsilon \subset \mathbb{R}^n$ is the compact set. Let η represents the distance between $W(t, t_0, W_0)$ and Υ, which is described as:

$$\eta(W(t, t_0, W_0), \Upsilon) = \inf_{Y \in \Upsilon} ||W(t, t_0, W_0) - Y||. \tag{4.2}$$

Suppose for each $\chi > 0$, $\Upsilon_\chi = \{W | \eta(W, \Upsilon) < \chi\}$, then we have $\Upsilon \subset \Upsilon_\chi$.
Definition: [29] Assume that there is a compact set $\Upsilon \in \mathbb{R}^n$ fulfilling the following criteria:

$$\lim_{t \to \infty} \eta(W(t), \Upsilon) = 0, \quad \forall \ W_0 \in \mathbb{R}^n / \Upsilon,$$

i.e., for each $\chi > 0$, there exists $T > t_0$ such that $W(t, t_0, W_0) \in \Upsilon_\chi$ for all $t \geq T$. The set Υ_χ is referred as ultimate bound set of system (4.1). To determine the bound set of system (4.1), let us rewrite the system (4.1) as:

$$\dot{W} = AW + \sum_{i=1}^{n} w_i H_i W + U, \tag{4.3}$$

where $W = (w_1, w_2, ..., w_n)^T \in \mathbb{R}^n$ are system state vectors. Also, $A \in \mathbb{R}^{n \times n}$, $U \in \mathbb{R}^n$, and $H_i = (h^i_{jk})_{n \times n} \in \mathbb{R}^{n \times n}$ with every element of H_i satisfying $h^j_{ik} = h^k_{ij}$, for all $i, j, k = 1, 2, ..., n$. Now, let us assume the quadratic function V as follows:

$$V(W) = (W + \psi)^T M (W + \psi), \tag{4.4}$$

where $M = M^T = (m_{ij})_{n \times n}$, $i, j = 1, 2, ..., n$ is a symmetric matrix and $\psi = (\psi_1, \psi_2, ..., \psi_n) \in \mathbb{R}^n$ are real parameters to be calculated. Taking the derivative of (4.4) we get,

$$\dot{V}(W) = \dot{W}^T M (W + \psi) + (W + \psi)^T M \dot{W}, \tag{4.5}$$

$$\begin{aligned} \dot{V}(W) &= W^T [A^T M + MA + 2(H_1^T M\psi, H_2^T M\psi, ..., H_n^T M\psi)^T] W \\ &+ \sum_{i=1}^{n} w_i W^T (H_i^T M + M H_i) + 2(\psi^T MA + U^T M) W + 2U^T M\psi. \end{aligned} \tag{4.6}$$

Denoting $Q = A^T M + MA + 2(H_1^T M\psi, H_2^T M\psi, ..., H_n^T M\psi)^T$, and $G = 2(\psi^T MA + U^T M)$.
We rewrite as follows:

$$\dot{V}(W) = W^T QW + \sum_{i=1}^{n} w_i W^T (H_i^T M + MH_i)W + GW + 2U^T M\psi. \tag{4.7}$$

Theorem 4.1. *[29] Suppose that $M > 0$ is a positive definite symmetric matrix and $\psi \in \mathbb{R}^n$ is a vector such that*

$$Q = A^T M + MA + 2(H_1^T M\psi, H_2^T M\psi, ..., H_n^T M\psi)^T < 0, \tag{4.8}$$

and for any $W = (w_1, w_2, ..., w_n)^T \in \mathbb{R}^n$

$$\sum_{i=1}^{n} w_i W^T (H_i^T M + MH_i)W = 0, \tag{4.9}$$

then, system (4.3) is bounded and defines its ultimate bound as follows:

$$\Upsilon = \{W \in \mathbb{R}^n : (W + \psi)^T M(W + \psi) \leq J\}, \tag{4.10}$$

where J is a real value that can be determined using the optimization problem described in the following manner:

$$\begin{aligned} &\textit{maximize } \ (W + \psi)^T M(W + \psi) \\ &\textit{subject to } \ W^T QW + 2(\psi^T MA + U^T M)W + 2U^T M\psi = 0. \end{aligned} \tag{4.11}$$

