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AN IMPROVISED B-SPLINE COLLOCATION ALGORITHM TO SOLVE NEWELL-WHITEHEAD-SEGEL EQUATION

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Abstract. In this paper, we introduce a new parallel iterative scheme and employ the same to investigate an altering points problem. Some consequent results are also discussed. The obtained results extend and generalize some relevant results of the existing literature. The usefulness and efficiency of our scheme is illustrated using numerical examples.

1. Introduction

Newell-Whitehead is an important nonlinear partial differential equation that describes the envelope of modulated roll-solution in systems having extended or unbounded space directions. The general form of Newell-Whitehead equation is as follows [14]:

$$\frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2} + aw - bw^3. \tag{1.1}$$

Further, Segel [20] modified this equation to the following form and named as Newell-Whitehead-Segal (NWS) equation:

$$\frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2} + aw - bw^p, \tag{1.2}$$

with the initial condition:

$$w(x,t_0) = G(x),\ x \in [x_L, x_R], \tag{1.3}$$

and the boundary conditions:

$$w(x_L,t) = f_1(t),\ w(x_R,t) = f_2(t),\ t \geq 0, \tag{1.4}$$

where $a,$ b and k are real constants with $k > 0$, p is a positive integer, and $w(x,t)$ represents the nonlinear conveyance of temperature in an infinitely long and thin rod or the flow velocity of a fluid in an infinitely long small diameter pipe. NWS equation plays a vital role in fluid dynamics, as it reports the emergence of the stripe pattern in two dimensions. Also, this equation provides the mathematical model for different systems, such as Rayleigh-Benard convection, Faraday instability, etc.

Researchers used different methods to solve NWS equation such as Aasaraai [1] applied differential transform method to obtain its analytical solution. Pue-on [18] applied the Laplace Adomian decomposition method, Hariharan [8] opted Legendre

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wavelet-based method, Zahra et al. [21] applied cubic B-spline collocation method, Patade and Bhalekar [15] applied iterative method, Jassim [10] opted combination of homotopy perturbation and Laplace transform method, Prakash and Kumar [16] solved the NWS equation using He ' s Variational iteration method, Akinlabi and Edekiwhich [2] applied a combination of Perturbation Iteration Algorithm and conventional Laplace Transform Method to solve this equation, Hilal et al. [9] presented two different approaches to solve NWS equation. One is implicit exponential finite difference method and the other is fully implicit exponential finite difference method and many more.

The paper is organized as: The improvised cubic B-spline collocation method is derived in Section 2. The proposed ICSCM and Crank-Nicolson scheme is implemented to the NWS equation in Section 3. In Section 4, a stability analysis of the technique is carried out and is shown to be stable. Convergence analysis of the technique is established in Section 5 and is shown to be fourth-order convergent in the space domain and second-order convergent in the time direction. Relevant examples are solved in Section 6, to illustrate the performance of the proposed technique. In Section 7, the conclusion part of the paper is discussed.

2. Improvised cubic B-spline collocation method

The improvised cubic B-spline collocation method is formed by making posteriori corrections to the cubic B-spline interpolant. With these corrections, better results are obtained as compared to the standard B-spline collocation method.

2.1. **Properties of cubic B-splines.** Uniform partition of the space domain $\Pi_x \equiv \{x_L = x_0 < x_1 < \cdots < x_{N-1} < x_N = x_R\}$ with the nodal points $x_k = x_L + kh,\ j = 0, 1, \ldots, N$ and spatial step size $h = (x_R - x_L)/N$ is taken. According to the behaviour of cubic B-splines, each cubic B-spline cover four elements of the domain and each finite element $[x_j, x_{j+1}]$ is occupied by four spline functions. So, for calculations four more nodal points are required outside the interval $[x_L, x_R]$, which are positioned as $x_{-2} < x_{-1} < x_0$ and $x_N < x_{N+1} < x_{N+2}$. The introduction to cubic B-spline functions were given by Prenter [17] as follows:

$$\mathbb{S}_{j,3}(x) = \frac{1}{h^3} \begin{cases} (x - x_{j-2})^3, & x \in [x_{j-2}, x_{j-1}] \\ h^3 + 3h^2(x - x_{j-1}) + 3h(x - x_{j-1})^2 - 3(x - x_{j-1})^3, & x \in [x_{j-1}, x_j] \\ h^3 + 3h^2(x_{j+1} - x) + 3h(x_{j+1} - x)^2 - 3(x_{j+1} - x)^3, & x \in [x_j, x_{j+1}] \\ (x_{j+2} - x)^3, & x \in [x_{j+1}, x_{j+2}] \\ 0, & Otherwise. \end{cases} \tag{2.1}$$

This collection of cubic B-splines $\mathbb{S}_{j,3}(x) = \{\mathbb{S}_{-1}(x), \mathbb{S}_0(x), \ldots, \mathbb{S}_{N+1}(x)\}$ forms the basis function for the subspace X of $C^2[x_L, x_R]$. Let $z(x,t)$ be the cubic B-spline approximate solution corresponding to the exact solution $w(x,t)$, then it can be expressed as follows:

$$z(x,t) = \sum_{j=-1}^{N+1} \delta_j(t)\mathbb{S}_j(x), \tag{2.2}$$

where $\delta_j(t)$'s are the time dependent unknown quantities to be determined.

