Journal of Nonlinear and Convex Analysis
Volume 26, Number 4, 2025, 1043–1055
Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

SOLVABILITY OF A NEW FRACTIONAL DIFFERENTIAL EQUATION IN HÖLDER SPACE BY MEASURES OF NONCOMPACTNESS

HOJJATOLLAH AMIRI KAYVANLOO AND MOHAMMAD MURSALEEN*

Abstract. We define a new fractional differential equation of order n with boundary value problem and prove the existence of solutions in the Hölder function space with Lipschitz condition

$$\begin{cases} {}^cD^{\jmath}w(\xi) = f(\xi),\ 0 \le \xi < E,\ E \ge 1, \\ w(0) = w'(0) = w^{(5)}(0) = w^{(6)}(0) = \cdots = w^{(n)}(0) = 0, \\ w''(E) - w''(0) = w''(\eta),\ w^{(3)}(1) = w^{(4)}(1),\ w''(0) = \beta w(\eta),\ \beta \in \mathbb{R},\ \eta \in (0,1). \end{cases}$$

We apply the technique of measures of noncompactness and Darbo's fixed point theorem, also we present one illustrative example in support of main results.

1. Introduction

Fractional calculus, is active field of mathematics analysis is as old as the classical calculus and occur in many scientific disciplines as the mathematical modeling of systems in the fields of biology, economy, engineering and many other fields ([1, 6, 12, 13, 16]).

Measure of noncompactness (MNC) ([8, 11, 18]) plays an outstanding pattern in nonlinear functional analysis. Recently, researchers studied the issues of existence of solutions of integral equations in various spaces [2, 3, 10, 19, 24, 26] . The problems for solving infinite systems of differential and fractional differential equations in sequence spaces have been addressed in [4, 14, 15, 20–23].

In this work we define a new MNC in the Hölder space $\mathcal{H}_1([0, E])$, then we define the new fractional differential equation with boundary value problem and we study the existence of solutions of new fractional equation in $\mathcal{H}_1([0, E])$ by using Darbo's fixed point theorem via MNC. Then, we present one example to show the performance of main results.

Let Λ be a real Banach space, and $\emptyset \neq \mathfrak{L} \subset \Lambda$. Then

- $\overline{\mathfrak{L}}$ the closure of $\mathfrak{L}$ and Conv$\mathfrak{L}$ closed convex hull.
- $D(\nu, \sigma)$ is a closed ball in Λ.
- $\mathfrak{N}_\Lambda \subseteq \Lambda$ is the family of relatively compact.
- $\mathfrak{M}_\Lambda \subseteq \Lambda$ is the family of bounded.

Definition 1.1 ([7]). The mapping $\mu : \mathfrak{M}_\Lambda \to \mathbb{R}_+$ is a measure of noncompactness (MNC) in Λ if for any $\mathcal{W}, \mathcal{Q} \in \mathfrak{M}_\Lambda$ we have:

(1) $\mathfrak{N}_\Lambda \supseteq \ker \mu = \{\mathcal{W} \in \mathfrak{M}_\Lambda : \mu(\mathcal{W}) = 0\} \neq \emptyset$.

2020 *Mathematics Subject Classification.* 47H10, 47H08, 34A08, 26A33.

Key words and phrases. Darbo's theorem, measures of noncompactness, fractional differential equations, integral equation.

(2) If $\mathcal{W} \subset \mathcal{Q}, \Rightarrow \mu(\mathcal{W}) \leq \mu(\mathcal{Q})$.
(3) $\mu(\mathcal{W}) = \mu(\overline{\mathcal{W}}) = \mu(\text{Conv}\mathcal{W})$.
(4) $\mu(\lambda\mathcal{W} + (1-\lambda)\mathcal{Q}) \leq \lambda\mu(\mathcal{W}) + (1-\lambda)\mu(\mathcal{Q})$ for each $\lambda \in [0,1]$.
(5) If for each $n \in \mathbb{N}$, $\overline{\mathcal{W}_n} = \mathcal{W}_n \subseteq \mathfrak{M}_\Lambda$, $\mathcal{W}_{n+1} \subset \mathcal{W}_n$ If $\lim\limits_{n\to\infty}\mu(\mathcal{W}_n) = 0$, $\Rightarrow$ $\emptyset \neq \mathcal{W}_\infty = \bigcap\limits_{n=1}^{\infty}\mathcal{W}_n$.

Theorem 1.2 ([11])**.** *Let $\emptyset \neq \mho = \overline{\mho} \subseteq \Lambda$ be convex, bounded and $G : \mho \to \mho$ be a continuous function and $\exists$ a constant $\ell \in [0,1)$ so that*

$$\mu(G\Im) \leq \ell\mu(\Im),$$

for any $\emptyset \neq \Im \subseteq \mho$. Then G has a fixed point in the set $\mho$.

2. Hölder space $\mathcal{H}_1([a,b])$

In this part, we present a (MNC) in the Hölder space $\mathcal{H}_1([a,b])$ (satisfying the Lipschitz condition). In what follows, this space will be denoted by $Lip(M)$. Observe that $\mathcal{H}_1([a,b])$ is Banach space by following norm

$$|w|_{Lip} = |w(a)| + \sup\Big\{\frac{|w(\xi_1) - w(\xi_2)|}{|\xi_1 - \xi_2|} : \xi_1, \xi_2 \in [a,b],\ \xi_1 \neq \xi_2\Big\}.$$

Remark 2.1. The $\emptyset \neq D \subset \mathcal{H}_1([a,b])$ is bounded if

$$\sup\{|w|_{Lip} :\ w \in D\} < \infty.$$

Theorem 2.2 ([9])**.** *Suppose that $D \subseteq \mathcal{H}_1([a,b])$ be bounded so that for every $\varepsilon > 0$ $\exists\ \delta > 0$, we get*

$$0 < |\xi_1 - \xi_2| < \delta \Rightarrow \frac{|w(\xi_1) - w(\xi_2)|}{|\xi_1 - \xi_2|} \leq \varepsilon,$$

for every $w \in D$ and $\xi_1, \xi_2 \in [a,b]$. Then D is relatively compact in $\mathcal{H}_1([a,b])$.