4.2. **Estimating the ultimate bound set of the 3D chaotic system.** We rewrite the system (2.2) in the form of equation (4.3) in order to apply Theorem 4.1 to establish the ultimate bound set of the proposed 3D chaotic system:

$$\dot{W} = AW + \sum_{i=1}^{3} w_i H_i W + U, \tag{4.12}$$

where

$$A = \begin{pmatrix} -1 & \mu & 0 \\ \kappa & -\beta & 0 \\ 0 & 0 & -\delta \end{pmatrix}, H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix},$$

$$H_3 = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } U = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Let $M = M^T = (m_{ij})_{3\times 3}$ for all $i.j = 1, 2, 3$. From Eq. (4.9), we have

$$\sum_{i=1}^{3} w_i W^T (H_i^T M + MH_i)W = 0, \tag{4.13}$$

and equation (4.13) holds for any $w_i \in \mathbb{R}$ with $i = 1, 2, 3$. Thus,

$$\begin{aligned} &2(m_{13} + m_{31})w_1^2 w_2 - 2(m_{21} + m_{12})w_1^2 w_3 + 2(m_{23} + m_{32})w_2^2 w_1 \\ &-2(m_{22} - m_{33})w_1 w_2 w_3 - 2(m_{23} + m_{32})w_3^2 w_1 + 2(m_{23} + m_{32})w_2^2 w_1 = 0. \end{aligned} \tag{4.14}$$

Let $m_{12} = m_{21} = m_{13} = m_{31} = m_{23} = m_{32} = 0$ and $m_{22} - m_{33} = 0$. Corresponding to these calculation the positive definite matrix M becomes,

$$M = \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{22} \end{pmatrix}.$$

Thus, $Q = \begin{pmatrix} -2m_{11} & \psi_3 - \mu m_{11} - \kappa m_{22} & -\psi_2 \\ \psi_3 - \mu m_{11} - \kappa m_{22} & -2\beta m_{22} & 0 \\ -\psi_2 & 0 & -2\delta m_{22} \end{pmatrix}.$

For simplification, choose $\psi = (\psi_1, \psi_2, \psi_3) = \Big(0, 0, \mu m_{11} + \kappa m_{22}\Big)$. Then

$$Q = \begin{pmatrix} -2m_{11} & 0 & 0 \\ 0 & -2\beta m_{22} & 0 \\ 0 & 0 & -2\delta m_{22} \end{pmatrix},$$

and

$$2(\psi^T MA + U^T M) = [0, 0, -2\delta m_{22}(\mu m_{11} + r m_{22})].$$

Therefore,

$$V(W) = (W + \psi)^T M(W + \psi), \tag{4.15}$$

$$V(W) = m_{11}w_1^2 + m_{22}w_2^2 + m_{22}(w_3 + (\mu m_{11} + \kappa m_{22}))^2, \tag{4.16}$$

and $\dot{V}(W) = 0$ gives,

$$-2m_{11}w_1^2 - 2\beta m_{22}w_2^2 - 2\delta m_{22}w_3^2 - 2\delta(\mu m_{11} + \kappa m_{22})w_3 = 0.$$

Theorem 4.2. *Suppose that $\mu > 0$, $\beta > 0$, $\kappa > 0$, $\delta > 0$, $Q < 0$ and $m_{ii} \in \mathbb{R}^+$. Denote*

$$\Upsilon = \Big\{W(t) \in \mathbb{R}^3 : m_{11}w_1^2 + m_{22}w_2^2 + m_{22}(w_3 + (\mu m_{11} + \kappa m_{22}))^2 \leq J\Big\}. \tag{4.17}$$