2.2. **Posteriori correction in second-order derivative of cubic B-spline interpolant.** Assume that the cubic B-spline interpolant satisfies the following conditions:

(I) the interpolatory condition, for $j = 0, 1, \ldots, N$:

$$z(x_j,t) = w(x_j,t), \tag{2.3}$$

(II) at end nodal points, for $j = 0$ and N:

$$z_{xx}(x_j,t) = w_{xx}(x_j,t) - \frac{h^2}{12} w_{xxxx}(x_j,t). \tag{2.4}$$

Theorem 2.1. *The following relations hold for the cubic B-spline interpolant (CSI) z(x,t) of w(x,t), where w(x,t) is sufficiently smooth function in spatial domain and satisfy Eqs. (2.3) and (2.4), for* $j = 0, 1, \ldots, N$*:*

$$z_{xx}(x_j,t) = w_{xx}(x_j,t) - \frac{h^2}{12} w_{xxxx}(x_j,t) + O(h^4), \tag{2.5}$$

$$z_x(x_j,t) = w_x(x_j,t) + O(h^4). \tag{2.6}$$

In addition,

$$\| w^{(k)} - z^{(k)} \|_\infty = O(h^{4-k}), \ k = 0, 1, 2, \tag{2.7}$$

where $w^{(k)}$ *and* $z^{(k)}$ *represents* k^{th} *derivative w.r.t 'x'.*

Proof. Given in [11]. □

Lemma 2.2. *For w(x,t)* $\in C^6[x_L, x_R]$*, the following relations hold:*

$$\begin{aligned}
w_{xxxx}(x_0,t) &= \frac{1}{h^2}[2z_{xx}(x_0,t) - 5z_{xx}(x_1,t) + 4z_{xx}(x_2,t) - z_{xx}(x_3,t)] + O(h^2),\\
w_{xxxx}(x_j,t) &= \frac{1}{h^2}[z_{xx}(x_{j-1},t) - 2z_{xx}(x_j,t) + z_{xx}(x_{j+1},t)] + O(h^2),\\
&\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \ldots, N-1,\\
w_{xxxx}(x_N,t) &= \frac{1}{h^2}[2z_{xx}(x_N,t) - 5z_{xx}(x_{N-1},t) + 4z_{xx}(x_{N-2},t) - z_{xx}(x_{N-3},t)]\\
&\quad + O(h^2).
\end{aligned}$$

Proof. The above mentioned relations can be proved by finite differences and Taylors expansion. □

Corollary 2.3. *For w(x,t)* $\in C^6[x_L, x_R]$*, the following relations hold:*

$$w_x(x_j,t) = z_x(x_k,t) + O(h^4), \ j = 0, 1, \ldots, N,$$

$$
\begin{aligned}
w_{xx}(x_0,t) &= \frac{1}{12}[14z_{xx}(x_0,t) - 5z_{xx}(x_1,t) + 4z_{xx}(x_2,t) - z_{xx}(x_3,t)] + O(h^4),\\
w_{xx}(x_j,t) &= \frac{1}{12}[z_{xx}(x_{j-1},t) + 10z_{xx}(x_j,t) + z_{xx}(x_{j+1},t)] + O(h^4),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1,2,\dots,N-1,\\
w_{xx}(x_N,t) &= \frac{1}{12}[14z_{xx}(x_N,t) - 5z_{xx}(x_{N-1},t) + 4z_{xx}(x_{N-2},t) - z_{xx}(x_{N-3},t)]\\
&\quad + O(h^4).
\end{aligned}
$$

3. Implementation of proposed technique

For the implementation of the technique, take uniform partitioning of the time domain $\Pi_t \equiv \{0 = t^0 < t^1 < \cdots < t^n < t^{n+1} < \cdots < T\}$, with $t^{n+1} = t^n + \Delta t$, for $n = 0,\ 1, \dots,$ where Δt is the temporal step size.

Applying Crank-Nicolson scheme to discretize Eq. (1.2):

$$
\frac{w^{n+1} - w^n}{\Delta t} = k\left[\frac{w_{xx}^{n+1} + w_{xx}^n}{2}\right] + a\left[\frac{w^{n+1} + w^n}{2}\right] - b\left[\frac{(w^p)^{n+1} + (w^p)^n}{2}\right]. \tag{3.1}
$$

Apply the Quasilinearization process to linearize the nonlinear terms, which was proposed by [3] as follows:

$$
(w^p)^{n+1} = (w^p)^n + (w^{n+1} - w^n)p(w^{p-1})^n. \tag{3.2}
$$

Substituting the above expression and combine the terms at $(n+1)^{th}$ and n^{th} time levels:

$$
\begin{aligned}
\left[\frac{1}{\Delta t} - \frac{a}{2} + \frac{bp}{2}(w^{p-1})^n\right]w^{n+1} - \frac{k}{2}w_{xx}^{n+1} &= \left[\frac{1}{\Delta t} + \frac{a}{2}\right]w^n + \frac{k}{2}w_{xx}^n\\
&\quad - \frac{b}{2}(2-p)(w^p)^n.
\end{aligned} \tag{3.3}
$$

At any 'j^{th}' nodal point, the above equation can be written as:

$$
A_j w_j^{n+1} + B(w_{xx})_j^{n+1} = D_j, \tag{3.4}
$$

where

$$
\begin{gathered}
A_j = \frac{1}{\Delta t} - \frac{a}{2} + \frac{bp}{2}(z_j^{p-1})^n,\ B = -\frac{k}{2}\\
\text{and}\\
D_j = \left[\frac{1}{\Delta t} + \frac{a}{2}\right]w_j^n + \frac{k}{2}(w_{xx})_j^n - \frac{b}{2}(2-p)(w^p)_j^n.
\end{gathered} \tag{3.5}
$$

Substitute the calculated improvised cubic B-spline values of w and w_{xx} in Eq. (3.4):

