Journal of Nonlinear and Convex Analysis
Volume 26, Number 4, 2025, 739–748

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

EXPLORING REFINEMENTS TO WEAKLY PICARD MAPPINGS OF POPESCU TYPE WITH AN APPLICATION TO DYNAMIC PROGRAMMING

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Abstract. In this paper, we consider and refine a single-valued analogue of the multivalued contraction due to Popescu and proved that such a map is weakly Picard mapping. In support of our fixed point theorem proved herein, we furnish several illustrative examples whereas the involved mapping have more than one fixed points. To demonstrate the degree of applicability of our result, we utilize the solvability of certain functional equation related to dynamic programming.

1. Introduction

In what follows, $\mathbf{F}(T)$ denotes the set of fixed points of a mapping T. Let us summarize several noted classes of mappings in metric fixed point theory frequently used in existing literature.

Definition 1.1 ([7–9, 13]). Let (X, d) be a metric space and T a self-mapping on X. Then

- a point $x \in X$ is called a *Picard point* of T if the sequence $\{T^n x\}$ is convergent.
- T is called *pre-Picard mapping* if every point of X is a Picard point. In other words, T is called pre-Picard mapping if for each $x \in X$, the sequence $\{T^n x\}$ is convergent.
- A pre-Picard mapping T is called *Picard mapping* if T has a unique fixed point x^* and for all $x \in X$, the sequence $\{T^n x\}$ converges to x^*.
- A pre-Picard mapping T is called *weakly Picard mapping* if $\mathbf{F}(T)$ is nonempty and for each $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point of T.

Obviously, we have the following implications:

$$\text{Picard maps} \Longrightarrow \text{weakly Picard maps} \Longrightarrow \text{pre} - \text{Picard maps}.$$

A contraction mapping defined on a complete metric space is an oblivious example of Picard mapping. The mapping involved in Caristi theorem is an example of pre-Picard mapping which is not weakly Picard. Now, we shall revisit to a noted class of weakly Picard mappings which is defined as below.

2020 *Mathematics Subject Classification.* 47H10, 54H25, 39B52, 30L99.

Key words and phrases. Fixed points, weakly Picard mappings, inevitable contractions, functional equations.

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Definition 1.2 ([9])**.** A self-mapping T on metric space (X, d) is called *graphic contraction* if there exists $k \in [0, 1)$ such that

$$d(Tx, T^2x) \leq kd(x, Tx), \quad \forall\, x \in X.$$

The name graphic contraction was given due to fact that the contraction condition is required to hold merely on the elements of Graph of T rather than all elements of X^2. Independently, Rus [6] and Subrahmanyam [11] first proved the graphic contraction principle, that runs as follows:

Theorem 1.3 ([6, 11])**.** *A continuous graphic contraction mapping on a complete metric space is a weakly Picard mapping.*

In 2013, Popescu [5] improved the Ćirić-type multivalued contraction [2] by introducing the concept of (λ, k)-contractive multivalued mappings with real constants $0 \leq k < 1$ and $\lambda > k$ and utilized the same to prove an interesting generalization of classical fixed point theorems of Nadler [4] as well as Ćirić [2]. In process, Popescu [5] derived a corresponding fixed point theorem in case of single valued (λ, k)-contractive mappings. In the last several years, this paper has attracted the attention of several researchers of this domain and by now there exist a considerable literature on and around this paper but all such results are proved for multivalued maps. The authors of the present paper are of the view that there is a considerable scope of proving similar result for single valued mappings besides exploring the possibilities of refinements in single-valued considerations. With this formation in mind, we consider a special but most natural version of the Popescu's single valued (λ, k)-contractive mapping, which turns out to be a weakly Picard mapping on a complete metric space.

In Section 2, we term the Popescu's single valued (λ, k)-contractive mapping as inevitable contraction and refine the non-unique fixed point theorem under inevitable contraction. Our result is a straightforward variant of Banach contraction principle. To testify the credibility of our results, we construct two examples with more than one fixed points. We also observe that the inevitable contraction remains a class of graphic contractions. However our result is different from Theorem 1.3 due to the removal of continuity requirement.