Then, the system (4.12) has an ultimate bound set Υ, where

$$J = \begin{cases} \frac{\delta^2}{(\delta-1)} \frac{m_{22}(\mu m_{11} + \kappa m_{22})^2}{4}, & (\mu, \beta, \kappa, \delta) \in E_1, \\ \frac{\delta^2}{\beta(\delta-\beta)} \frac{m_{22}(\mu m_{11} + \kappa m_{22})^2}{4}, & (\mu, \beta, \kappa, \delta) \in E_2, \\ m_{22}(\mu m_{11} + \kappa m_{22})^2, & (\mu, \beta, \kappa, \delta) \in E_3, \end{cases} \tag{4.18}$$

and J is a real value that can be determined by using the problem described in the following manner:

$$\begin{aligned} &\textit{maximize } \ (W + \psi)^T M(W + \psi) \\ &\textit{subject to } \ W^T QW + 2(\psi^T MA + U^T M)W + 2U^T M\psi = 0, \end{aligned} \tag{4.19}$$

and

$$E_1 = \{(\mu, \beta, \kappa, \delta) \in \mathbb{R}^{3+} |\ \beta > 1,\ \beta \leq (\delta - 1)\},$$

$$E_2 = \{(\mu, \beta, \kappa, \delta) \in \mathbb{R}^{3+} |\ 0 < \beta < 1,\ \beta \leq (\delta - 1)\},$$

$$E_3 = \mathbb{R}^{3+} - (E_1 \cup E_2).$$

Proof. Consider the following problem in order to optimize:

$$\begin{aligned} &\textit{maximize } \ V = (W + \psi)^T M(W + \psi) \\ &\textit{subject to } \ W^T QW + 2(\psi^T MA + U^T M)W + 2U^T M\psi = 0. \end{aligned} \tag{4.20}$$

This implies that,

$$\begin{aligned} &max \;\; V = m_{11}w_1^2 + m_{22}w_2^2 + m_{22}(w_3 + (\mu m_{11} + \kappa m_{22}))^2, \\ &s.t. \;\; m_{11}w_1^2 + \beta m_{22}w_2^2 + \delta m_{22}w_3^2 + \delta(\mu m_{11} + \kappa m_{22})w_3 = 0. \end{aligned} \tag{4.21}$$

Expressing $\sqrt{m_{22}}\left(\frac{\mu m_{11}+\kappa m_{22}}{2}\right) = \tau$. Let $\sqrt{m_{11}}w_1 = y_1$, $\sqrt{m_{22}}w_2 = y_2$ and $\sqrt{m_{33}}w_3 = y_3$. Then equation (4.21) can be rewritten as follows:

$$\begin{aligned} &max \;\; V = y_1^2 + y_2^2 + y_3^2, \\ &s.t \;\; y_1^2 + \beta y_2^2 + \delta(y_3 + \tau)^2 = \delta\tau^2. \end{aligned} \tag{4.22}$$

Using the Lagrange method, we defined Θ as,

$$\Theta = y_1^2 + y_2^2 + y_3^2 - \zeta\Bigg[y_1^2 + \beta y_2^2 + \delta(y_3 + \tau)^2 - \delta\tau^2\Bigg].$$

Differentiate both sides,

$$\begin{aligned} \frac{\partial\Theta}{\partial y_1} &= 2y_1 - \zeta[2\mu y_1] = 0, \\ \frac{\partial\Theta}{\partial y_2} &= 2y_2 - \zeta[2\beta y_2] = 0, \\ \frac{\partial\Theta}{\partial y_3} &= 2y_3 - \zeta[2\delta(y_3 + \tau)] = 0, \\ \frac{\partial\Theta}{\partial \zeta} &= -\Bigg[y_1^2 + \beta y_2^2 + \delta(y_3 + \tau)^2 - \delta\tau^2\Bigg] = 0. \end{aligned} \tag{4.23}$$

(i) If $\zeta = 1$ and $\delta - 2 > 0$, $\delta \neq 1$ then

$$\begin{aligned} (y_1^\star, y_2^\star, y_3^\star) =& \Bigg(\pm \frac{\delta\tau}{(1-\delta)}\sqrt{(\delta-2)}, 0, \frac{\delta\tau}{(1-\delta)}\Bigg), \\ \text{we get } V_1 =& V(y_1^\star, y_2^\star, y_3^\star) = \frac{\delta^2\tau^2}{(\delta-1)}. \end{aligned} \tag{4.24}$$