For $j = 0$:

$$
\begin{aligned}
A_0(\delta_{-1}^{n+1} + 4\delta_0^{n+1} + \delta_1^{n+1}) + \frac{B}{2h^2}(14\delta_{-1}^{n+1} - 33\delta_0^{n+1} + 28\delta_1^{n+1}\\
- 14\delta_2^{n+1} + 6\delta_3^{n+1} - \delta_4^{n+1}) = D_0 + O(h^4).
\end{aligned}
$$

For $j = 1, 2, \ldots, N-1$:

$$A_j(\delta_{j-1}^{n+1} + 4\delta_j^{n+1} + \delta_{j+1}^{n+1}) + \frac{B}{2h^2}(\delta_{j-2}^{n+1} + 8\delta_{j-1}^{n+1} - 18\delta_j^{n+1} + 8\delta_{j+1}^{n+1} + \delta_{j+2}^{n+1}) = D_j + O(h^4).$$

For $j = N$:

$$\begin{aligned} &A_N(\delta_{N-1}^{n+1} + 4\delta_N^{n+1} + \delta_{N+1}^{n+1}) \\ &\quad + \frac{B}{2h^2}(-\delta_{N-4}^{n+1} + 6\delta_{N-3}^{n+1} - 14\delta_{N-2}^{n+1} + 28\delta_{N-1}^{n+1} - 33\delta_N^{n+1} + 14\delta_{N+1}^{n+1}) \\ &\qquad = D_N + O(h^4). \end{aligned}$$

Clubbing up the coefficients of δ_j^{n+1}'s, the following equations are obtained:
For $j = 0$:

$$\begin{aligned} &\left(A_0 + \frac{7B}{h^2}\right)\delta_{-1}^{n+1} + \left(4A_0 - \frac{33B}{2h^2}\right)\delta_0^{n+1} + \left(A_0 + \frac{14B}{h^2}\right)\delta_1^{n+1} - \frac{7B}{h^2}\delta_2^{n+1} \\ &\qquad + \frac{3B}{h^2}\delta_3^{n+1} - \frac{B}{2h^2}\delta_4^{n+1} = D_0 + O(h^4), \end{aligned}$$

$$p_0\delta_{-1}^{n+1} + q_0\delta_0^{n+1} + r_0\delta_1^{n+1} + s_0\delta_2^{n+1} + v_0\delta_3^{n+1} + y_0\delta_4^{n+1} = D_0 + O(h^4). \tag{3.6}$$

For $j = 1, 2, \ldots, N-1$:

$$\begin{aligned} &\frac{B}{2h^2}\delta_{j-2}^{n+1} + \left(A_j + \frac{4B}{h^2}\right)\delta_{j-1}^{n+1} + \left(4A_j - \frac{9B}{h^2}\right)\delta_j^{n+1} \\ &\qquad + \left(A_j + \frac{4B}{h^2}\right)\delta_{j+1}^{n+1} + \frac{B}{2h^2}\delta_{j+2}^{n+1} = D_j + O(h^4), \end{aligned}$$

$$p_j\delta_{j-2}^{n+1} + q_j\delta_{j-1}^{n+1} + r_j\delta_j^{n+1} + s_j\delta_{j+1}^{n+1} + v_j\delta_{j+2}^{n+1} = D_j + O(h^4). \tag{3.7}$$

For $j = N$:

$$\begin{aligned} &-\frac{B}{2h^2}\delta_{N-4}^{n+1} + \frac{3B}{h^2}\delta_{N-3}^{n+1} - \frac{7B}{h^2}\delta_{N-2}^{n+1} + \left(A_N + \frac{14B}{h^2}\right)\delta_{N-1}^{n+1} \\ &\qquad + \left(4A_N - \frac{33B}{2h^2}\right)\delta_N^{n+1} + \left(A_N + \frac{7B}{h^2}\right)\delta_{N+1}^{n+1} = D_N + O(h^4), \end{aligned}$$

$$p_N\delta_{N-4}^{n+1} + q_N\delta_{N-3}^{n+1} + r_N\delta_{N-2}^{n+1} + s_N\delta_{N-1}^{n+1} + v_N\delta_N^{n+1} + y_N\delta_{N+1}^{n+1} = D_N + O(h^4). \tag{3.8}$$

A system of $(N+1)$ differential equations is obtained in $(N+3)$ unknowns. To deal with the two remaining unknowns, the boundary conditions given by Eq. (1.4) are used. Collection of all the equations can be represented as follows:

$$\mathcal{P}\mathcal{C} = \mathcal{Q} \tag{3.9}$$

where $\mathcal{P}$ is $(N+3)\times(N+3)$ matrix, $\mathcal{C}$ and $\mathcal{Q}$ are $(N+3)\times 1$ column vectors given below:

$$\begin{bmatrix} 1 & 4 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ p_0 & q_0 & r_0 & s_0 & v_0 & y_0 & 0 & \dots & 0 \\ p_1 & q_1 & r_1 & s_1 & v_1 & 0 & 0 & \dots & 0 \\ 0 & p_2 & q_2 & r_2 & s_2 & v_2 & 0 & \dots & 0 \\ 0 & 0 & p_3 & q_3 & r_3 & s_3 & v_3 & \dots & 0 \\ . & . & . & . & . & . & . & \dots & . \\ . & . & . & . & . & . & . & \dots & . \\ 0 & \dots & p_{N-3} & q_{N-3} & r_{N-3} & s_{N-3} & v_{N-3} & 0 & 0 \\ 0 & \dots & 0 & p_{N-2} & q_{N-2} & r_{N-2} & s_{N-2} & v_{N-2} & 0 \\ 0 & \dots & 0 & 0 & p_{N-1} & q_{N-1} & r_{N-1} & s_{N-1} & v_{N-1} \\ 0 & \dots & 0 & p_N & q_N & r_N & s_N & v_N & y_N \\ 0 & \dots & 0 & 0 & 0 & 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} \delta_{-1}^{n+1} \\ \delta_0^{n+1} \\ \delta_1^{n+1} \\ \delta_2^{n+1} \\ \delta_3^{n+1} \\ . \\ . \\ \delta_{N-3}^{n+1} \\ \delta_{N-2}^{n+1} \\ \delta_{N-1}^{n+1} \\ \delta_N^{n+1} \\ \delta_{N+1}^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} f_1[(n+1)\Delta t] \\ D_0 \\ D_1 \\ D_2 \\ D_3 \\ . \\ . \\ D_{N-3} \\ D_{N-2} \\ D_{N-1} \\ D_N \\ f_2[(n+1)\Delta t] \end{bmatrix}$$

3.1. **Initial Condition:** To find the value of δ^0 which is required to find the solution at the next time levels, the initial condition (1.3) is used at every nodal point *i.e.*, $w(x_j,t_0) = G(x_j)$. Two more conditions $w_x(x_0,t_0) = G_x(x_0)$ and $w_x(x_N,t_0) = G_x(x_N)$ are used. By using these equations following system is obtained.

$$\begin{bmatrix} \frac{-3}{h} & 0 & \frac{3}{h} & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 & 0 & \dots & 0 \\ . & . & . & . & . & . & . & \dots & . \\ . & . & . & . & . & . & . & \dots & . \\ 0 & \dots & 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & \dots & 0 & 0 & 0 & 0 & \frac{-3}{h} & 0 & \frac{3}{h} \end{bmatrix} \begin{bmatrix} \delta_{-1}^0 \\ \delta_0^0 \\ \delta_1^0 \\ \delta_2^0 \\ . \\ . \\ \delta_{N-2}^0 \\ \delta_{N-1}^0 \\ \delta_N^0 \\ \delta_{N+1}^0 \end{bmatrix} = \begin{bmatrix} G_x(x_0) \\ G(x_0) \\ G(x_1) \\ G(x_2) \\ . \\ . \\ G(x_{N-2}) \\ G(x_{N-1}) \\ G(x_N) \\ G_x(x_N) \end{bmatrix}$$

4. Stability analysis

Von-Neumann method is applied to discuss the stability analysis of the proposed improvised collocation technique. Substitute w as a local constant $l = \max\,(w)$ to linearize the nonlinear term in the equation (1.2) and then apply the Crank-Nicolson scheme to discretize the temporal domain:

$$\frac{w_j^{n+1} - w_j^n}{\Delta t} = k\left[\frac{(w_{xx})_j^{n+1} + (w_{xx})_j^n}{2}\right] + (a - bl^{(p-1)})\left[\frac{w_j^{n+1} + w_j^n}{2}\right]. \tag{4.1}$$

Combining the $(n+1)^{th}$ and n^{th} time level terms:

$$\left[\frac{1}{\Delta t} - \frac{(a - bl^{(p-1)})}{2}\right] w_j^{n+1} - \frac{k}{2}(w_{xx})_j^{n+1} = \left[\frac{1}{\Delta t} + \frac{(a - bl^{(p-1)})}{2}\right] w_j^n + \frac{k}{2}(w_{xx})_j^n. \tag{4.2}$$

For simplification, write above equation as follows:

$$f\ w_j^{n+1} - \frac{k}{2}(w_{xx})_j^{n+1} = g\ w_j^n + \frac{k}{2}(w_{xx})_j^n. \tag{4.3}$$

where

$$f = \frac{1}{\Delta t} - \frac{(a - bl^{(p-1)})}{2},\ g = \frac{1}{\Delta t} + \frac{(a - bl^{(p-1)})}{2}. \tag{4.4}$$

Using improvised cubic B-splines, substitute the values of w and w_{xx}:

$$\begin{aligned} &f(\delta_{j-1}^{n+1} + 4\delta_j^{n+1} + \delta_{j+1}^{n+1}) - \frac{k}{4h^2}(\delta_{j-2}^{n+1} + 8\delta_{j-1}^{n+1} - 18\delta_j^{n+1} + 8\delta_{j+1}^{n+1} + \delta_{j+2}^{n+1}) \\ &= g(\delta_{j-1}^n + 4\delta_j^n + \delta_{j+1}^n) + \frac{k}{4h^2}(\delta_{j-2}^n + 8\delta_{j-1}^n - 18\delta_j^n + 8\delta_{j+1}^n + \delta_{j+2}^n). \end{aligned} \tag{4.5}$$

Simplifying Eq. (4.5) yields:

$$\begin{aligned} &-\frac{k}{4h^2}\delta_{j-2}^{n+1} + \left(f - \frac{2k}{h^2}\right)\delta_{j-1}^{n+1} \\ &+ \left(4f + \frac{9k}{2h^2}\right)\delta_j^{n+1} + \left(f - \frac{2k}{h^2}\right)\delta_{j+1}^{n+1} - \frac{k}{4h^2}\delta_{j+2}^{n+1} \\ &= \frac{k}{4h^2}\delta_{j-2}^n + \left(g + \frac{2k}{h^2}\right)\delta_{j-1}^n + \left(4g - \frac{9k}{2h^2}\right)\delta_j^n \\ &+ \left(g + \frac{2k}{h^2}\right)\delta_{j+1}^n + \frac{k}{4h^2}\delta_{j+2}^n. \end{aligned} \tag{4.6}$$