Suppose $W \in \mathfrak{M}_{\mathcal{H}_1([a,b])}$. For $w \in W$ and $\varepsilon > 0$, define

$$\mu(w,\varepsilon) = \sup\big\{\frac{|w(\xi_1) - w(\xi_2)|}{|\xi_1 - \xi_2|} : \xi_2, \xi_1 \in [a,b],\ \xi_2 \neq \xi_1,\ |\xi_1 - \xi_2| \leq \varepsilon\big\},$$

$$\mu(W,\varepsilon) = \sup\{\mu(w,\varepsilon),\ w \in W\},$$

and

$$\mu(W) = \lim_{\varepsilon\to 0}\mu(W,\varepsilon). \tag{2.1}$$

Theorem 2.3. *The function $\mu : \mathfrak{M}_{\mathcal{H}_1([a,b])} \to \mathbb{R}_+$ given by (2.1), fulfils the hypothesis $1^\circ - 5^\circ$ of Definition 1.1.*

Proof. Its proof is similar to [5] and [9]. □

3. New class of n-order FDE with BVP

In this section, we define the new class of fractional deferential equation of order $\jmath \in (n-1, n]$, $(n \geq 5)$ with boundary value problems as following:

$$\begin{cases} {}^cD^\jmath w(\xi) = f(\xi),\ 0 \leq \xi < E,\ E \geq 1, \\ w(0) = w'(0) = w^{(5)}(0) = w^{(6)}(0) = \cdots = w^{(n)}(0) = 0, \\ w''(E) - w''(0) = w''(\eta),\ w^{(3)}(1) = w^{(4)}(1),\ w''(0) = \beta w(\eta), \\ \beta \in \mathbb{R},\ \eta \in (0,1). \end{cases} \tag{3.1}$$

Definition 3.1 ([25])**.** The fractional integral of order $\jmath$ is

$$I^\jmath f(\xi) = \frac{1}{\Gamma(\jmath)} \int_0^\xi \frac{f(\rho)}{(\xi-\rho)^{1-\jmath}} d\rho,\ \jmath > 0,$$

Definition 3.2 ([25])**.** Let $f : [0,\infty) \to \mathbb{R}$, then Caputo fractional derivative of order $\jmath > 0$ is

$${}^cD^\jmath f(\xi) = \frac{1}{\Gamma(n-\jmath)} \int_0^\xi \frac{f^{(n)}(\rho)}{(\xi-\rho)^{\jmath-n+1}} d\rho,$$

where $n = [\jmath] + 1$.

Lemma 3.3 ([17])**.** *Let $w \in C([0,\infty)) \cap L^1([0,\infty))$ with the Caputo fractional derivative of order $\jmath$ that belongs to $C([0,\infty)) \cap L^1([0,\infty))$. So*

$$I^\jmath\ {}^cD^\jmath w(\xi) = w(\xi) + c_1 + c_2\xi + c_3\xi^2 + \cdots + c_n\xi^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ and $n = [\jmath]$.

Lemma 3.4. *Let $f \in l^1([0,\infty))$ be continuous function and $n-1 < \jmath \leq n$, $(n \geq 5)$. Then the BVP problem of fractional differential equation*

$$\begin{cases} {}^cD^\vartheta w(\xi) = f(\xi),\ 0 \leq \xi < E,\ E \geq 1, \\ w(0) = w'(0) = w^{(5)}(0) = w^{(6)}(0) = \cdots = w^{(n)}(0) = 0, \\ w''(E) - w''(0) = w''(\eta),\ w^{(3)}(1) = w^{(4)}(1),\ w''(0) = \beta w(\eta),\ \beta \in \mathbb{R},\ \eta \in (0,1). \end{cases}$$

has a unique solution

$$\begin{aligned} w(\xi) = & \int_0^\xi \frac{(\xi-\wp)^{\jmath-1}}{\Gamma(\jmath)} f(\wp) d\wp + \frac{\beta\xi^2}{2-\beta\eta^2}\Bigg(\int_0^\eta \frac{(\eta-\wp)^{\jmath-1}}{\Gamma(\jmath)} f(\wp) d\wp \\ & + \frac{\eta^3}{6}\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)} f(\wp) d\wp - \int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)} f(\wp) d\wp \Big) \\ & + \frac{\eta^4}{24\eta - 12E^2}\Big(\int_0^E \frac{(E-\wp)^{\jmath-2}}{\Gamma(\jmath-1)} f(\wp) d\wp - \int_0^\eta \frac{(\eta-\wp)^{\jmath-4}}{\Gamma(\jmath-3)} f(\wp) d\wp \\ & + (E-1)\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)} f(\wp) d\wp - \int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)} f(\wp) d\wp \Big)\Big)\Bigg) \\ & + \frac{\xi^3}{6}\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)} f(\wp) d\wp - \int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)} f(\wp) d\wp \Big) \end{aligned}$$

$$+\frac{\xi^4}{24\eta-12E^2}\Bigg(\int_0^E \frac{(E-\wp)^{\jmath-2}}{\Gamma(\jmath-1)}f(\wp)d\wp-\int_0^\eta \frac{(\eta-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp$$

$$+(E-1)\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp)d\wp-\int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp\Big)\Bigg).$$

Proof. By Lemma 3.3, the equation (3.1) is equivalent to the integral form

$$w(\xi)=I^{\vartheta}f(\xi)+c_1+c_2\xi+c_3\xi^2+c_4\xi^3+\cdots+c_{n+1}\xi^n,$$

for some $c_i\in\mathbb{R}$, $i=1,2,3,4,\ldots,n+1$.

