Journal of Nonlinear and Convex Analysis
Volume 26, Number 4, 2025, 1027–1041

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

CONTROLLABILITY OF ψ-HILFER FRACTIONAL DIFFERENTIAL EQUATION WITH NON-LOCAL CONDITIONS VIA MEASURE OF NONCOMPACTNESS

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ABSTRACT. In this paper, we prove controllability results for a class of ψ-Hilfer fractional differential equations with almost sectorial operator and non-local conditions. By means of measure of noncompactness and semigroup theory, the Mönch fixed point theorem is used to derive certain sufficient requirements for controllability. We specifically are not presuming the evolution system's compactness. An illustration is provided as an example of how well our findings work.

1. Introduction

In 1960, Kalman introduced the idea of controllability. Controllability plays a momentous role in development of modelling problems expressed by differential equations, neutral differential equations, impulsive equations, delay differential equations, integrodifferential equations and differential inclusions in Banach spaces. In the study of controllability, the operator semigroup $\mathcal{F}(z)$ must be assumed to be compact for the majority of the earlier findings. Balachandran and Park [4] proved their results by using a compact analytic semigroup for fractional integrodifferential systems in Banach spaces. Yan [23] developed necessary conditions for the controllability of fractional order partial neutral functional integrodifferential inclusion. Chang [8] looked into a mixed Volterra-Fredholm type integrodifferential inclusion controllability result in Banach spaces using a compact semigroup and Bohnenblust-Karlin fixed point theorem.

When the compactness of a semigroup and several other requirements are met, O'Regan and Hernandez [10] noted that the outcomes of controllability will be constrained to the finite dimensional space. Determining adequate criteria that ensures the controllability of the outcomes of diverse systems without needing the compactness of a semigroup has therefore been a major goal of many researchers. Later, scholars have always attempted to use the measure of noncompactness to avoid a semigroup's compactness.

Ji et al. [13] focused their study on the controllability of impulsive functional differential equations with nonlocal circumstances by utilizing Mönch fixed point theorem and the measure of noncompactness.

2020 *Mathematics Subject Classification.* 93B05, 34A08, 47H08, 47H10.

Key words and phrases. Controllabillity, fractional differential equation, measure of noncompactness, fixed point theorem.

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By employing Mönch fixed point theorem, Machado et al. [14] also obtained sufficient conditions for controllability and controllability results for a class of impulsive functional integrodifferential equations with finite delay in Banach spaces.

Restrictive requirements on the estimated parameters and the measure of noncompactness are necessary, however the results in [13, 14] do not presume the compactness of the evolution system. Under the presumptions of the measure of noncompactness in a separable and uniformly smooth Banach space, Xue [22] achieved the existence results of integral solutions for nonlinear first order differential equations with nonlocal initial value conditions. In [9], Haque et al. studied the controllability of ψ-Hilfer fractional differential equation via measure of noncompactness.

Ahmad et al. [2] shown that first order impulsive integrodifferential equations with nonlinearity of the form $f(t, u(t), Gu(t))$ have mild solutions where $Gu(t)$ denotes a Volterra-type integral operator. Bose and Udhayakumar [7] studied the controllability of Hilfer fractional neutral differential equations with almost sectorial opertor via mnc.

Motivated by the earlier results of above authors, we study the controllability of the following ψ-Hilfer fractional differential equation with almost sectorial operator.