Above equation can be written as follows:

$$a_1\delta_{j-2}^{n+1} + a_2\delta_{j-1}^{n+1} + a_3\delta_j^{n+1} + a_2\delta_{j+1}^{n+1} + a_1\delta_{j+2}^{n+1} = -a_1\delta_{j-2}^n + a_4\delta_{j-1}^n + a_5\delta_j^n + a_4\delta_{j+1}^n - a_1\delta_{j+2}^n,$$

where

$$a_1 = -\frac{k}{4h^2},\ a_2 = f - \frac{2k}{h^2},\ a_3 = 4f + \frac{9k}{2h^2},\ a_4 = g + \frac{2k}{h^2},\ a_5 = 4g - \frac{9k}{2h^2}.$$

Put $\delta_j^n = A\alpha^n \exp(ij\sigma h)$, where A is the amplitude, $i = \sqrt{-1}$, h is the spatial step length and σ is a mode number, we get:

$$\alpha = \frac{-a_1\exp(-2i\sigma h) + a_4\exp(-i\sigma h) + a_5 + a_4\exp(i\sigma h) - a_1\exp(2i\sigma h)}{a_1\exp(-2i\sigma h) + a_2\exp(-i\sigma h) + a_3 + a_2\exp(i\sigma h) + a_1\exp(2i\sigma h)},$$

$$= \frac{-2a_1\cos(2\sigma h) + a_5 + 2a_4\cos(\sigma h)}{2a_1\cos(2\sigma h) + a_3 + 2a_2\cos(\sigma h)},$$

$$= \frac{X_1}{X_2},$$

where

$$X_1 = -2a_1\cos(2\sigma h) + a_5 + 2a_4\cos(\sigma h),$$
$$X_2 = 2a_1\cos(2\sigma h) + a_3 + 2a_2\cos(\sigma h.$$

It can be easily observed that $|\alpha| \leq 1$, i.e., $X_1^2 \leq X_2^2$. Hence the technique is unconditionally stable.

5. Convergence analysis

Green's function approach is followed to establish the convergence analysis which is based on the work of [4, 6, 7] etc.

Take Eq. (1.2) in the following operator form:

$$\mathfrak{T} \equiv kw_{xx} - w_t + aw - \Psi(x,t,w), \tag{5.1}$$

with the boundary conditions as:

$$\mathfrak{B}w = \Phi_j, \ j = 0, N, \tag{5.2}$$

where $\Psi(x,t,w) = w^p$. Take $\hat{\mathfrak{T}}$ and $\hat{\mathfrak{B}}$ to be the perturbed form of the operators $\mathfrak{T}$ and $\mathfrak{B}$ respectively. Then the following relations hold between above defined operators for the CSI $z(x,t)$.

$$\begin{aligned}
\hat{\mathfrak{T}}z_j(t) &\equiv \mathfrak{T}(z_j(t), (z_x)_j(t), (z_{xx})_j(t) \\
&\quad + \frac{1}{12}[(z_{xx})_{j-1}(t) - 2(z_{xx})_j(t) + (z_{xx})_{j+1}(t)]), \ j = 1, 2, \ldots, N-1, \\
\hat{\mathfrak{T}}z_0(t) &\equiv \mathfrak{T}(z_0(t), (z_x)_0(t), (z_{xx})_0(t) \\
&\quad + \frac{1}{12}[2(z_{xx})_0(t) - 5(z_{xx})_1(t) + 4(z_{xx})_2(t) - (z_{xx})_3(t)]), \\
\hat{\mathfrak{T}}z_N(t) &\equiv \mathfrak{T}(z_N(t), (z_x)_N(t), (z_{xx})_N(t) \\
&\quad + \frac{1}{12}[2(z_{xx})_N(t) - 5(z_{xx})_{N-1}(t) + 4(z_{xx})_{N-2}(t) - (z_{xx})_{N-3}(t)]), \\
\hat{\mathfrak{B}}z_j(t) &= \mathfrak{B}z_j(t), \ j = 0, N.
\end{aligned} \tag{5.3}$$

Lemma 5.1. *The following relations hold at the nodal points, for the unique CSI $z(x,t)$ of $w(x,t)$, where $w(x,t) \in C^6[x_L, x_R]$,*

$$\hat{\mathfrak{T}}z_j(t) = O(h^4), \ j = 0, 1, \ldots, N, \ \hat{\mathfrak{B}}z_j(t) = O(h^4), \ j = 0, N. \tag{5.4}$$

Let $\hat{z}(x,t)$ be the unique CSI of $w(x,t)$ such that,

$$\hat{\mathfrak{T}}\hat{z}_j(t) = 0,\ j = 0, 1, \ldots, N,\ \hat{\mathfrak{B}}\hat{z}_j(t) = 0,\ j = 0, N. \tag{5.5}$$

Lemma 5.2. *The coefficient matrix corresponding to the problem $w_{xx} = \sigma(x,t)$ having homogenous boundary conditions is invertible with finite norm of inverse matrix.*

Proof. The coefficient matrix $\mathcal{M}$ of the problem $w_{xx} = \sigma(x,t)$ is given below:

$$\mathcal{M} = \frac{1}{12}\begin{bmatrix} 14 & -5 & 4 & -1 & 0 & \ldots & 0 \\ 1 & 10 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 10 & 1 & 0 & \ldots & 0 \\ . & . & . & . & . & \ldots & . \\ . & . & . & . & . & \ldots & . \\ 0 & \ldots & 0 & 1 & 10 & 1 & 0 \\ 0 & \ldots & 0 & 0 & 1 & 10 & 1 \\ 0 & \ldots & 0 & -1 & 4 & 15 & 14 \end{bmatrix}$$