In Section 3, we utilize our result to discuss the existence of bounded solutions of a class of functional equations arising in dynamic programming.

Section 4 is devoted to highlight the role and the meaning of the concept of inevitable contractions. This work and the work of this kind can inspire new researches refinements and improvements.

2. Main results

With a view to emphasize the role of Popescu [5] type single-valued mapping, let us agree to re-name it as "inevitable contraction mapping".

Definition 2.1. Let (X, d) be a metric space and T a self-mapping on X. If there exist real numbers k and λ verifying $0 \leq k < 1$ and $k < \lambda$ such that for all $x, y \in X$,

$$d(y, Tx) \leq \lambda d(y, x) \Longrightarrow d(Tx, Ty) \leq kd(x, y),$$

then T is called a (λ, k)-inevitable contraction or simply, an inevitable contraction.

Clearly, every contraction mapping is an inevitable contraction, but not conversely.

Proposition 2.2. *Every inevitable contraction is a graphic contraction.*

Proof. Take an arbitrary $x \in X$. Then for every $\lambda \geq 0$, the inequality

$$d(y, Tx) \leq \lambda d(y, x)$$

will be satisfied automatically for $y = T(x)$. Consequently, for such pair, the inevitable contraction condition reduces to

$$d(Tx, T^2x) \leq kd(x, Tx).$$

As x is an arbitrary, above inequality holds for all $x \in X$. This completes the proof. $\square$

Now, we prove a metrical fixed point theorem under inevitable contraction, which generalizes the Banach contraction principle. Apart from Theorem 1.3, in the hypotheses of our result, there is no need continuity of T. The existence part of this result follows from Theorem 2.5 of Popescu [5]. However, for the sake of brevity and self-containment, we present a complete proof.

Theorem 2.3. *Every inevitable contraction on a complete metric space is a weakly Picard mapping.*

Proof. Let (X, d) be a complete metric space and $T : X \longrightarrow X$ an inevitable contraction. Then there exist real numbers k and λ verifying $0 \leq k < 1$ and $k < \lambda$ such that for all $x, y \in X$,

$$d(y, Tx) \leq \lambda d(y, x) \Longrightarrow d(Tx, Ty) \leq kd(x, y). \tag{2.1}$$

Take an arbitrary $x_0 \in X$ and construct the sequence $\{x_n\}$ of successive approximations of T, *i.e*, $x_n := T^n(x_0)$ so that

$$x_n = T(x_{n-1}), \quad \forall\, n \in \mathbb{N}.$$

Setting $x = x_{n-1}$ and $y = x_n$, we get

$$d(y, Tx) = d(x_n, x_n) = 0 \leq \lambda d(x_n, x_{n-1}) = \lambda d(y, x).$$

Hence, by applying the contractivity condition (2.1) on these points, we deduce that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq kd(x_{n-1}, x_n), \quad \forall\, n \in \mathbb{N} \tag{2.2}$$

which using induction on n, reduces to

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1), \quad \forall\, n \in \mathbb{N}. \tag{2.3}$$

For $n < m$, using triangular inequality and (2.3), we obtain

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq [k^n + k^{n+1} + \cdots + k^{m-1}]d(x_0, x_1) \\ &= \frac{k^n - k^m}{1-k} d(x_0, x_1) \\ &\leq \frac{k^n}{1-k} d(x_0, x_1) \end{aligned}$$

so that

$$d(x_n, x_m) \leq \frac{k^n}{1-k} d(x_0, x_1).$$

As $0 \leq k < 1$ and $d(x_0, x_1)$ is fixed, taking the limit as n tends to ∞ in above inequality, the right hand side approaches to 0 consequently $d(x_n, x_m) \xrightarrow{\mathbb{R}} 0$. Thus, the sequence $\{x_n\}$ is Cauchy. Owing to completeness of metric space X, there exists $\overline{x} \in X$ such that $x_n \xrightarrow{d} \overline{x}$.