$$D^{k,\epsilon,\psi}_{0^+}[w(z)] = Aw(z) + h(z, w(z), \chi w(z)) + Bv(z), \; z \in J = (0, d] \tag{1.1}$$

$$I^{1-\delta;\psi}_{0^+} w(z)|_{z=0} = w_0 + \Sigma^{m}_{j=1} c_j w(\tau_j), \; \tau_j \in (0, d) \tag{1.2}$$

where $0 < k < 1$, $0 \leq \epsilon \leq 1$, $\delta = k + \epsilon - k\epsilon$, $D^{k,\epsilon,\psi}_{0^+}$ is a ψ-Hilfer fractional derivative operator and $I^{1-\delta;\psi}_{0^+}$ is ψ-Riemann–Liouville fractional integral operator. A is almost sectorial operator, $h(z, w(.), \chi w(.)) \in C_{1-\delta;\psi}(J, \mathcal{C}, \mathbb{R})$, the control function is $v \in L^2([0, d], U), U$ is Hilbert space and $B : U \to \mathbb{R}$ is bounded linear operator with $\| B \| \leq K_1$. For $\chi : D \times \mathbb{R} \to \mathbb{R}$, $\chi w(z) = \int_0^z \mathcal{K}(z, s, w(s))ds$, $D = \{(t, s) : 0 \leq s \leq t \leq d\}$, c_k are real numbers and $\tau_j, j = 1, 2, \dots, m$ are given points satisfying $0 < \tau_1 < \tau_2 < \cdots < \tau_m < d$.

This type of fractional differential equations arise in the financial crisis model represented in the work of Norouzi [16], where author solved same kind of problem without discussing controllability.

If we consider all $c_j = 0$, then system reduces to problem of Bose and Udhayakumar [7] without delay for a particular value of function $\psi(t)$. We establish the existence of mild solution for evolution equations, and give a set of sufficient conditions for the controllability result for ψ-Hilfer fractional differential equation through the measure of noncompactness approach and almost sectorial operator.

2. Preliminaries and basic results

Basic definitions and lemmas from the fractional calculus along with some results for measure of noncompactness (mnc) are recalled in this section.

Let $J = [0, d]$ be an interval and $\psi : J \to \mathbb{R}^+$ be an increasing and positive function for all $z \in J$.

The space $C_{1-\delta,\psi}(J, \mathcal{C}, \mathbb{R})$ denotes the Banach space of weighted functions defined on J, i.e.

$$C_{1-\delta,\psi}(J, \mathcal{C}, \mathbb{R}) = \{w : J \to \mathbb{R} | (\psi(.) - \psi(0))^{1-\delta} w(.) \in C(J, \mathbb{R})\},$$

with norm $\|w\|_{C_{1-\delta,\psi}(J,\mathcal{C},\mathbb{R})} = \max_{z\in J}\{|(\psi(.)-\psi(0))^{1-\delta}w(z)|\}$.

Definition 2.1 ([11]). Let f be a integrable function defined on $(0,d)$ and ψ be an increasing function having a continuous derivative ψ' on $(0,d)$ such that $\psi'(t)\neq 0$ for all $t\in J$ and $k>0$ is a constant. The left-sided fractional integral of order k of function f with respect to ψ is defined by

$$I_{0^+}^{k,\psi}f(t) = \frac{1}{\Gamma(k)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{k-1}f(s)ds. \tag{2.1}$$

If we take $\psi(t)=t$, we get a well known classical Riemann-Liouville fractional integral.

Definition 2.2 ([3]). Let $n-1<k<n\in\mathbb{N},\ 0\leq\epsilon\leq 1$ and $f,\psi\in C^n(J,\mathbb{R})$ two functions such that ψ is an increasing and $\psi'(t)\neq 0$ for all $t\in J$. The left-sided ψ-Hilfer fractional derivative of order k and type ϵ of function f is defined as

$$D_{0^+}^{k,\epsilon;\psi}f(t) = I_{0^+}^{\epsilon(n-k);\psi}\left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n I_{0^+}^{(1-\epsilon)(n-k);\psi}f(t).$$

Or we can write it as

$$D_{0^+}^{k,\epsilon;\psi}f(t) = I_{0^+}^{\epsilon(n-k);\psi}\mathrm{D}_{0^+}^{\delta;\psi}f(t),\ \delta = k+\epsilon(n-k)$$

where $D_{0^+}^{\delta;\psi}f(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n I_{0^+}^{(1-\epsilon)(n-k);\psi}f(t)$.