(ii) If $\zeta = \frac{1}{\beta}$ and $\delta - 2\beta > 0$, $\beta \neq \delta$ then

$$\begin{aligned} (y_1^\star, y_2^\star, y_3^\star) =& \Bigg(0, \pm\frac{\delta\tau}{(\beta-\delta)}\sqrt{\frac{(\delta-2\beta)}{\beta}}, \frac{\delta\tau}{(\beta-\delta)}\Bigg), \\ \text{we get } V_2 =& V(y_1^\star, y_2^\star, y_3^\star) = \frac{\delta^2\tau^2}{\beta(\delta-\beta)}. \end{aligned} \tag{4.25}$$

(iii) If $\zeta \neq 1$ and $\zeta \neq \frac{1}{\beta}$, then

$$\text{(a)} \qquad (y_1^\star, y_2^\star, y_3^\star) = (0, 0, -2\tau), \text{ we get, } V_3 = V(y_1^\star, y_2^\star, y_3^\star) = 4\tau^2. \tag{4.26}$$

$$\text{(b)} \qquad (y_1^\star, y_2^\star, y_3^\star) = (0, 0, 0), \text{ we obtain, } V_4 = V(y_1^\star, y_2^\star, y_3^\star) = 0. \tag{4.27}$$

Defining the sets:

$$\begin{aligned} E_1 &= \{(\mu, \beta, \kappa, \delta) \in \mathbb{R}^{3+} | \;\; \beta > 1, \beta \geq (\delta - 1)\}, \\ E_2 &= \{(\mu, \beta, \kappa, \delta) \in \mathbb{R}^{3+} | \;\; 0 < \beta < 1, \beta \leq (\delta - 1)\}, \end{aligned}$$

$$E_3 = \mathbb{R}^{3+} - (E_1 \cup E_2).$$

One can have,

$$V_{max} = \begin{cases} \frac{\delta^2}{(\delta-1)}\frac{m_{22}(\mu m_{11}+\kappa m_{22})^2}{4}, & (\mu,\beta,\kappa,\delta) \in E_1, \\ \frac{\delta^2}{\beta(\delta-\beta)}\frac{m_{22}(\mu m_{11}+\kappa m_{22})^2}{4}, & (\mu,\beta,\kappa,\delta) \in E_2, \\ m_{22}(\mu m_{11}+\kappa m_{22})^2, & (\mu,\beta,\kappa,\delta) \in E_3, \end{cases} \tag{4.28}$$

and, $V_{max} = J$. Thus, the proof is concluded. □

4.3. **Simulation Results.** Theoretical results are verified by the numerical simulations. The phase portraits and ultimate bound sets, described by $\Upsilon_{\mu,\beta,\kappa,\delta}$ of the system (4.12) are shown as below when the various value of $\mu > 0$, $\beta > 0$, $\kappa > 0$ and $\delta > 0$ are taken under consideration. If we choose $\mu = 2.5$, $\beta = 0.3$, $\kappa = 4$ and $\delta = 0.1$, then we have,

$$\Upsilon_{2.5,0.3,4,0.1} = \left\{ (w_1, w_2, w_3) | 0.7w_1^2 + 0.7w_2^2 + 0.7(w_3 + 4.55)^2 \le (0.7)(4.55)^2 \right\}.$$

The ultimate bound set $\Upsilon_{2.5,0.3,4,0.1}$, which is displayed in Fig.6, comprised with various space trajectories namely chaotic attractors. If we select the parameter as: $\mu = 2$, $\beta = 0.1$, $\kappa = 3$ and $\delta = 1.2$, then we have,

$$\Upsilon_{2,0.1,3,1.2} = \left\{ (w_1, w_2, w_3) | 0.7w_1^2 + 0.7w_2^2 + 0.7(w_3 + 3.5)^2 \le (3.2727)(3.5)^2 \right\}.$$

The ultimate bound set $\Upsilon_{2,0.1,3,1.2}$, which is displayed in Fig.7, comprised each of the various space trajectories namely phase portraits.