Now, we claim that

$$d(\overline{x}, Tx_n) = d(\overline{x}, x_{n+1}) \leq \lambda d(\overline{x}, x_n), \quad \forall\, n \in \mathbb{N}. \tag{2.4}$$

On contrary, assume that there exists a positive integer N such that

$$d(\overline{x}, x_{n+1}) > \lambda d(\overline{x}, x_n), \quad \forall\, n \geq N.$$

By easy induction, we obtain for all $n \geq N$ and $r \geq 1$ that

$$d(\overline{x}, x_{n+r}) > \lambda^r d(\overline{x}, x_n)$$

so that

$$d(\overline{x}, x_n) < \lambda^{-r} d(\overline{x}, x_{n+r}) \tag{2.5}$$

On the other hand, using (2.2) and the triangular inequality, we get for all $n \geq N$ and $s \geq 1$ that

$$\begin{aligned} d(x_{n+s}, x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+s-1}, x_{n+s}) \\ &\leq [1 + k + k^2 + \cdots + k^{s-1}] d(x_n, x_{n+1}) \\ &= \frac{1-k^s}{1-k} d(x_n, x_{n+1}). \end{aligned}$$

Letting $s \to \infty$ in above inequality, we obtain

$$d((\overline{x}, x_n) \leq \frac{1}{1-k} d(x_n, x_{n+1}), \quad \forall\, n \in \mathbb{N}$$

which on replacing n by $n + r$ gives rise to

$$d((\overline{x}, x_{n+r}) \leq \frac{1}{1-k} d(x_{n+r}, x_{n+r+1}), \quad \forall\, n \in \mathbb{N} \text{ and } r \geq 1. \tag{2.6}$$

Making use of (2.2) and by an easy induction, we get for all $n \in \mathbb{N}$ and $r \geq 1$ that

$$\begin{aligned} d(x_{n+r}, x_{n+r+1}) &\leq kd(x_{n+r-1}, x_{n+r}) \\ &\leq k^2 d(x_{n+r-2}, x_{n+r-1}) \\ &\;\;\vdots \\ &\leq k^r d(x_n, x_{n+1}). \end{aligned} \tag{2.7}$$

On combining (2.5), (2.6) and (2.7), we obtain for all $n \geq N$ and $r \geq 1$ that

$$d(\overline{x}, x_n) \leq \frac{(k\lambda^{-1})^r}{1-k} d(x_n, x_{n+1}).$$

As $k < \lambda$, we have $k\lambda^{-1} < 1$. Therefore on taking the limit as $r \to \infty$, we obtain $d(\overline{x}, x_n) = 0$, for all $n \geq N$, which contradicts to (2.5). Hence, (2.4) holds. Using (2.1) and (2.4), we obtain

$$d(x_{n+1}, T\overline{x}) = d(Tx_n, T\overline{x}) \leq kd(x_n, \overline{x})$$

which by using $n \to \infty$ gives rise to $x_{n+1} \xrightarrow{d} T(\overline{x})$. Owing to uniqueness of limit, we get $T(\overline{x}) = \overline{x}$. This completes the proof. □

Remark 2.4. Under the restriction $\lambda \geq 1$, any (λ, k)-inevitable on a complete metric space is a Picard mapping. Indeed, for every pair $\overline{x}, \overline{y} \in \mathbf{F}(T)$, we have

$$d(\overline{y}, T\overline{x}) = d(\overline{y}, \overline{x}) \leq \lambda d(\overline{y}, \overline{x}).$$

Therefore, applying (λ, k)-inevitable contraction condition on these points, we get

$$d(\overline{x}, \overline{y}) = d(T\overline{x}, T\overline{y}) \leq kd(\overline{x}, \overline{y})$$

implying thereby

$$(1 - k)d(\overline{x}, \overline{y}) \leq 0 \Longrightarrow d(\overline{x}, \overline{y}) = 0, \text{ as } 0 \leq k < 1.$$

Hence, we get the uniqueness of fixed points.