In particular, the ψ-Hilfer fractional derivative of order $0<k<1$ and type $0\leq\epsilon\leq 1$ can be written in the following form :

$$D_{0^+}^{k,\epsilon;\psi}f(t) = \frac{1}{\Gamma(\delta-k)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\delta-k-1}\mathrm{D}_{0^+}^{\delta;\psi}f(s)ds = I_{0^+}^{\delta-k;\psi}D_{0^+}^{\delta;\psi}f(t)$$

where $\delta = k+\epsilon(1-k)$, $I_{0^+}^{\delta-k;\psi}(\cdot)$ are defined by equation (2.1) and $D_{0^+}^{\delta;\psi}f(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n I_{0^+}^{1-\delta;\psi}f(t)$.

Lemma 2.3 ([11]). *Let $\alpha>0$ and $\beta>0$. The following semigroup properties hold:*

(i) *If $f\in \mathrm{L}^{\mathrm{p}}(J,\mathbb{R})(p\geq 1)$, then $I_{0^+}^{\alpha;\psi}I_{0^+}^{\beta;\psi}f(t) = I_{0^+}^{\alpha+\beta;\psi}f(t)$, a.e. $t\in J$.*

(ii) *If $f\in C_{\gamma;\psi}(J,\mathbb{R})$, then $I_{0^+}^{\alpha;\psi}I_{0^+}^{\beta;\psi}f(t) = I_{0^+}^{\alpha+\beta;\psi}f(t),\ t\in[a,b],\ 0\leq\gamma<1$.*

(iii) *If $f\in \mathrm{C}(J,\mathbb{R})$, then $I_{0^+}^{\alpha;\psi}I_{0^+}^{\beta;\psi}f(t) = I_{0^+}^{\alpha+\beta;\psi}f(t),\ t\in J$.*

Lemma 2.4 ([20]). *Let $0<k<1,\ 0\leq\epsilon\leq 1$ and $0\leq\delta<1$. If $f\in\mathrm{L}^1(J,\mathbb{R})$ and $D_{0^+}^{\epsilon(1-k)}f$ is well-defined as an element of $\mathrm{L}^1(J,\mathbb{R})$, then*

$$D_{0^+}^{k,\epsilon;\psi}I_{0^+}^{k,\psi}f(t) = I_{0^+}^{\epsilon(1-k),\psi}D_{0^+}^{\epsilon(1-k),\epsilon;\psi}f(t).$$

Moreover, if $f\in C^1(J,\mathbb{R})$

$$D_{0^+}^{k,\epsilon;\psi}I_{0^+}^{k,\psi}h(t) = h(t).$$

For $0<k<1,\ 0\leq\epsilon\leq 1$ and $\delta = k+\epsilon(1-k)$, we introduce the weighted spaces

$$\mathrm{C}_{1-\delta;\psi}^{k,\epsilon}(J,\mathbb{R}) = \{w\in\mathrm{C}_{1-\delta;\psi}|D_{0^+}^{k,\epsilon;\psi}w\in\mathrm{C}_{1-\delta;\psi}(J,\mathbb{R})\}$$

and

$$\mathrm{C}_{1-\delta;\psi}^{\delta}(J,\mathbb{R}) = \{w\in\mathrm{C}_{1-\delta;\psi}|D_{0^+}^{\delta;\psi}w\in\mathrm{C}_{1-\delta;\psi}(J,\mathbb{R})\}.$$

Since $D_{0^+}^{k,\epsilon;\psi}w = I_{0^+}^{\epsilon(1-k),\psi}D_{0^+}^{\delta;\psi}w$, it is obvious that $\mathrm{C}_{1-\delta;\psi}^{\delta}(J,\mathbb{R})\subset\mathrm{C}_{1-\delta;\psi}^{k,\epsilon}(J,\mathbb{R})$.