By the boundary value conditions for (3.1), we find that

$$c_1=c_2=c_6=c_7=\cdots=c_{n+1}=0,$$

and

$$w(\xi)=\int_0^\xi \frac{(\xi-\wp)^{\jmath-1}}{\Gamma(\jmath)}f(\wp)d\wp+c_3\xi^2+c_4\xi^3+c_5\xi^4. \tag{3.2}$$

So we get

$$\begin{aligned}
w'(\xi) &= \int_0^\xi \frac{(\xi-\wp)^{\jmath-2}}{\Gamma(\jmath-1)}f(\wp)d\wp+2c_3\xi+3c_4\xi^2+4c_5\xi^3,\\
w''(\xi) &= \int_0^\xi \frac{(\xi-\wp)^{\jmath-3}}{\Gamma(\jmath-2)}f(\wp)d\wp+2c_3+6c_4\xi+12c_5\xi^2,\\
w'''(\xi) &= \int_0^\xi \frac{(\xi-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp+6c_4+24c_5\xi,\\
w^{(4)}(\xi) &= \int_0^\xi \frac{(\xi-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp)d\wp+24c_5.
\end{aligned}$$

Applying, $w^{(4)}(1)=w'''(1)$ we get

$$\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp)d\wp+24c_5=\int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp+6c_4+24c_5$$

which imply that

$$6c_4=\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp)d\wp-\int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp.$$

Consequently,

$$c_4=\frac{1}{6}\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp)d\wp-\int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp\Big),$$

By $w''(E)-w''(0)=w'''(\eta)$ we have

$$\begin{aligned}
\int_0^E \frac{(E-\wp)^{\jmath-2}}{\Gamma(\jmath-1)}f(\wp)d\wp+2c_3+6c_4E+12c_5E^2-2c_3 = {}&\int_0^\eta \frac{(\eta-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp\\
&+6c_4+24c_5\eta.
\end{aligned}$$

Then, we have

$$\int_0^E \frac{(E-\wp)^{\jmath-2}}{\Gamma(\jmath-1)}f(\wp)d\wp - \int_0^\eta \frac{(\eta-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp + (6E-6)c_4 = (24\eta-12E^2)c_5.$$

Consequently,

$$\begin{aligned} c_5 &= \frac{1}{24\eta-12E^2}\Bigg(\int_0^E \frac{(E-\wp)^{\jmath-2}}{\Gamma(\jmath-1)}f(\wp)d\wp - \int_0^\eta \frac{(\eta-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp \\ &\quad +(E-1)\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp)d\wp - \int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp\Big)\Bigg). \end{aligned}$$

By $w''(0) = \beta w(\eta)$ we have

$$2c_3 = \beta\Big(\int_0^\eta \frac{(\eta-\wp)^{\jmath-1}}{\Gamma(\jmath)}f(\wp)d\wp + c_3\eta^2 + c_4\eta^3 + c_5\eta^4\Big),$$

so we get

$$(2-\beta\eta^2)c_3 = \beta\Big(\int_0^\eta \frac{(\eta-\wp)^{\jmath-1}}{\Gamma(\jmath)}f(\wp)d\wp + c_4\eta^3 + c_5\eta^4\Big),$$

Consequently,

$$\begin{aligned} c_3 = \frac{\beta}{2-\beta\eta^2}\Bigg(&\int_0^\eta \frac{(\eta-\wp)^{\jmath-1}}{\Gamma(\jmath)}f(\wp)d\wp + \frac{\eta^3}{6}\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp)d\wp \\ &- \int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp\Big) + \frac{\eta^4}{24\eta-12E^2}\Big(\int_0^E \frac{(E-\wp)^{\jmath-2}}{\Gamma(\jmath-1)}f(\wp)d\wp \\ &- \int_0^\eta \frac{(\eta-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp + (E-1)\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp)d\wp \\ &- \int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp\Big)\Big)\Bigg). \end{aligned}$$

Substituting the value of c_3, c_4 and c_5 in (3.2), it yields

$$\begin{aligned} w(\xi) = &\int_0^\xi \frac{(\xi-\wp)^{\jmath-1}}{\Gamma(\jmath)}f(\wp)d\wp + \frac{\beta\xi^2}{2-\beta\eta^2}\Bigg(\int_0^\eta \frac{(\eta-\wp)^{\jmath-1}}{\Gamma(\jmath)}f(\wp)d\wp \\ &+ \frac{\eta^3}{6}\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp)d\wp - \int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp\Big) \\ &+ \frac{\eta^4}{24\eta-12E^2}\Big(\int_0^E \frac{(E-\wp)^{\jmath-2}}{\Gamma(\jmath-1)}f(\wp)d\wp - \int_0^\eta \frac{(\eta-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp \\ &+ (E-1)\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp)d\wp - \int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp\Big)\Big)\Bigg) \\ &+ \frac{\xi^3}{6}\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp)d\wp - \int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp\Big) \end{aligned}$$

$$+\frac{\xi^4}{24\eta-12E^2}\Bigg(\int_0^E \frac{(E-\wp)^{\jmath-2}}{\Gamma(\jmath-1)}f(\wp)d\wp-\int_0^\eta \frac{(\eta-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp$$
$$+(E-1)\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp)d\wp-\int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp\Big)\Bigg).$$

□

4. Application

Now, we study the solvability of E.q (3.1) in $\mathcal{H}_1([0,E])$ and we present one example to performance main results.