5. Application of bound set in synchronization

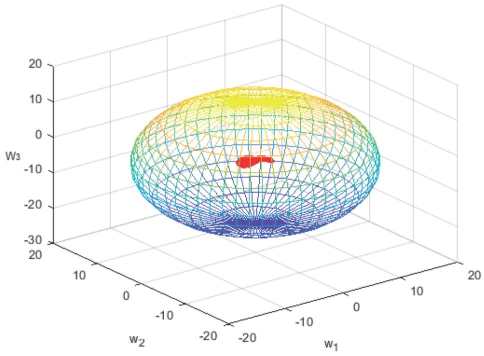
The conclusions that are produced in Theorem 4.2 have been used in order to accomplish the goal of synchronization between two 3D chaotic systems that are identical. It has been explained in this section that an exponential synchronization scheme with linear feedback control is being used. First, we will provide the lemma as follows:

Lemma 5.1. *For given any $\mu > 0$, w_1, $w_2 \in \mathbb{R}$, then the inequality $2w_1w_2 \le \mu w_1^2 + \frac{1}{\mu}w_2^2$ holds.*

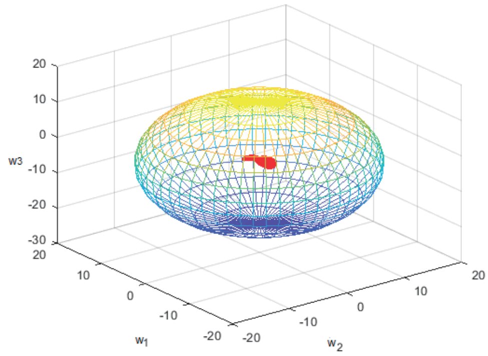
Consider the 3D chaotic system (2.2) as master (drive) system and the corresponding slave (response) system is defined as follows:

$$\begin{cases} \dot{z}_1 = -z_1 - \mu z_2 + p_1, \\ \dot{z}_2 = -z_1 z_3 - \beta z_2 - \kappa z_1 + p_2, \\ \dot{z}_3 = z_1 z_2 - \delta z_3 + p_3, \end{cases} \tag{5.1}$$

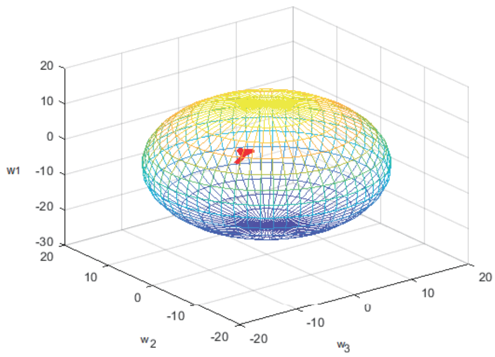
where z_1, z_2 and z_3 are the state vectors of the system and μ, β, κ and δ are parameters, while p_1, p_2 and p_3 are the controllers to be designed to obtain the



(A) UBS with $w_1 - w_2 - w_3$ space



(B) UBS with $w_2 - w_1 - w_3$ space



(C) $w_3 - w_2 - w_1$ space

FIGURE 6. (A), (B) and (C):UBS with chaotic attractors of system (4.12) with initial conditions $(-0.4, 0.1, -4)$, where $\mu = 2.5$, $\beta = 0.3$, $\kappa = 4$, and $\delta = 0.1$.

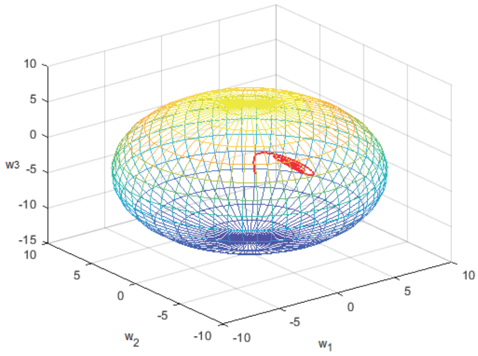
synchronization between the systems (2.2) and (5.1). On the other hand, by using Theorem 4.2, we can get,

$$\begin{cases} |w_1| \leq \sqrt{\frac{J}{m_{11}}} = \Delta_1, \\ |w_2| \leq \sqrt{\frac{J}{m_{22}}} = \Delta_2, \\ |w_3| \leq \sqrt{\frac{J}{m_{22}}} + |\mu m_{11} + \kappa m_{22}| = \Delta_3, \end{cases} \tag{5.2}$$