Lemma 2.5 ([1]). *Let* $0<k<1,\ 0\leq\epsilon\leq 1$ *and* $\delta=k+\epsilon(1-k)$. *If* $f\in \mathrm{C}^{\delta}_{1-\delta;\psi}(J,\mathbb{R})$, *then*

$$I^{\delta;\psi}_{0^+}D^{\delta;\psi}_{0^+}f(t)=I^{k;\psi}_{0^+}D^{k,\epsilon;\psi}_{0^+}f(t) \tag{2.2}$$

and

$$D^{\delta;\psi}_{0^+}I^{k;\psi}_{0^+}f(t)=D^{\epsilon(1-k);\psi}_{0^+}f(t). \tag{2.3}$$

Lemma 2.6 ([11]). *Let* $k>0$, *and* $\delta>0$, *then* ψ*-fractional integral and derivative of a power function are given by*

$$I^{k;\psi}_{0^+}(\psi(t)-\psi(0))^{\delta-1}=\frac{\Gamma(\delta)}{\Gamma(\delta+k)}(\psi(t)-\psi(0))^{k+\delta-1}$$

and

$$D^{k;\psi}_{0^+}(\psi(t)-\psi(0))^{k-1}=0,\ \ 0<k<1.$$

Lemma 2.7 ([19]). *Let* $k>0$ *and* $0\leq\delta<1$. *Then* $I^{k;\psi}_{0^+}(.)$ *is bounded from* $\mathrm{C}_{1-\delta;\psi}(J,\mathbb{R})$ *into* $\mathrm{C}_{1-\delta;\psi}(J,\mathbb{R})$. *In particular, if* $\delta\leq k$, *then* $I^{k;\psi}_{0^+}(.)$ *is bounded from* $\mathrm{C}_{1-\delta;\psi}(J,\mathbb{R})$ *into* $\mathrm{C}(J,\mathbb{R})$.

Lemma 2.8 ([20]). *Let* $k>0$ *and* $0\leq\delta<1$, *and* $\mathrm{C}_{1-\delta;\psi}([a,b],\mathbb{R})$. *If* $k>\delta$, *then* $I^{k;\psi}_{a^+}f\in C([a,b],\mathbb{R})$ *and*

$$I^{k;\psi}_{a^+}f(a)=\lim_{t\to a^+}I^{k;\psi}_{a^+}f(t)=0.$$

Theorem 2.9 ([20]). *Let* $0<k<1$ *and* $0\leq\epsilon\leq 1$. *If* $\mathrm{C}_{1-\delta,\psi}(J,\mathbb{R})$, *then*

$$I^{k;\psi}_{0^+}D^{k,\epsilon;\psi}_{0^+}f(t)=f(t)-\frac{I^{(1-\epsilon)(1-k),\psi}_{0^+}f(0)}{\Gamma(k+\epsilon(1-k))}[\psi(t)-\psi(0)]^{k+\epsilon(1-k)-1}.$$

Moreover, if $\delta=k+\epsilon(1-k),\ f\in\mathrm{C}^{\delta}_{1-\delta;\psi}(J,\mathbb{R})$ *and* $I^{(1-\delta);\psi}_{0^+}f\in\mathrm{C}^{1}_{1-\delta;\psi}(J,\mathbb{R})$, *then*

$$I^{\delta;\psi}_{0^+}D^{\delta;\psi}_{0^+}f(t)=f(t)-\frac{I^{(1-\delta);\psi}_{0^+}f(0)}{\Gamma(\delta)}[\psi(t)-\psi(0)]^{\delta-1}.$$

Definition 2.10 ([5]). For any bounded subset Z of a metric space (X,d), the Kuratowski [12] measure of noncompactness is defined as:

$\omega(\mathrm{Z})=\inf\{\epsilon>0:\mathrm{Z}=\bigcup_{i=1}^{n}\mathrm{Z}_i,\ \mathrm{diam}(\mathrm{Z}_i)\leq\epsilon,\ 1\leq i\leq n<\infty\}$, where $\mathrm{diam}(\mathrm{Z}_i)$ represents the diameter of set $\mathrm{Z}_i\subset X$.