The above coefficient matrix is diagonally dominant and so is invertible.

$$\|\mathcal{M}^{-1}\|_\infty \leq \max_{0\leq j\leq N} \frac{1}{\Delta_j \mathcal{M}},\ \text{where } \Delta_j\mathcal{M} = \mid \mathcal{M}_{jj} \mid - \sum_{k\neq j} \mid \mathcal{M}_{jk} \mid > 0,\ j = 0, 1, \ldots, N.$$

So,

$$\|\mathcal{M}^{-1}\|_\infty \leq \frac{1}{\min_{0\leq j\leq N} \Delta_j(\mathcal{M})} = \frac{12}{14 - (5 + 4 + 1)} = 3.$$

According to Russell and Shampine [19], if the equation $w^{(2)} = 0$ with the boundary conditions $\mathfrak{B}(w) = 0$ is uniquely solvable then there exists a Green's function $\mathcal{G}(x,t)$ corresponding to given problem. Let $w^{(2)} = \hat{u}$ and $\hat{z}^{(2)} = \hat{y}$ such that $\hat{u}$ and y satisfies the B.C's. Then using Green's function w and $\hat{z}$ can be expressed as follows:

$$w^{(k)}(x,t) = \int_{x_L}^{x_R} \frac{\partial^k \mathcal{G}(x,t,s)}{\partial x^k} \hat{u}(s,t)ds,\ k = 0, 1, \tag{5.6}$$

$$\hat{z}^{(k)}(x,t) = \int_{x_L}^{x_R} \frac{\partial^k \mathcal{G}(x,t,s)}{\partial x^k} \hat{y}(s,t)ds,\ k = 0, 1. \tag{5.7}$$

Let $\Omega = [x_L, x_R] \times [0, T]$ and ρ be any continuous function. Operators required to establish the convergence analysis are defined below.

$$\mathcal{R} : \mathbb{C}(\Omega) \longrightarrow \mathbb{C}(\Omega) \text{ such that } \mathcal{R}\rho = \frac{1}{k}\left[G_0\rho_t - aG_0\rho + \Psi(x,t,G_0\rho)\right], \tag{5.8}$$

where $G_0\rho = \int_{x_L}^{x_R} \frac{\mathcal{G}(x,t,s)}{\rho}(s,t)ds$, is the operator from Ω onto Ω. Let $\mathcal{P}$ be the unique piecewise linear interpolation operator at the nodal points $\{(x_j, t)\}_{j=0}^N$. Next projection operator is defined as follows:

$$\mathcal{Q} : \mathbb{C}(\Omega) \longrightarrow \mathbb{R}^{n+1} \text{ such that } \mathcal{Q}\rho = [\rho(x_0,t), \rho(x_1,t), \ldots, \rho(x_N,t)]^T. \tag{5.9}$$

Using the definition of above operator Eqs. (1.2) and (5.5) can be written as:

$$(I - \mathcal{R})\hat{u} = 0. \tag{5.10}$$

$$(\mathcal{M}\mathcal{Q}-\mathcal{R})\hat{y}=0. \tag{5.11}$$

As the matrix $\mathcal{M}$ is invertible, so

$$(\mathcal{Q}-\mathcal{M}^{-1}\mathcal{R})\hat{y}=0. \tag{5.12}$$

As $\hat{y}$ is a linear polynomial, so $\mathcal{P}\mathcal{Q}\hat{y}=\hat{y}$,

$$(I-\mathcal{P}\mathcal{M}^{-1}\mathcal{R})\hat{y}=0. \tag{5.13}$$

□

Lemma 5.3. *For the equally spaced partitioning Π_x of $[x_L, x_R]$ and any continuous function ρ, $\| \mathcal{P}\mathcal{M}^{-1}\mathcal{R}\rho - \mathcal{R}\rho \|_\infty \to 0$ as $h \to 0$.*

Proof.

$$\begin{aligned}\| \mathcal{P}\mathcal{M}^{-1}\mathcal{R}\rho - \mathcal{R}\rho \|_\infty &\leq \| \mathcal{P}\mathcal{M}^{-1}\mathcal{R}\rho - \mathcal{P}\mathcal{Q}\mathcal{R}\rho \|_\infty + \| \mathcal{P}\mathcal{Q}\mathcal{R}\rho - \mathcal{R}\rho \|_\infty \\ &\leq \| \mathcal{P} \|_\infty \| \mathcal{M}^{-1} \|_\infty \| \mathcal{R}\rho - \mathcal{M}\mathcal{Q}\mathcal{R}\rho \|_\infty + \| \mathcal{P}\mathcal{Q}\mathcal{R}\rho - \mathcal{R}\rho \|_\infty \\ &\leq \| \mathcal{R}\rho - \mathcal{M}\mathcal{Q}\mathcal{R}\rho \|_\infty + O(h^2).\end{aligned}$$

(As $\| \mathcal{M}^{-1} \|_\infty$ is finite and $\| \mathcal{P} \|_\infty = 1$). By the modulus of continuity of functions ρ and Green's function $\mathcal{G}$ over a width of $6h$, the term $\| \mathcal{R}\rho - \mathcal{M}\mathcal{Q}\mathcal{R}\rho \|_\infty$ can be dominated. So $\| \mathcal{P}\mathcal{M}^{-1}\mathcal{R}\rho - \mathcal{R}\rho \|_\infty \to 0$ as $h \to 0$. □