To demonstrate Theorem 2.3, we furnish the following examples.

Example 2.5. Consider $X = [0, \frac{9}{10}]$ with usual metric d. Then (X, d) is complete metric space. Define a self-mapping T on X by

$$T(x) = \begin{cases} 2/3, & x \in (0, \frac{9}{10}]; \\ 0, & x = 0. \end{cases}$$

Let us choose $\lambda = \frac{7}{10}$ and $k = \frac{2}{3}$. Now, we consider the following cases:

Case-1: If $x = y = 0$, then $\lambda d(x, y) = 0$. In this case, we have $d(y, Tx) = 0$ and hence the inequality $d(y, Tx) \leq \lambda d(y, x)$ will be satisfied for $x = y = 0$. Now, for the given pair of points, we have

$$d(T0, T0) = 0 = kd(0, 0).$$

Case-2: If $x = 0$ and $y \neq 0$, then we have $d(y, Tx) = y$ and $\lambda d(y, x) = \dfrac{7y}{10}$. Thus, the inequality $d(y, Tx) \leq \lambda d(y, x)$ will never be satisfied for any values of $y \in \big(0, \frac{9}{10}\big]$.

Case-3: If $y = 0$ and $x \neq 0$, then we have $d(y, Tx) = \dfrac{2}{3}$ and and $\lambda d(y, x) = \dfrac{7x}{10}$. Thus, the inequality $d(y, Tx) \leq \lambda d(y, x)$ will never be satisfied for any values of $x \in \big(0, \frac{9}{10}\big]$.

Case-4: Denote $\mathcal{S} := \{(x, y) \in X^2 : x \neq 0, y \neq 0, d(y, Tx) \leq \lambda d(y, x)\}$. Now for each pair $x, y \in X$ with $x \neq 0$, $y \neq 0$, we have

$$d(Tx, Ty) = 0 < \frac{2}{3}|x - y| = kd(x, y).$$

As a consequence, the above inequality will be satisfied for every $(x, y) \in \mathcal{S}$.

Thus, T is an inevitable contraction mapping. Consequently, by Theorem 2.3, T is a weakly Picard mapping. Notice that $\mathbf{F}(T) = \left\{0, \frac{2}{3}\right\}$.

Example 2.6. Consider $X = \{a, b, c, d, e, f\}$ equipped with discrete metric d, i.e.,

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y. \end{cases}$$

Clearly, (X, d) is complete metric space. Define a self-mapping T on X by

$$T := \begin{pmatrix} a & b & c & d & e & f \\ a & b & c & a & b & c \end{pmatrix}$$

Let $k \in (0,1)$ and $\lambda \in (k,1)$ be fixed. Now, to substantiate the contractive condition of Theorems 2.3, we need to discuss the following cases.

Case-1: If $(x,y) \in \{(a,a),(b,b),(c,c)\}$, then $d(y,Tx) = 0 = \lambda d(y,x)$ and $d(Tx,Ty) = 0 = kd(x,y)$.

Case-2: If $(x,y) \in \{(d,a),(e,b),(f,c)\}$, then $d(y,Tx) = 0 < \lambda \cdot 1 = \lambda d(y,x)$ and $d(Tx,Ty) = 0 < k \cdot 1 \leq kd(x,y)$.

Case-3: In all remaining pairs (x,y), we have $d(y,Tx) = 1$ and $\lambda d(y,x) = \lambda$. Consequently, the inequality $d(y,Tx) \leq \lambda d(y,x)$ never holds.

Thus in view of Theorem 2.3, T is a weakly Picard mapping. Notice that $\mathbf{F}(T) = \{a, b, c\}$.