Definition 2.11. Let Z be the bounded subset of a metric space (X,d), the Hausdorff measure of non-compactness μ is defined by

$$\mu(Z)=inf\{\theta>0: Z\ can\ be\ covered\ by\ a\ finite\ number\ of\ balls\ with\ radii\ \theta\}.$$

There are only in a few Banach spaces, which able to express Hausdorff's measure of noncompactness with the help of these definitions. So, Banaś and Goebel introduced the definition of mnc in axiomatic way.

Definition 2.12 ([5]). Let E be any Banach space and $\mathfrak{M}_{\mathcal{E}}$ denotes the set of all bounded subsets of E.
A mapping $\varsigma:\mathfrak{M}_{\mathcal{E}}\to\mathbb{R}_+$ *is called measure of noncompactness* (**mnc**) *in* E *if the following conditions hold:*

(A_1) *A non-empty family* $\ker\varsigma := \{\mathcal{B} \in \mathfrak{M}_{\mathcal{E}} : \varsigma(\mathcal{B}) = 0\} \subseteq \mathfrak{N}_{\mathcal{E}}$;
(A_2) $\mathcal{B}_1 \subset \mathcal{B}_2 \implies \varsigma(\mathcal{B}_1) \leq \varsigma(\mathcal{B}_2)$;
(A_3) $\varsigma(\overline{\mathcal{B}}) = \varsigma(\mathcal{B})$;
(A_4) $\varsigma(Conv\mathcal{B}) = \varsigma(\mathcal{B})$.

Definition 2.13. Suppose P^+ is the positive cone of an ordered Banach space $(P, \leq)$. Let ψ be the function defined on the set of all bounded subsets of the Banach space E with values in P^+ is known as mnc on E iff $\psi(c\bar{o}nv(\Omega)) = \psi(\Omega)$ for every bounded subset $\Omega \subset E$, where $conv(\Omega)$ denoted the closed convex hull of Ω.

Lemma 2.14 ([5])**.** *Suppose E is a Banach space and $G_1, G_2 \subseteq E$ are bounded. Then, the following properties satisfy.*

(i) *G_1 is precompact iff $\varsigma(G_1) = 0$;*
(ii) *$\varsigma(G_1) = \varsigma(\bar{G_1}) = \varsigma(conv(G_1))$, where $conv(G_1)$ and $\bar{G_1}$ denote the convex hull and closure of G_1,respectively;*
(iii) *If $G_1 \subseteq G_2$, then $\varsigma(G_1) \leq \varsigma(G_2)$;*
(iv) *$\varsigma(G_1+G_2) \leq \varsigma(G_1)+\varsigma(G_2)$, such that $G_1+G_2 = \{b_1+b_2 : b_1 \in G_1, b_2 \in G_2\}$;*
(v) *$\varsigma(G_1 \cup G_2) \leq \max\{\varsigma(G_1), \varsigma(G_2)\}$;*
(vi) *$\varsigma(\lambda G_1) =\mid \lambda \mid \varsigma(G_1)$ for every $\lambda \in \mathbb{R}$, where E is a real Banach space.*
(vii) *If the operator $S : D(S) \subseteq E \to E_1$ is Lipschitz continuous and η is the constant then we know $\mu(S(G_1)) \leq \eta\varsigma(G_1)$ for any bounded subset $G_1 \subset D(S)$, where μ represent the Hausdorff mnc in the Banach space E_1.*