Theorem 5.4 ([5])**.** *Consider the curve $\mathrm{C} = (x,t,w) \in \mathbb{R}^4, (x,t) \in \bar{\Omega}$ and let $w(x,.) \in C^6[x_L, x_R]$ be the solution of the problem (1.2) with boundary condition (1.4), $\Psi(x,t,U)$ be sufficiently smooth function near w and the following linear problem*

$$w_{xx} - \frac{d}{dU}\frac{1}{k}[U_t - aU + \Psi(x,t,U)]w = 0, \tag{5.14}$$

with the boundary conditions (1.4) is uniquely solvable and possesses Green's function $\mathcal{G}(x,t,s)$. Then, there exist constants $\beta,\ \gamma > 0$ such that

(I) *there does not exists any other solution $\hat{w}$ corresponding to the problem (1.2) with B.C (1.4) satisfying $\| w_{xx} - \hat{w}_{xx} \| < \gamma$.*
(II) *For $h < \beta$ the Eq. (5.13) has a unique solution $z(x,.) \in \mathbb{S}_{j,3}(\Pi_x)$ in the same neighbourhood of w.*
(III) *Newton's method converges quadratically in some neighborhood of w for $h < \beta$ which is used for numerically solving the Eq. (5.13).*

Theorem 5.5. *Suppose the assumptions of Theorem 5.4 hold, then the following error bounds exists:*

The global error bounds:

$$\| (w^{(k)}(x,.) - \hat{z}^{(k)}(x,.) \|_\infty = O(h^{4-k}), \quad k = 0,1,2.$$

The local error bounds:

$$\begin{aligned}| w^{(k)}(x,.) - \hat{z}^{(k)}(x,.) |_{x_j} &= O(h^4), \quad k = 0,1. \\ | (w^{(2)}(x,.) - \hat{z}^{(2)}(x,.) |_{x_j} &= O(h^2).\end{aligned}$$

Proof. Consider the problem $z^{(2)} = \hat{\alpha}$, $\mathfrak{B}z = O(h^4)$. By Theorem 5.4, there exists a linear polynomial $\bar{u}$, such that

$$\mathfrak{B}\bar{u} = \mathfrak{B}z = O(h^4),\ \| \bar{u}^{(k)} \|_\infty = O(h^4),\ k = 0, 1. \tag{5.15}$$

As $(z - \bar{u})^{(2)} = \hat{\alpha}$, $\mathfrak{B}(z - \bar{u}) = 0$ is uniquely solvable. Therefore by Theorem 5.4,

$$(I - \mathcal{P}\mathcal{M}^{-1}\mathcal{R})(z^{(2)} - \bar{u}^{(2)}) = O(h^4). \tag{5.16}$$

Subtracting Eq. (5.13) from Eq. (5.16),

$$(I - \mathcal{P}\mathcal{M}^{-1}\mathcal{R})(z^{(2)} - \bar{u}^{(2)} - \hat{z}^{(2)}) = O(h^4). \tag{5.17}$$

As $(I - \mathcal{P}\mathcal{M}^{-1}\mathcal{R})$ is bounded, therefore

$$\| z^{(2)} - \bar{u}^{(2)} - \hat{z}^{(2)} \|_\infty = O(h^4). \tag{5.18}$$

By Theorem 5.4, the problem $(z - \bar{u} - \hat{z})^{(2)} = \bar{\eta}$, $\mathfrak{B}(z - \bar{u} - \hat{z}) = 0$ is uniquely solvable, so there exists a Green's function such that,

$$| (z - \bar{u} - \hat{z})^{(k)} | = \int_{x_L}^{x_R} \frac{\partial^k \mathcal{G}(x, t, s)}{\partial x^k} (z^{(2)} - \bar{u}^{(2)} - \hat{z}^{(2)}) ds,\ k = 0, 1.$$

Thus,

$$\| (z - \bar{u} - \hat{z})^{(k)} \|_\infty = O(h^4),\ k = 0, 1.$$

So,

$$\| (z - \hat{z})^{(k)} \|_\infty \leq \| (z - \bar{u} - \hat{z})^{(k)} \|_\infty + \| \bar{u}^{(k)} \|_\infty = O(h^4),\ k = 0, 1, 2. \tag{5.19}$$

Using Theorem 2.1, Eq. (5.19) and triangular inequality,

$$\| (w - \hat{z})^{(k)} \|_\infty \leq \| (w - z)^{(k)} \|_\infty + \| (z - \hat{z})^{(k)} \|_\infty = O(h^{4-k}) \quad fork = 0, 1, 2.$$

Using Theorem 2.1, local error bounds can be obtained, which completes the proof. Therefore, the proposed technique is fourth-order convergent in space direction. As the Crank-Nicolson scheme is used to discretize the time direction which is second-order convergent in time [13], hence the order of convergence of ICSCM is $O(h^4 + \Delta t^2)$. □

6. Numerical examples

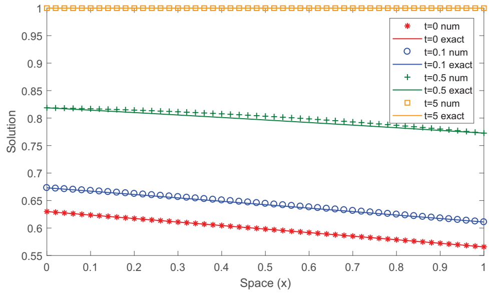
Several examples of the Newell-Whitehead-Segel equation are solved in this section to demonstrate the applicability and good performance of the proposed technique. L_∞ and L_2 error norms are calculated using the following formulas:

$$L_\infty = \max_{0 \leq j \leq N} |w_j^{exact} - w_j^{num}|,\ L_2 = \sqrt{h \sum_{j=0}^{N} (w_j^{exact} - w_j^{num})^2}, \tag{6.1}$$

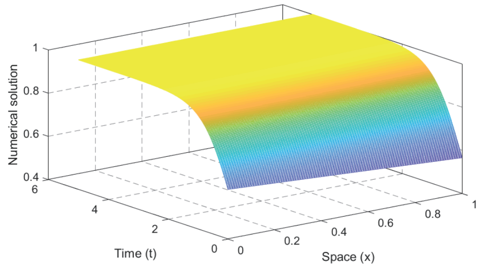
where w_j^{exact} and w_j^{num} are the exact and improvised cubic B-spline solutions respectively at the nodal point 'x_j' for some fixed time.