3. An application in dynamic programming

Various methods of fixed point theory are widely used in the field of mathematical optimization. It is commonly recognized that dynamic programming offers practical resources for computer programming and mathematical optimization. Under this scenario, the dynamic programming problem associated with a multistage process reduces to solution of specific functional equation described in the forthcoming lines.

Throughout the section, we assume that A and B are Banach spaces, $\mathcal{Z} \subset A$ is the state space and $\mathcal{S} \subset B$ is the decision space. If $\theta : \mathcal{Z} \times \mathcal{S} \to \mathcal{Z}$, $\hbar : \mathcal{Z} \times \mathcal{S} \to \mathbb{R}$ and $F : \mathcal{Z} \times \mathcal{S} \times \mathbb{R} \to \mathbb{R}$ are known functions, then the return function $f : \mathcal{Z} \to \mathbb{R}$ of the continuous decision process is defined by the functional equation:

$$f(t) = \sup_{s \in \mathcal{S}} \{\hbar(t,s) + F(t,s,f(\theta(t,s)))\}, \quad \text{for each } t \in \mathcal{Z}. \tag{3.1}$$

Let us consider the following hypothesis:

$(*)$: *If for a given $\epsilon > 0$, there exist $\delta \geq 0$, $t_1, t_2 \in \mathcal{Z}$ and $s_1, s_2 \in \mathcal{S}$ such that for all $t \in \mathcal{Z}$, for some real number $\lambda > 0$ and for some pair of real-valued functions α, β defined on $\mathcal{Z}$, the following inequalities hold:*

$$\begin{cases} -\hbar(t,s_1) - F(t,s_1,\alpha(t_1)) + \beta(t) < 2\epsilon + \lambda\delta \\ \hbar(t,s_2) + F(t,s_2,\alpha(t_2)) - \beta(t) < \epsilon + \lambda\delta, \end{cases} \tag{3.2}$$

then there exists $k \in (0,1)$ *verifying* $k < \lambda$ *such that for all* $t \in \mathcal{Z}$ *and for all* $p, q \in \mathbb{R}$, *we have*

$$|\mathcal{F}(t, s_i, p) - \mathcal{F}(t, s_i, q)| \leq k|p-q|, \quad \textit{for each } i \in \{1,2\}.$$

Now, using Theorem 2.3, we prove the existence of bounded solution of the functional equation (3.1). In doing so, we are essentially motivated by Bhakta and Mitra [1].

Theorem 3.1. *Assume that the functions* $\hbar$ *and* $\mathcal{F}$ *are bounded and the assumption* $(*)$ *holds. Then the functional equation* (3.1) *has a bounded solution.*

Proof. Let $\mathbb{B}(\mathcal{Z})$ denote the set of all bounded real-valued functions on $\mathcal{Z}$. On $\mathbb{B}(\mathcal{Z})$, define a metric d:

$$d(x,y) = \sup_{t \in \mathcal{Z}} |x(t) - y(t)|, \quad \forall\ x, y \in \mathcal{B}(\mathcal{Z}).$$

As the convergence in the space $\mathcal{B}(\mathcal{Z})$ is uniform, therefore for any Cauchy sequence $\{x_n\} \subset \mathcal{B}(\mathcal{Z})$, $\{x_n\}$ converges uniformly to a bounded function x^*, we conclude that $x^* \in \mathcal{B}(\mathcal{Z})$. It follows that $(\mathcal{B}(\mathcal{Z}), d)$ is a complete metric space.