Lemma 2.15 ([5])**.** *Assume that S is any equicontinuous and bounded subset of $C(J, E)$, then function $z \to \varsigma(S(z))$ is continuous on $[0, d]$. $\varsigma(S) = \sup\{\varsigma(S(z))\}$ where $S(z) = \{w(z) : w \in S, z \in [0, d]\}$.*

Lemma 2.16 ([15])**.** *Let $\mathfrak{D}$ be a closed convex subset of a Banach space E and $0 \in \mathfrak{D}$. Assume that $S : \mathfrak{D} \to \mathfrak{D}$ be a continuous map which satisfies Mönch's condition, i.e., if $\mathfrak{D}_1 \subset \mathfrak{D}$ is countable and $\mathfrak{D}_1 \subset conv(0 \cup S(\mathfrak{D}_1))$ implies $\mathfrak{D}_1$ is compact. Then S has a fixed point in $\mathfrak{D}$.*

Definition 2.17 ([17])**.** Let $0 < v < 1, 0 < \mu < \frac{\pi}{2}$, we define Θ_μ^{-v} be the family of closed linear operators, the sector $S_\mu = \{z \in \mathbb{C}\backslash\{0\}\ with\ |argz| \leq \mu\}$ and $A : D(A) \subset E \to E$ such that

(i) $\sigma(A) \subseteq S_\mu$;
(ii) There exists a constant C_μ,

$$\|(z - A)^{-1}\| \leq C_\mu |z|^{-v},\ for\ all\ z \notin S_\mu,$$

then $A \in \Theta_\mu^{-v}$ is called almost sectorial operator on E.

Lemma 2.18 ([17])**.** *Consider $0 < v < 1, 0 < \mu < \frac{\pi}{2}$ and $A \in \Theta_\mu^{-v}$. Then*

(i) *$\mathcal{F}(z_1 + z_2) = \mathcal{F}(z_1) + \mathcal{F}(z_2)$, for all $z_1, z_2 \in S^0_{\frac{\pi}{2}-\mu}$;*
(ii) *There exists a constant $\mathfrak{L} > 0$ such that $\|\mathcal{F}(z)\|_C \leq \mathfrak{L}z^{v-1}$, for all $z > 0$;*
(iii) *The range $R(\mathcal{F}(z))$ of $\mathcal{F}(z)$, $z \in S^0_{\frac{\pi}{2}-\mu}$ which belongs $D(A^\infty)$. Especially, $R(\mathcal{F}(z)) \subset D(A^\theta)$ for all $\theta \in \mathbb{C}$ with $Re(\theta) > 0$,*

$$A^\theta \mathcal{F}(z) \leq \frac{1}{2\pi i}\int_{\Gamma_E} z^\theta e^{-zs} R(z; A) y ds,\ for\ every\ y \in E.$$

(iv) *If* $\theta > 1 - v$, *then* $D(A^{\theta}) \subset \Sigma_{\mathcal{F}} = \{y \in E : \lim_{z\to 0} \mathcal{F}(z)y = y\}$;

(v) $R(z; A) = \int_0^{\infty} e^{-k'z}\mathcal{F}(z)dz$, *for every* $k' \in \mathbb{C}$ *with* $Re(k') > 0$.

Definition 2.19 ([17])**.** Define the wright function $M_k(\zeta)$ by

$$M_k(\zeta) = \sum_{n=1}^{\infty} \frac{(-\zeta)^{n-1}}{(n-1)!\,\Gamma(1-kn)}$$

which holds the following properties such that, $-1 < \lambda < \infty, r > 0$:

(a) $M_k(\zeta) \geq 0, z > 0$;

(b) $\int_0^{\infty} \zeta^{\lambda} M_k(\zeta)d\zeta = \frac{\Gamma(1+\lambda)}{\Gamma(1+k\lambda)}$, for $\lambda \geq 0$;

(c) $\int_0^{\infty} \frac{k}{\zeta^{(k+1)}} e^{-r\zeta} M_k(\frac{1}{\zeta^k})d\zeta = e^{-r^k}$.