$(A1)$ The function $f:[0,E]\times\mathbb{R}\to\mathbb{R}$ is continuous and $\exists$ increasing functions $\psi:[0,E]\to[0,+\infty)$ so that $\psi(\wp)\to 0$, as $\wp\to 0$ and $\forall\ w,v\in\mathbb{R}$, $\wp\in[0,T]$ the inequality

$$|f(\wp,w)-f(\wp,v)|\le\psi(|w-v|),$$

is satisfied and

$$\overline{N}=\sup\{|f(\wp,0)|:\ \wp\in[0,E]\}<\infty.$$

$(A2)$ $\exists$ a solution $r_0>0$ for the inequality

$$(\psi(r)+\overline{N})(\frac{2E^{\jmath-1}}{\jmath\Gamma(\jmath)})\le r.$$

Theorem 4.1. *By conditions* $(A1)$ *and* $(A2)$ *the E.q* (3.1) *has at least one solution in* $\mathcal{H}_1([0,E])$.

Proof. Define $F:\mathcal{H}_1([0,E])\to\mathcal{H}_1([0,E])$ as

$$F(w)(\xi)=\int_0^\xi \frac{(\xi-\wp)^{\jmath-1}}{\Gamma(\jmath)}f(\wp)d\wp+\frac{\beta\xi^2}{2-\beta\eta^2}\Bigg(\int_0^\eta \frac{(\eta-\wp)^{\jmath-1}}{\Gamma(\jmath)}f(\wp)d\wp$$
$$+\frac{\eta^3}{6}\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp)d\wp-\int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp\Big)$$
$$+\frac{\eta^4}{24\eta-12E^2}\Big(\int_0^E \frac{(E-\wp)^{\jmath-2}}{\Gamma(\jmath-1)}f(\wp)d\wp-\int_0^\eta \frac{(\eta-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp$$
$$+(E-1)\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp)d\wp-\int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp\Big)\Big)\Bigg)$$
$$+\frac{\xi^3}{6}\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp)d\wp-\int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp\Big)$$
$$+\frac{\xi^4}{24\eta-12E^2}\Bigg(\int_0^E \frac{(E-\wp)^{\jmath-2}}{\Gamma(\jmath-1)}f(\wp)d\wp-\int_0^\eta \frac{(\eta-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp$$
$$+(E-1)\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp)d\wp-\int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp)d\wp\Big)\Bigg).$$

For any $w\in\mathcal{H}_1([0,E])$, we show that $F(w)\in\mathcal{H}_1([0,E])$. For this choose $\xi_2,\xi_1\in[0,E]$ with $\xi_2\neq\xi_1$ and $\xi_1<\xi_2$. By $(A1)$, we have

$$\frac{\Big|(Fw)(\xi_2)-(Fw)(\xi_1)\Big|}{|\xi_2-\xi_1|}$$

$$\leq \frac{\Big|\int_0^{\xi_2}\frac{(\xi_2-\wp)^{\jmath-1}}{\Gamma(\jmath)}f(\wp,w(\wp))d\wp-\int_0^{\xi_1}\frac{(\xi_1-\wp)^{\jmath-1}}{\Gamma(\jmath)}f(\wp,w(\wp))d\wp\Big|}{|\xi_2-\xi_1|}$$

$$+\frac{\Big|\frac{\beta(\xi_2^2-\xi_1^2)}{2-\beta\eta^2}\Big(\int_0^{\eta}\frac{(\eta-\wp)^{\jmath-1}}{\Gamma(\jmath)}f(\wp,w(\wp))d\wp+\frac{\eta^3}{6}\big(\int_0^1\frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp,w(\wp))d\wp-\int_0^1\frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp,w(\wp))d\wp\big)\Big|}{|\xi_2-\xi_1|}$$

$$+\frac{\Big|\frac{\eta^4}{24\eta-12E^2}\Big(\int_0^{E}\frac{(E-\wp)^{\jmath-2}}{\Gamma(\jmath-1)}f(\wp,w(\wp))d\wp-\int_0^{\eta}\frac{(\eta-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp,w(\wp))d\wp\Big|}{|\xi_2-\xi_1|}$$

$$+\frac{\Big|(E-1)\big(\int_0^1\frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp,w(\wp))d\wp-\int_0^1\frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp,w(\wp))d\wp)\Big)\Big|}{|\xi_2-\xi_1|}$$

$$+\frac{\Big|\frac{\xi_2^3-\xi_1^3}{6}\Big(\int_0^1\frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp,w(\wp))d\wp-\int_0^1\frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp,w(\wp))d\wp\Big)\Big|}{|\xi_2-\xi_1|}$$

$$+\frac{\Big|\frac{\xi_2^4-\xi_1^4}{24\eta-12E^2}\Big(\int_0^{E}\frac{(E-\wp)^{\jmath-2}}{\Gamma(\jmath-1)}f(\wp,w(\wp))d\wp-\int_0^{\eta}\frac{(\eta-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp,w(\wp))d\wp\Big|}{|\xi_2-\xi_1|}$$

$$+\frac{\Big|(E-1)\big(\int_0^1\frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}f(\wp,w(\wp))d\wp-\int_0^1\frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}f(\wp,w(\wp))d\wp\Big)\Big|}{|\xi_2-\xi_1|}$$