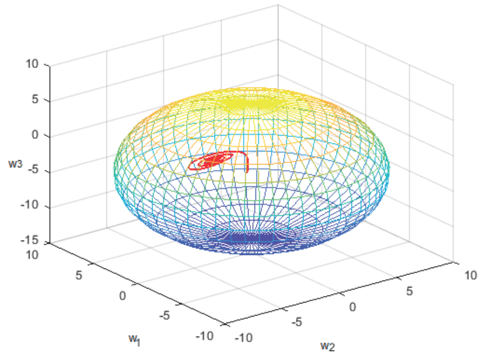
where J is defined in equation (4.18).

Theorem 5.2. *To obtain the exponential synchronization between the drive system (2.2) and the response system (5.1) with exponential manner. The value of the controllers are chosen as follows:*

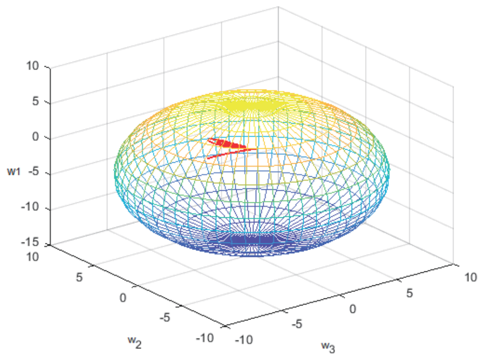
$$p_1 = -k_1 e_1, \;\; p_2 = -k_2 e_2 \;\; \textit{and} \;\; p_3 = -k_3 e_3, \tag{5.3}$$



(A) UBS with $w_1 - w_2 - w_3$ space



(B) UBS with $w_2 - w_1 - w_3$ space



(C) UBS with $w_3 - w_2 - w_1$ space

FIGURE 7. (A), (B) and (C): UBS with phase portraits of system (4.12) with initial conditions $(-0.4, 0.1, -4)$, where $\mu = 2$, $\beta = 0.1$, $\kappa = 3$, and $\delta = 1.2$.

with $k_1 > \frac{\Delta_2}{2\sqrt{\rho}} - \frac{\kappa}{2\sqrt{\rho}} - \frac{\Delta_3}{2\sqrt{\rho}} - \frac{\mu}{2\sqrt{\rho}} - 1$, $\quad k_2 + \frac{\Delta_3\sqrt{\rho}}{2} + \beta + \frac{\kappa\sqrt{\rho}}{2} + \frac{\mu\sqrt{\rho}}{2} > 0$, $\quad k_3 + \frac{\Delta_2\sqrt{\rho}}{2} + \delta > 0$ *and* ρ *is a real positive parameter.*

Proof. Define the error system as follows:

$$e_1 = z_1 - w_1, \;\; e_2 = z_2 - w_2, \;\; \textit{and} \;\; e_3 = z_3 - w_3,$$

then the error dynamics becomes, $\;\dot{e}_i = \dot{z}_i - \dot{w}_i, \;\; for \;\; i = 1, 2, 3.$

$$\begin{cases} \dot{e}_1 = -e_1 - \mu e_2 + p_1, \\ \dot{e}_2 = -e_1 e_3 - e_1 w_3 - e_3 w_1 - \beta e_2 - \kappa e_1 + p_2, \\ \dot{e}_3 = e_1 e_2 + e_1 w_2 + e_2 w_1 - \delta e_3 + p_3. \end{cases} \tag{5.4}$$

Take the Lyapunov function as follows:

$$\nu(e_1, e_2, e_3) = \frac{1}{2}\rho e_1^2 + \frac{1}{2}\rho e_2^2 + \frac{1}{2}\rho e_3^2. \tag{5.5}$$