Example 1. Consider the following NWS equation [9]:

$$w_t = w_{xx} + w - w^4,\ (x, t) \in [0, 1] \times [0, T], \tag{6.2}$$



(A) Comparison of numerical and exact solution



(B) 3-D solution profile of numerical solution

FIGURE 1. Solutions of Example 1 with $N = 100$ and $\Delta t = 0.01$.

with the exact solution:

$$w(x,t) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left\{-\frac{3}{2\sqrt{10}}\left(x - \frac{7t}{\sqrt{10}}\right)\right\}\right]^{\frac{2}{3}}. \tag{6.3}$$

Table 1 gives the comparison of L_∞ and L_2 error norms. The comparison shows that the results are better than the implicit exponential finite difference (I-EFD) method [9] and fully implicit exponential finite difference (FI-EFD) method [9]. Figure 1(a) gives the comparison of the numerical and exact solution with $N = 100$ and $\Delta t = 0.01$ at different time levels. Figure 1(b) represents the 3-D profile of the numerical solution.

TABLE 1. Comparison of L_∞ and L_2 error norms of Example 1 for $h = 0.05$ and $\Delta t = 0.001$ at different time levels.

Time	ICSCM		I-EFD [?]		FI-EFD [?]	
	L_∞	L_2	L_∞	L_2	L_∞	L_2
$t = 0.01$	2.3399E-04	1.7057E-04	3.131E-04	1.260E-04	3.129E-04	1.254E-04
$t = 0.1$	1.3077E-04	9.5760E-05	4.134E-04	2.954E-04	4.101E-04	2.889E-04
$t = 5$	1.9745E-06	1.4459E-06	9.100E-06	6.600E-06	9.200E-06	6.700E-06
$t = 10$	5.4363E-11	3.9805E-11	2.511E-10	1.830E-10	2.529E-10	1.843E-10

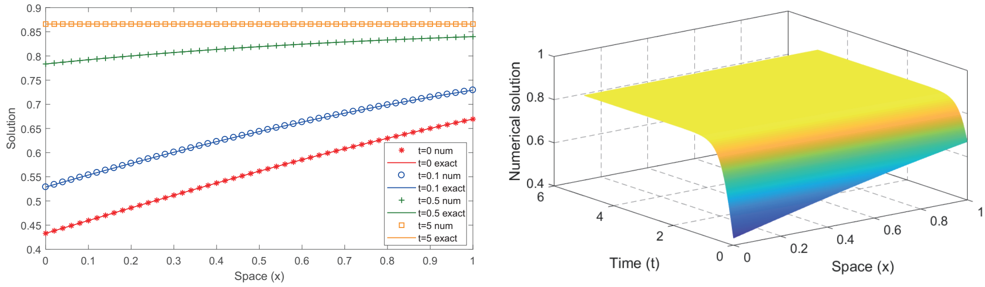
Example 2. Consider the following NWS equation [9]:

$$w_t = w_{xx} + 3w - 4w^3, \ (x,t) \in [0,1] \times [0,T], \tag{6.4}$$

with the exact solution:

$$w(x,t) = \sqrt{\frac{3}{4}\frac{\exp(\sqrt{6}x)}{\exp(\sqrt{6}x) + \exp(\frac{\sqrt{6}x}{2} - \frac{9t}{2})}}. \tag{6.5}$$

In Table 2, a comparison of L_∞ and L_2 error norms with I-EFD and FI-EFD is reported with $h = 0.05$ and $\Delta t = 0.001$ at different time levels.It is clear from the table that the error decreases with the increase in time. In Figure 2(a), a comparison



(A) Comparison of numerical and exact solution

(B) 3-D solution profile of numerical solution

FIGURE 2. Solutions of Example 2 with $N = 100$ and $\Delta t = 0.01$.

Time	ICSCM		I-EFD [?]		FI-EFD [?]	
	L_∞	L_2	L_∞	L_2	L_∞	L_2
$t = 0.01$	1.4906E-04	4.0606E-05	6.927E-04	2.387E-04	6.911E-04	2.404E-04
$t = 0.1$	1.8823E-04	7.8405E-05	8.403E-04	4.591E-04	8.515E-04	5.057E-04
$t = 5$	1.6553E-13	8.8671E-14	6.759E-11	4.898E-11	6.895E-11	4.996E-11
$t = 10$	8.1157E-14	2.0216E-14	4.219E-15	3.042E-15	3.220E-15	2.229E-15

of the numerical and exact solution with $N = 100$ and $\Delta t = 0.01$ at different time levels is represented and Figure 2(b) gives the 3-D profile of the numerical solution.

7. Conclusion

In this work, the improvised cubic B-spline collocation method has been successfully applied to solve the nonlinear Newell-Whitehead-Segel equation. This technique is found to be unconditionally stable. The proposed combination of techniques is shown to be fourth-order convergent in the space domain and second-order convergent in the time domain. Results in terms of accuracy have been observed to be better than many existing techniques. The numerical results are in good agreement with the exact values, which is shown graphically.

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