Since, $\xi_2>\xi_1$ so $|\xi_2^2-\xi_1^2|\to 0$, $|\xi_2^3-\xi_1^3|\to 0$ and $|\xi_2^4-\xi_1^4|\to 0$, then we get

$$\frac{\Big|(Fw)(\xi_2)-(Fw)(\xi_1)\Big|}{|\xi_2-\xi_1|}\leq\frac{\Big|\int_0^{\xi_2}\frac{(\xi_2-\wp)^{\jmath-1}}{\Gamma(\jmath)}f(\wp,w(\wp))d\wp-\int_0^{\xi_1}\frac{(\xi_1-\wp)^{\jmath-1}}{\Gamma(\jmath)}f(\wp,w(\wp))d\wp\Big|}{|\xi_2-\xi_1|}$$

By condition $(A1)$, we obtain

$$\frac{\Big|(Fw)(\xi_2)-(Fw)(\xi_1)\Big|}{|\xi_2-\xi_1|}$$

$$\leq\frac{\Big|\int_0^{\xi_1}\big(\frac{(\xi_2-\wp)^{\jmath-1}}{\Gamma(\jmath)}-\frac{(\xi_1-\wp)^{\jmath-1}}{\Gamma(\jmath)}\big)f(\wp,w(\wp))d\wp+\int_{\xi_1}^{\xi_2}\frac{(\xi_2-\wp)^{\jmath-1}}{\Gamma(\jmath)}f(\wp,w(\wp))d\wp\Big|}{|\xi_2-\xi_1|}$$

$$\leq \frac{\left| \int_0^{t_1} \left(\frac{(\xi_2-\wp)^{\jmath-1}}{\Gamma(\jmath)} - \frac{(\xi_1-\wp)^{\jmath-1}}{\Gamma(\jmath)}\right)\big(f(\wp,w(\wp)) - f(\wp,0) + f(\wp,0)\big)d\wp + \int_{\xi_1}^{\xi_2} \frac{(\xi_2-\wp)^{\jmath-1}}{\Gamma(\jmath)}(f(\wp,w(\wp)) - f(\wp,0) + f(\wp,0))d\wp \right|}{|\xi_2-\xi_1|}$$

$$\leq \frac{\left|(\psi(|w|_{Lip}) + \overline{N})\frac{1}{\Gamma(\jmath)}\left(\frac{\xi_1^{\jmath}}{\jmath} + \frac{(\xi_2-\xi_1)^{\jmath}}{\jmath} - \frac{\xi_2^{\jmath}}{\jmath} + \frac{(\xi_2-\xi_1)^{\jmath}}{\jmath}\right)\right|}{|\xi_2-\xi_1|}.$$

Since, $\xi_2 > \xi_1$, using the facts that $\left|\frac{\xi_1^{\jmath}}{\jmath} - \frac{\xi_2^{\jmath}}{\jmath}\right| \leq 0$, we have

$$\begin{aligned}\frac{\left|(Fw)(\xi_2) - (Fw)(\xi_1)\right|}{|\xi_2-\xi_1|} &\leq 2(\psi(|w|_{Lip}) + \overline{N})\frac{(\xi_2-\xi_1)^{\jmath}}{\jmath\Gamma(\jmath)|\xi_2-\xi_1|} \\ &\leq 2(\psi(|w|_{Lip}) + \overline{N})\Big(\frac{E^{\jmath-1}}{\jmath\Gamma(\jmath)}\Big).\end{aligned}$$

Consequently, we obtain

$$|Fw|_{Lip} \leq (\psi(|w|_{Lip}) + \overline{N})\Big(\frac{2E^{\jmath-1}}{\jmath\Gamma(\jmath)}\Big). \tag{4.1}$$

By (4.1) F is well defined. We define the $D \subseteq \mathcal{H}_1([0,E])$ as

$$D = \{w \in \mathcal{H}_1([0,E]) : \ |w|_{Lip} \leq r\ \}.$$

Clearly $\emptyset \neq D = \overline{D}$ is convex and bounded in $\mathcal{H}_1([0,E])$ by $(A2)$ gives that $F : D \to D$. We show that F is continuous, let $\varepsilon > 0$ and $w, v \in D$, so that $|w-v|_{Lip} \leq \varepsilon$. Then, for arbitrary $\xi_1, \xi_2 \in [0,E]$ with $\xi_2 > \xi_1$, we can write

$$\frac{\Big|[(Fw)(\xi_2)-(Fv)(\xi_2)]-[(Fw)(\xi_1)-(Fv)(\xi_1)]\Big|}{|\xi_2-\xi_1|}$$

$$\leq \frac{\left| \int_0^{\xi_2} \frac{(\xi_2-\wp)^{\jmath-1}}{\Gamma(\jmath)}(f(\wp,w(\wp)) - f(\wp,v(\wp)))d\wp - \int_0^{\xi_1} \frac{(\xi_1-\wp)^{\jmath-1}}{\Gamma(\jmath)}(f(\wp,w(\wp)) - f(\wp,v(\wp)))d\wp \right|}{|\xi_2-\xi_1|}$$

$$+ \frac{\left|\frac{\beta(\xi_2^2-\xi_1^2)}{2-\beta\eta^2}\left(\int_0^{\eta} \frac{(\eta-\wp)^{\jmath-1}}{\Gamma(\jmath)}(f(\wp,w(\wp)) - f(\wp,v(\wp)))d\wp\right| + \left|\frac{\eta^3}{6}\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}(f(\wp,w(\wp)) - f(\wp,v(\wp)))d\wp - \int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}(f(\wp,w(\wp)) - f(\wp,v(\wp)))d\wp\Big)\right|}{|\xi_2-\xi_1|}$$