Define a map $T : \mathcal{B}(\mathcal{Z}) \to \mathcal{B}(\mathcal{Z})$ by

$$(Tx)(t) = \sup_{s \in \mathcal{S}} \{\hbar(t,s) + \mathcal{F}(t, s, x(\theta(t,s)))\}, \quad \forall\ x \in \mathcal{B}(\mathcal{Z}) \text{ and } t \in \mathcal{Z}.$$

Take $x, y \in \mathcal{B}(\mathcal{Z})$ such that

$$d(y, Tx) \leq \lambda d(y,x). \tag{3.3}$$

Since $x, y \in \mathcal{B}(\mathcal{Z})$, therefore, for all $t \in \mathcal{Z}$, we have

$$(Tx)(t) = \sup_{s \in \mathcal{S}} \{\hbar(t,s) + \mathcal{F}(t, s, x(\theta(t,s)))\}$$

and

$$(Ty)(t) = \sup_{s \in \mathcal{S}} \{\hbar(t,s) + \mathcal{F}(t, s, y(\theta(t,s)))\}.$$

As ϵ is an arbitrary positive real number, there exist $s_1, s_2 \in \mathcal{S}$ such that

$$(Tx)(t) < \hbar(t, s_1) + \mathcal{F}(t, s_1, x(t_1)) + \epsilon, \tag{3.4}$$

$$(Ty)(t) < \hbar(t, s_2) + \mathcal{F}(t, s_2, y(t_2))) + \epsilon, \tag{3.5}$$

where $t_1 := \theta(t, s_1)$ and $t_2 := \theta(t, s_2)$ are parameters. Also, we have

$$(Tx)(t) \geq \hbar(t, s_2) + \mathcal{F}(t, s_2, x(t_2)), \tag{3.6}$$

$$(Ty)(t) \geq \hbar(t, s_1) + \mathcal{F}(t, s_1, y(t_1)). \tag{3.7}$$

On using (3.4), for all $t \in \mathcal{Z}$, we get

$$-\hbar(t, s_1) - \mathcal{F}(t, s_1, x(t_1)) + y(t) - \epsilon < y(t) - (Tx)(t) \leq |(Tx)(t) - y(t)| \leq d(y, Tx)$$

which using (3.3) reduces to

$$-\hbar(t, s_1) - \mathcal{F}(t, s_1, x(t_1)) + y(t) < \epsilon + \lambda d(y,x), \quad \forall\ t \in \mathcal{Z}. \tag{3.8}$$

Further, by using (3.6), for all $t \in \mathcal{Z}$, we get

$$\hbar(t, s_2) + \mathcal{F}(t, s_2, x(t_2)) - y(t) \leq (Tx)(t) - y(t) \leq |(Tx)(t) - y(t)| \leq d(y, Tx)$$

which using (3.3) reduces to

$$\hbar(t, s_2) + F(t, s_2, x(t_2)) - y(t) \leq \lambda d(x, y), \quad \forall\ t \in \mathcal{Z}. \tag{3.9}$$

Note that

$$\lambda d(x, y) = \sup_{t \in \mathcal{Z}} |\lambda x(t) - \lambda y(t)|.$$

As $\epsilon > 0$ be given arbitrarily, there exists $t_\epsilon \in \mathcal{Z}$ such that

$$\lambda d(x, y) < \epsilon + |\lambda x(t_\epsilon) - \lambda y(t_\epsilon)| = \epsilon + \lambda |x(t_\epsilon) - y(t_\epsilon)|.$$

Denote $\delta = \delta(\epsilon) := |x(t_\epsilon) - y(t_\epsilon)|$. Then the last inequality becomes

$$\lambda d(x, y) < \epsilon + \lambda \delta.$$

Hence, the inequalities (3.8) and (3.9) becomes, respectively

$$-\hbar(t, s_1) - F(t, s_1, x(t_1)) + y(t) < 2\epsilon + \lambda \delta, \quad \forall\ t \in \mathcal{Z}$$

and

$$\hbar(t, s_2) + F(t, s_2, x(t_2)) - y(t) < \epsilon + \lambda \delta, \quad \forall\ t \in \mathcal{Z}.$$