Using (5.4) and (5.5), one can get

$$\begin{aligned}\dot{\nu}(e_1,e_2,e_3) &= \rho e_1[-e_1-\mu e_2-k_1e_1]+\rho e_2[-e_1e_3-e_1w_3\\ &\quad -e_3w_1-\beta e_2-\rho\kappa e_1-k_2e_2]+\rho e_3[e_1e_2+e_1w_2+e_2w_1-\delta e_3-k_3e_3],\\ \dot{\nu}(e_1,e_2,e_3) &= -(\rho+\rho k_1)e_1^2-(\rho\beta+\rho k_2)e_2^2\\ &\quad -(\rho\delta+\rho k_3)e_3^2-\rho\mu e_1e_2-\rho e_1e_2w_3-\rho\kappa e_1e_2+\rho e_1e_3w_2. \end{aligned} \tag{5.6}$$

By using Lemma 5.1 and the equation (5.2), we obtain

$$\begin{cases} \rho\mu e_1e_2 \le \mu\Big(\rho^{\frac14}|e_1|\rho^{\frac34}|e_2|\Big) \le \Big(\frac{\mu\sqrt{\rho}}{2}e_1^2+\frac{\mu\rho\sqrt{\rho}}{2}e_2^2\Big),\\ \rho e_1e_2w_3 \le \Delta_3\Big(\rho^{\frac14}|e_1|\rho^{\frac34}|e_2|\Big) \le \Big(\frac{\Delta_3\sqrt{\rho}}{2}e_1^2+\frac{\Delta_3\rho\sqrt{\rho}}{2}e_2^2\Big),\\ \rho e_1e_2\kappa \le \kappa\Big(\rho^{\frac14}|e_1|\rho^{\frac34}|e_2|\Big) \le \Big(\frac{\kappa\sqrt{\rho}}{2}e_1^2+\frac{\kappa\rho\sqrt{\rho}}{2}e_2^2\Big),\\ \rho e_1e_3w_2 \le \Delta_2\Big(\rho^{\frac14}|e_1|\rho^{\frac34}|e_3|\Big) \le \Big(\frac{\Delta_2\sqrt{\rho}}{2}e_1^2+\frac{\Delta_2\rho\sqrt{\rho}}{2}e_3^2\Big). \end{cases} \tag{5.7}$$

From (5.6) and (5.7), we obtain

$$\begin{aligned}\dot{\nu}(e_1,e_2,e_3) \le &-\Big(\rho k_1+\rho-\frac{\Delta_2\sqrt{\rho}}{2}+\frac{\kappa\sqrt{\rho}}{2}+\frac{\Delta_3\sqrt{\rho}}{2}+\frac{\mu\sqrt{\rho}}{2}\Big)e_1^2\\ &-\Big(\rho k_2+\frac{\Delta_3\rho\sqrt{\rho}}{2}+\beta\rho+\frac{\kappa\rho\sqrt{\rho}}{2}+\frac{\mu\rho\sqrt{\rho}}{2}\Big)e_2^2\\ &-\Big(\rho k_3+\rho\delta+\frac{\Delta_2\rho\sqrt{\rho}}{2}\Big)e_3^2. \end{aligned} \tag{5.8}$$

Set

$$\begin{cases} \rho k_1+\rho-\frac{\Delta_2\sqrt{\rho}}{2}+\frac{\kappa\sqrt{\rho}}{2}+\frac{\Delta_3\sqrt{\rho}}{2}+\frac{\mu\sqrt{\rho}}{2}=\beta_1>0,\\ \rho k_2+\frac{\Delta_3\rho\sqrt{\rho}}{2}+\beta\rho+\frac{\kappa\rho\sqrt{\rho}}{2}+\frac{\mu\rho\sqrt{\rho}}{2}=\beta_2>0,\\ \rho k_3+\rho\delta+\frac{\Delta_2\rho\sqrt{\rho}}{2}=\beta_3>0,\\ \hat{\alpha}=\min\{\beta_1,\beta_2,\beta_3,\}>0. \end{cases} \tag{5.9}$$