3. An auxillary result

We represent $C(J, \mathbb{R})$ the Banach space of all continuous functions $w : J \to \mathbb{R}$ where $J = [0, d]$. Consider the set $\Pi = \{w \in C : \lim_{z\to 0} z^{-1+\epsilon-k\epsilon+k} w(z)$ exists and finite$\}$ which is a Banach space w.r.t the norm defined as $\| w \|_{\Pi} = \sup_{z\in J}\{z^{-1+\epsilon-k\epsilon+k} \| w(z) \|\}$.

Theorem 3.1. *Let* $0 < k < 1, 0 \leq \epsilon \leq 1$ *and* $\delta = k + \epsilon - k\epsilon$. *Assume that* $h(\cdot, w(\cdot), \chi(\cdot)) \in C_{1-\delta;\psi}[0, d]$ *for any* $w \in C_{1-\delta;\psi}[0, d]$. *If* $w \in C^{\delta}_{1-\delta;\psi}[0, d]$, *then* w *satisfies the problem* (1.1)-(1.2) *if and only if* w *satisfies the integral equation*

$$\begin{aligned} w(z) = \frac{(\psi(z) - \psi(0))^{\delta-1}}{B_1} \Big[& w_0 + \Sigma \frac{c_j}{\Gamma(k)} \int_0^{\tau_j} \wedge^k_{\psi}(\tau_j, s)Aw(s)ds \\ & + \Sigma \frac{1}{\Gamma(k)} c_j \int_0^{\tau_j} \wedge^k_{\psi}(\tau_j, s)F(s)ds + \Sigma \frac{c_j}{\Gamma(k)} \int_0^{\tau_j} \wedge^k_{\psi}(\tau_j, s)Bv(s)ds \Big] \\ & + \frac{1}{\Gamma(k)} \Big(\int_0^{z} \wedge^k_{\psi}(z, s)(Aw(s) + F(s) + Bv(s))ds \Big) \end{aligned}$$

where $\wedge^k_{\psi}(\tau_j, s) = \psi'(s)(\psi(\tau_j) - \psi(s))^{k-1}$; $F(z) = h(z, w(z), \chi w(z))$ *and* $0 \neq B_1 = [\Gamma(\delta) - \Sigma c_j(\psi(\tau_j) - \psi(0))^{\delta-1}]$.

Proof.

$$D^{k,\epsilon,\psi}_{0^+}[w(z)] = Aw(z) + h(z, w(z), \chi(z)) + Bv(z),\ z \in J = (0, d], \tag{3.1}$$

$$I^{1-\delta;\psi}_{0^+} w(z)|_{z=0} = w_0 + \Sigma^m_{j=1} c_j w(\tau_j),\ \tau_j \in (0, d), \tag{3.2}$$

where $\delta = k + \epsilon - k\epsilon$.

By using Theorem 2.9, we have

$$I^{\delta;\psi}_{0^+} D^{\delta,\epsilon,\psi}_{0^+}[w(z)] = w(z) - \frac{I^{1-\delta;\psi}_{0^+} w(0)}{\Gamma(\delta)} (\psi(z) - \psi(0))^{\delta-1}. \tag{3.3}$$

Now by using Lemma 2.5 and applying $I^{\delta,\psi}_{0^+}(.)$ to equation (3.1) both sides

$$I^{\delta;\psi}_{0^+} D^{\delta,\epsilon,\psi}_{0^+}[w(z)] = I^{k;\psi}_{0^+} D^{k,\epsilon,\psi}_{0^+}[w(z)] = I^{k;\psi}_{0^+} Aw(z) + I^{k;\psi}_{0^+} F(z) + I^{k;\psi}_{0^+} Bv(z). \tag{3.4}$$