$$+\frac{\left|\frac{\eta^4}{24\eta-12E^2}\Big(\int_0^E \frac{(E-\wp)^{\jmath-2}}{\Gamma(\jmath-1)}(f(\wp,w(\wp))-f(\wp,v(\wp)))d\wp - \int_0^\eta \frac{(\eta-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}(f(\wp,w(\wp))-f(\wp,v(\wp)))d\wp\right| + \left|(E-1)\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}(f(\wp,w(\wp))-f(\wp,v(\wp)))d\wp - \int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}(f(\wp,w(\wp))-f(\wp,v(\wp)))d\wp\Big)\right)\right|}{|\xi_2-\xi_1|}$$

$$+\frac{\left|\frac{\xi_2^3-\xi_1^3}{6}\left(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}(f(\wp,w(\wp))-f(\wp,v(\wp)))d\wp - \int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}(f(\wp,w(\wp))-f(\wp,v(\wp)))d\wp\right)\right|}{|\xi_2-\xi_1|}$$

$$+\frac{\left|\frac{\xi_2^4-\xi_1^4}{24\eta-12E^2}\left(\int_0^E \frac{(E-\wp)^{\jmath-2}}{\Gamma(\jmath-1)}(f(\wp,w(\wp))-f(\wp,v(\wp)))d\wp - \int_0^\eta \frac{(\eta-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}(f(\wp,w(\wp))-f(\wp,v(\wp)))d\wp\right| + \left|(E-1)\Big(\int_0^1 \frac{(1-\wp)^{\jmath-5}}{\Gamma(\jmath-4)}(f(\wp,w(\wp))-f(\wp,v(\wp)))d\wp - \int_0^1 \frac{(1-\wp)^{\jmath-4}}{\Gamma(\jmath-3)}(f(\wp,w(\wp))-f(\wp,v(\wp)))d\wp\Big)\right)\right|}{|\xi_2-\xi_1|}.$$

So, we get

$$\left\{\frac{\Big|[(Fw)(\xi_2)-(Fv)(\xi_2)]-[(Fw)(\xi_1)-(Fv)(\xi_1)]\Big|}{|\xi_2-\xi_1|} : \xi_1,\xi_2\in[0,E], \xi_1\neq\xi_2\right\}$$

$$\begin{aligned}\leq\ & \frac{\frac{\psi(|w-v|_{Lip})}{\Gamma(\jmath)}\left(\frac{\xi_1^\jmath}{\jmath}+\frac{(\xi_2-\xi_1)^\jmath}{\jmath}-\frac{\xi_2^\jmath}{\jmath}+\frac{(\xi_2-\xi_1)^\jmath}{\jmath}\right)}{|\xi_2-\xi_1|} \\ & +\psi(|w-v|_{Lip})\left(\frac{\frac{\beta(\xi_2^2-\xi_1^2)}{2-\beta\eta^2}\left(\frac{\eta^\jmath}{\jmath\Gamma(\jmath)}+\frac{\eta^3}{6}\big(\frac{1}{(\jmath-4)\Gamma(\jmath-4)}-\frac{1}{(\jmath-3)\Gamma(\jmath-3)}\big)\right.}{|\xi_2-\xi_1|}\right. \\ & \left.+\frac{\frac{\eta^4}{24\eta-12E^2}\big(\frac{E^{\jmath-1}}{(\jmath-1)\Gamma(\jmath-1)}-\frac{\eta^{\jmath-3}}{(\jmath-3)\Gamma(\jmath-3)}\big)+(E-1)\big(\frac{1}{(\jmath-4)\Gamma(\jmath-4)}-\frac{1}{(\jmath-3)\Gamma(\jmath-3)}\big)\Big)}{|\xi_2-\xi_1|}\right) \\ & +\psi(|w-v|_{Lip})\left(\frac{\frac{\xi_2^3-\xi_1^3}{6}\big(\frac{1}{(\jmath-4)\Gamma(\jmath-4)}-\frac{1}{(\jmath-3)\Gamma(\jmath-3)}\big)}{|\xi_2-\xi_1|}\right)\end{aligned}$$

$$+\psi(|w-v|_{Lip})\left(\frac{\frac{\xi_2^4-\xi_1^4}{24\eta-12E^2}\Big(\frac{E^{\jmath-1}}{(\jmath-1)\Gamma(\jmath-1)}-\frac{\eta^{\jmath-3}}{(\jmath-3)\Gamma(\jmath-3)}+(E-1)\big(\frac{1}{(\jmath-4)\Gamma(\jmath-4)}-\frac{1}{(\jmath-3)\Gamma(\jmath-3)}\big)\Big)}{|\xi_2-\xi_1|}\right).$$

Finally,

$$\begin{aligned}
&|Fw-Fv|_{Lip}\\
&\leq \frac{\frac{\psi(|w-v|_{Lip})}{\Gamma(\jmath)}\left(\frac{\xi_1^\jmath}{\jmath}+\frac{(\xi_2-\xi_1)^\jmath}{\jmath}-\frac{\xi_2^\jmath}{\jmath}+\frac{(\xi_2-\xi_1)^\jmath}{\jmath}\right)}{|\xi_2-\xi_1|}\\
&+\psi(|w-v|_{Lip})\left(\frac{\frac{\beta(\xi_2^2-\xi_1^2)}{2-\beta\eta^2}\left(\frac{\eta^\jmath}{\jmath\Gamma(\jmath)}+\frac{\eta^3}{6}\big(\frac{1}{(\jmath-4)\Gamma(\jmath-4)}-\frac{1}{(\jmath-3)\Gamma(\jmath-3)}\big)\right.}{|\xi_2-\xi_1|}\right.\\
&\left.+\frac{\frac{\eta^4}{24\eta-12E^2}\big(\frac{E^{\jmath-1}}{(\jmath-1)\Gamma(\jmath-1)}-\frac{\eta^{\jmath-3}}{(\jmath-3)\Gamma(\jmath-3)}\big)+(E-1)\big(\frac{1}{(\jmath-4)\Gamma(\jmath-4)}-\frac{1}{(\jmath-3)\Gamma(\jmath-3)}\big)\Big)}{|\xi_2-\xi_1|}\right)\\
&+\psi(|w-v|_{Lip})\left(\frac{\frac{\xi_2^3-\xi_1^3}{6}\big(\frac{1}{(\jmath-4)\Gamma(\jmath-4)}-\frac{1}{(\jmath-3)\Gamma(\jmath-3)}\big)}{|\xi_2-\xi_1|}\right)\\
&+\psi(|w-v|_{Lip})\left(\frac{\frac{\xi_2^4-\xi_1^4}{24\eta-12E^2}\Big(\frac{E^{\jmath-1}}{(\jmath-1)\Gamma(\jmath-1)}-\frac{\eta^{\jmath-3}}{(\jmath-3)\Gamma(\jmath-3)}+(E-1)\big(\frac{1}{(\jmath-4)\Gamma(\jmath-4)}-\frac{1}{(\jmath-3)\Gamma(\jmath-3)}\big)\Big)}{|\xi_2-\xi_1|}\right).
\end{aligned}$$