Substituting the values from Eq. (5.9), equation (5.8) becomes

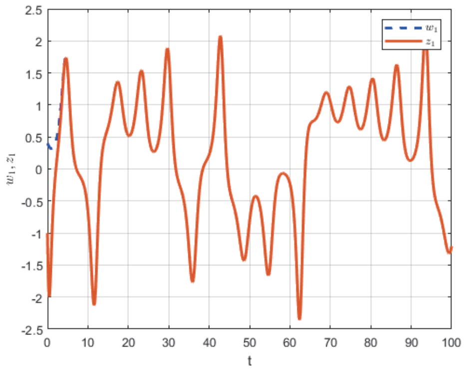
$$\dot{\nu}(e_1,e_2,e_3) \le -\beta_1e_1^2-\beta_2e_2^2-\beta_3e_3^2, \tag{5.10}$$

$$\text{that is,}\qquad \dot{\nu}(e_1,e_2,e_3) \le -\hat{\alpha}\nu(e_1,e_2,e_3). \tag{5.11}$$

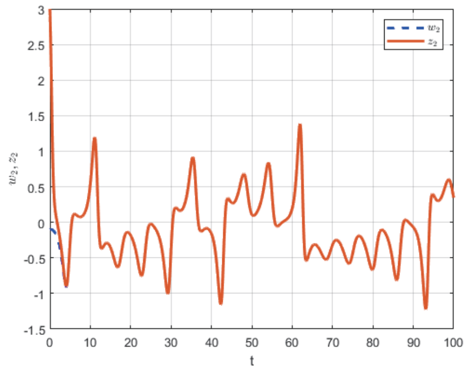
Hence, $\nu(t) \le \nu(t_0)e^{-\hat{\alpha}(t-t_0)}$. It demonstrates that the master system (2.2) and the slave system (5.1) are synchronized with one another in a globally exponential manner, as shown in Fig. 8. □

6. Conclusion

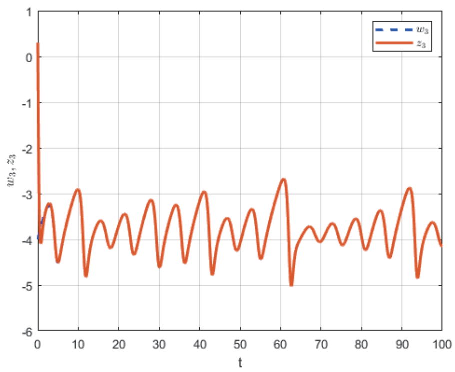
This study analyzes a new 3D Lorenz-like chaotic system from the perspective of dynamical analysis. An intriguing fact is that the Hamilton energy function changes in response to changes in the features and behavior of the chaotic system under consideration, and changes to the energy function have an impact on the system's behavior. We have also determined explicit bound sets for the solutions of



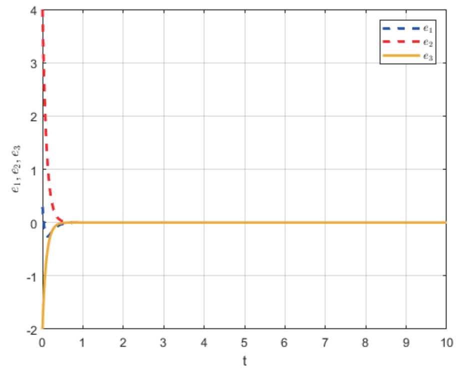
(A) Synchronization between w_1 and z_1



(B) Synchronization between w_2 and z_2



(C) Synchronization between w_3 and z_3



(D) Synchronized errors dynamics e_j, for $j = 1, 2, 3$

FIGURE 8. (A), (B), (C) Synchronized state-trajectories and (D) Errors dynamics synchronization of the systems (2.2) and (5.1) with initial conditions $(-0.4, 0.1, -4)$, $(-4, -1, 3)$ and $k_1 = 1$, $k_2 = 2$, $k_3 = 3$.

proposed system by solving an optimization problem. We have observed that given approach is effective under certain circumstances, and it has the potential to broaden the scope to point where any kind of attractors may be found inside the bound set. Additionally, we have discussed the use of bound sets in the synchronization process. Several simulations demonstrate the technical feasibility of the proposed procedures. It is possible to employ this chaotic system in the future to encrypt images, deal with secure communications, and solve circuit difficulties.

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