Thus, the two inequalities represented by (3.2) holds for $\alpha = x$ and $\beta = y$. Consequently, by assumption $(*)$, there exists $k \in (0, 1)$ verifying $k < \lambda$ such that for all $t \in \mathcal{Z}$ and for all $p, q \in \mathbb{R}$, we have

$$|F(t, s_1, p) - F(t, s_1, q)| \leq k|p - q|. \tag{3.10}$$

and

$$|F(t, s_2, p) - F(t, s_2, q)| \leq k|p - q|. \tag{3.11}$$

Using (3.4), (3.7) and (3.10), it follows that

$$\begin{aligned}
(Tx)(t) - (Ty)(t) &< F(t, s_1, x(t_1)) - F(t, s_1, y(t_1)) + \epsilon \\
&\leq |F(t, s_1, x(t_1)) - F(t, s_1, y(t_1))| + \epsilon \\
&\leq k|x(t_1) - y(t_1)| + \epsilon \\
&\leq kd(x, y) + \epsilon.
\end{aligned}$$

so that

$$(Tx)(t) - (Ty)(t) < kd(x, y) + \epsilon. \tag{3.12}$$

Now, from (3.5), (3.6) and (3.10), we get

$$\begin{aligned}
(Tx)(t) - (Ty)(t) &> F(t, s_2, x(t_2)) - F(t, s_2, y(t_2))) - \epsilon \\
&\geq -|F(t, s_2, x(t_2)) - F(t, s_2, y(t_2)))| - \epsilon \\
&\geq -k|x(t_2) - y(t_2)| - \epsilon \\
&\geq -kd(x, y) - \epsilon
\end{aligned}$$

so that

$$(Tx)(t) - (Ty)(t) > -kd(x, y) - \epsilon. \tag{3.13}$$

Combining the inequalities (3.12) and (3.13), we conclude that

$$|(Tx)(t) - (Ty)(t)| < kd(x, y) + \epsilon, \tag{3.14}$$

which, by taking supremum over $t \in \mathcal{Z}$ on both the sides, reduces to

$$d(Tx, Ty) < kd(x, y) + \epsilon. \tag{3.15}$$

As $\epsilon > 0$ is taken arbitrary, we conclude immediately that

$$d(Tx, Ty) \leq kd(x, y).$$

It follows that the operator T is an inevitable contraction. Consequently, by Theorem 2.3, T has a fixed point, say $\bar{x} \in \mathcal{B}(\mathcal{Z})$ and hence $\bar{x}$ forms a bounded solution of the functional equation (3.1). □

4. Conclusion

In his paper, Popescu [5] imposed a conjecture that weather Theorem 2.3 is still valid in the case where $k = \lambda$ instead of $k < \lambda$. A negative answer to this conjecture was given by Suzuki [12]. One more conjecture was raised in [5] regarding well-posedness of a fixed point problem for $(1, k)$-inevitable contraction, which was answered affirmatively by Khojasteh [3].

Indeed, the inevitable contraction condition in its right hand side contains the inequality $d(Tx, Ty) \leq kd(x, y)$, which by definition are required to satisfy merely for those pair of elements x, y that verify the inequality $d(y, Tx) \leq \lambda d(y, x)$ as implied by left hand side. Due to natural behaviour of $d(Tx, Ty) \leq kd(x, y)$, we are able to obtain further generalizations of inevitable contractions adopting the idea involved in the quasi-contractions, Boyd-Wong's φ-contractions, Matkowski's φ-contractions, weak ψ-contractions, (ϕ, ψ)-contractions, F-contractions, shifting distance admissible contractions, $\mathcal{Z}$-contractions, Meir-Keeler Contractions, CJM-Contractions, implicit contractions, etc.

In this article, we applied Theorem 2.3 to investigate a bounded solution of certain functional equations associated with dynamic programming. For further aspects of applicability, our result (and possible similar future results) can also be utilized to the domains of linear positive approximation operators, difference equations with deviating argument and functional-integral equations on the lines of Rus [10], which remains a very important and applicable area on its own.

Acknowledgement

All authors are grateful to three guest editors and two referees for their lucrative remarks and pertinent comments that helped us improve the depth of present paper.

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Manuscript received April 30, 2024
revised August 13, 2024

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