By $(A1)$, since $\psi(s)\to 0$ as $s\to 0$, we get F is continuous. We show that F is a condensing operator. Let $\emptyset\neq W\subseteq\mathcal{H}_1([0,E])$ be bounded, take $w\in W$, $\varepsilon>0$ and $\xi_2,\xi_1\in[0,E]$ with $\xi_2\neq\xi_1$ and $|\xi_2-\xi_1|\leq\varepsilon$, by (4.1) we obtain

$$\begin{aligned}
\mu(Fw,\varepsilon) &= \sup\Big\{\frac{|(Fw)(\xi_2)-(Fw)(\xi_1)|}{|\xi_2-\xi_1|},\ \xi_2,\xi_1\in[0,E],\ \xi_1\neq\xi_2,\ |\xi_1-\xi_2|\leq\varepsilon\Big\}\\
&\leq (\psi(|w|_{Lip})+\overline{N})\Big(\frac{2\varepsilon^{\jmath-1}}{\jmath\Gamma(\jmath)}\Big).
\end{aligned}$$

This implies that

$$\mu(FW,\varepsilon)=\sup\{\mu(Fw,\varepsilon),\ w\in W\}\leq(\psi(|w|_{Lip})+\overline{N})\Big(\frac{2\varepsilon^{\jmath-1}}{\jmath\Gamma(\jmath)}\Big).$$

By taking $\varepsilon \to 0$, we get

$$\mu(FW) = \lim_{\varepsilon\to 0}\mu(FW,\varepsilon) = 0,$$

which is equivalent to,

$$\mu(FW) \leq L\mu(W),$$

where $L = 0$. By Theorem 1.2, we get F has a fixed point w in the ball D so the E.q (3.1) has at least one solution in $\mathcal{H}_1([0,E])$. □

Example 4.2. Consider the equation

$$\begin{cases} {}^cD^{\frac{13}{2}}w(\xi) = \frac{e^{-4\xi}\arctan(w(\xi)+4)\sin(\xi^4+1)}{\sqrt{12+\xi^2}}, \ 0 \leq \xi < E, \ E \geq 1, \\ w(0) = w'(0) = w^{(5)}(0) = w^{(6)}(0) = w^{(7)}(0) = 0, \\ w''(E) - w''(0) = w''(\frac{1}{3}), \ w^{(3)}(1) = w^{(4)}(1), \ w''(0) = 5w(\frac{1}{3}). \end{cases} \tag{4.2}$$

Observe that (4.2) is a special case of (3.1) when $\jmath = \frac{13}{2}$, $\beta = 5$, $\eta = \frac{1}{3}$, and $f(\wp, w) = \frac{e^{-4\wp}\arctan(w+4)\sin(\wp^4+1)}{\sqrt{12+\wp^2}}$, $\wp \in (0, E]$. The condition $(A1)$ of Theorem 4.1 with $\psi(\wp) = \frac{1}{\sqrt{12}}\wp$ hold. Indeed, we have

$$\begin{aligned} |f(\wp,w) - f(\wp,v)| &= \Big|\frac{e^{-4\wp}\arctan(w+4)\sin(\wp^4+1)}{\sqrt{12+\wp^2}}\Big| \\ &\leq \frac{1}{\sqrt{12}e^{4\wp}}|\arctan(w+4)\sin(\wp^4+1)| \\ &\leq \frac{1}{\sqrt{12}}|w+4-v-4|)| \leq \frac{1}{\sqrt{12}}|w-v| \end{aligned}$$

and

$$\overline{N} = \sup\Big\{\Big|\frac{e^{-4\wp}\sin(\wp^4+1)arctan(2)}{\sqrt{\wp^2+12}}\Big|, \ \wp \in [0,E]\Big\} = \frac{arctan(2)}{\sqrt{12}}.$$

Also, $\exists\ r > 0$ so that fulfils the inequality of $(A2)$ of Th 4.1, that is,

$$\Big(\frac{r}{\sqrt{12}} + \frac{arctan(2)}{\sqrt{12}}\Big)\Big(\frac{2E^{\frac{11}{2}}}{\frac{13}{2}\Gamma(\frac{13}{2})}\Big) \leq r.$$

Now, Theorem 4.1 guarantees that E.q (4.2) has at least one solution in $\mathcal{H}_1([0,E])$.

References

[1] M. Al-Smadi, O.A. Arqub and S. Hadid, *An attractive analytical technique for coupled system of fractional partial differential equations in shallow water waves with conformable derivative*, Commun. Theor. Phys. **72** (2020): 085001.

[2] H. Amiri Kayvanloo, M. Khanehgir and R. Allahyari, *A family of measures of noncompactness in the space* $\boldsymbol{L^p_{loc}}(\mathbb{R}^{\boldsymbol{N}})$ *and its application to some nonlinear convolution type integral equations*, Cogent Math. Stat. **6** (2019): Art. ID 1592276.

[3] H. Amiri Kayvanloo, M. Khanehgir and R. Allahyari, *A family of measures of noncompactness in the Hölder space* $C^{n,\gamma}(\mathbb{R}_+)$ *and its application to some fractional differential equations and numerical methods*, J. Comput. Appl. Math. **363** (2020), 256–272.

[4] H. Amiri Kayvanloo, H. Mehravaran, M. Mursaleen, R. Allahyari and A. Allahyari, *Solvability of infinite systems of Caputo–Hadamard fractional differential equations in the triple sequence space* $c^3(\Delta)$, J. Pseudo-Differ. Oper. Appl. **15** (2024): 26.

[5] H. Amiri Kayvanloo, M. Mursaleen, M. Mehrabinezhad and F. Pouladi Najafabadi, *Solvability of some fractional differential equations in the Hölder space $\mathcal{H}_\gamma(\mathbb{R}_+)$ and their numerical treatment via measures of noncompactness*, Mathematical Sciences **17** (2022), 387–397.

[6] A. Atangana, *Modelling the spread of COVID-19 with new fractal-fractional operators: Can the lockdown save mankind before vaccination*, Chaos, Solitons Fractals **136** (2020): 109860.

[7] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, 60, Dekker, New York, 1980.

[8] J. Banaś and M. Mursaleen, *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations*, Springer, New Delhi, 2014.

[9] J. Banaś and R. Nalepa, *On a measure of noncompactness in the space of functions with tempered increments*, J. Math. Anal. Appl. **435** (2016), 1634–1651.

[10] A. Das, B. Hazarika, V. Parvaneh and M. Mursaleen, *Solvability of generalized fractional order integral equations via measures of noncompactness*, Mathematical Sciences **15** (2021), 241–251.

[11] G. Darbo, *Punti uniti in trasformazioni a codominio non compatto*, Rend. Sem. Mat. Uni. Padova **24** (1955), 84–92.

[12] R. M. Ganji, H. Jafari and D. Baleanu, *A new approach for solving multi variable orders differential equations with MittagaLeffler kernel*, Chaos, Solitons Fractals **130** (2020): 109405.

[13] J. Harjani, B. López and K. Sadarangani, *Existence and uniqueness of mild solutions for a fractional differential equation under Sturm-Liouville boundary conditions when the data function is of Lipschitzian type*, Demonstr. Math. **53** (2020), 167–173.

[14] I. Haque, J. Ali and M. Mursaleen, *Existence of solutions for an infinite system of Hilfer fractional boundary value problems in tempered sequence spaces*, Alexandria Engineering Journal **65** (2023), 575–583.

[15] I. Haque, J. Ali and M. Mursaleen, *Solvability of infinite system of Langevin fractional differential equation in a new tempered sequence space*, Fractional Calculus and Applied Analysis, DOI: 10.1007/s13540-023-00175-y.

[16] M. D. Johansyah, A. K. Supriatna, E. Rusyaman and J. Saputra, *Application of fractional differential equation in economic growth model: A systematic review approach*, AIMS Math. **6** (2021), 10266–10280.

[17] A. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, (vol. 204), Elsevier Science Limited, 2006.

[18] K. Kuratowski, *Sur les espaces complets*, Fund. Math. 15(1930) 301-309.

[19] H. Mehravaran, H. A. Kayvanloo and R. Allahyari, *Measures of noncompactness in the space of regulated functions $R(J, \mathbb{R}^\infty)$ and its application to some nonlinear infinite systems of fractional differential equations*, Mathematical Sciences **17** (2023), 223–232.

[20] H. Mehravaran, H. Amiri Kayvanloo and M. Mursaleen, *Solvability of infinite systems of fractional differential equations in the double sequence space $2^c(\Delta)$*, Fractional Calculus and Applied Analysis **25** (2022), 2298–2312.

[21] M. Mursaleen, B. Bilalov and S. M. H. Rizvi, *Applications of measures of noncompactness to infinite system of fractional differential equations*, Filomat **31** (2017), 3421–3432.

[22] M. Mursaleen and S. M. H. Rizvi, *Solvability of infinite systems of second order differential equations in c_0 and l_1 by Meir 蹇適 eeler condensing operators*, Proc. Am. Math. Soc. **144** (2016), 4279–4289.

[23] M. Mursaleen and E. Savaş, *Solvability of an infinite system of fractional differential equations with p−Laplacian operator in a new tempered sequence space*, J. Pseudo-Differ. Oper. Appl. **14** (2023): 57.

[24] F. P. Najafabadi, J. J. Nieto and H. Amiri Kayvanloo, *Measure of noncompactness on weighted Sobolev space with an application to some nonlinear convolution type integral equations*, J. Fixed Point Theory Appl. **22** (2020): article number 75.

[25] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, vol. 198, Elsevier, 1998.

[26] S. Saiedinezhad, *On a measure of noncompactness in the HHölder space $Ck, \gamma(\Omega)$ and its application*, J. Comput. Applied Math. **346** (2019), 566–571.

Manuscript received April 17, 2024
revised July 29, 2024

H. Amiri Kayvanloo
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran
E-mail address: amiri.hojjat93@mshdiau.ac.ir

M. Mursaleen
Department of Mathematical Sciences, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, Tamilnadu, India;
Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan
E-mail address: mursaleenm@gmail.com