Comparing (3.3) and (3.4), we get

$$w(z)=\frac{I_{0^+}^{1-\delta;\psi}w(0)}{\Gamma(\delta)}(\psi(z)-\psi(0))^{\delta-1}+I_{0^+}^{k;\psi}Aw(z)+I_{0^+}^{k;\psi}F(z)+I_{0^+}^{k;\psi}Bv(z). \tag{3.5}$$

Substitute $z=\tau_j$ and multiply c_j to both sides

$$\begin{aligned}\Sigma c_j w(\tau_j)&=\Sigma\frac{c_j I_{0^+}^{1-\delta;\psi}w(0)}{\Gamma(\delta)}(\psi(\tau_j)-\psi(0))^{\delta-1}\\&\quad+\Sigma c_j I_{0^+}^{k;\psi}Aw(\tau_j)+\Sigma c_j I_{0^+}^{k;\psi}F(\tau_j)+\Sigma c_j I_{0^+}^{k;\psi}Bv(\tau_j)\end{aligned}$$

which implies

$$\begin{aligned}\Gamma(\delta)\Sigma c_j w(\tau_j)&=\Sigma c_j I_{0^+}^{1-\delta;\psi}w(0)(\psi(\tau_j)-\psi(0))^{\delta-1}\\&\quad+\Gamma(\delta)\Sigma c_j I_{0^+}^{k;\psi}Aw(\tau_j)+\Gamma(\delta)\Sigma c_j I_{0^+}^{k;\psi}F(\tau_j)+\Gamma(\delta)\Sigma c_j I_{0^+}^{k;\psi}Bv(\tau_j).\end{aligned}$$

We can write above expression as

$$\begin{aligned}\Gamma(\delta)(I_{0^+}^{1-\delta;\psi}w(0)-w_0)&=\Sigma c_j I_{0^+}^{1-\delta;\psi}w(0)(\psi(\tau_j)-\psi(0))^{\delta-1}\\&\quad+\Gamma(\delta)\Sigma c_j I_{0^+}^{k;\psi}Aw(\tau_j)+\Gamma(\delta)\Sigma c_j I_{0^+}^{k;\psi}F(\tau_j)\\&\quad+\Gamma(\delta)\Sigma c_j I_{0^+}^{k;\psi}Bv(\tau_j),\end{aligned}$$

$$\begin{aligned}&I_{0^+}^{1-\delta;\psi}w(0)[\Gamma(\delta)-\Sigma c_j(\psi(\tau_j)-\psi(0))^{\delta-1}]\\&\quad=\Gamma(\delta)w_0+\Gamma(\delta)\Sigma c_j I_{0^+}^{k;\psi}Aw(\tau_j)+\Gamma(\delta)\Sigma c_j I_{0^+}^{k;\psi}F(\tau_j)+\Gamma(\delta)\Sigma c_j I_{0^+}^{k;\psi}Bv(\tau_j),\end{aligned}$$

$$I_{0^+}^{1-\delta;\psi}w(0)=\frac{\Gamma(\delta)}{B_1}\Big[w_0+\Sigma c_j I_{0^+}^{k;\psi}Aw(\tau_j)+\Sigma c_j I_{0^+}^{k;\psi}F(\tau_j)+\Sigma c_j I_{0^+}^{k;\psi}Bv(\tau_j)\Big].$$

Put this term in equation (3.5)

$$\begin{aligned}w(z)&=\frac{(\psi(z)-\psi(0))^{\delta-1}}{B_1}\Big[w_0+\Sigma c_j I_{0^+}^{k;\psi}Aw(\tau_j)+\Sigma c_j I_{0^+}^{k;\psi}F(\tau_j)+\Sigma c_j I_{0^+}^{k;\psi}Bv(\tau_j)\Big]\\&\quad+I_{0^+}^{k;\psi}Aw(z)+I_{0^+}^{k;\psi}F(z)+I_{0^+}^{k;\psi}Bv(z),\end{aligned}$$

$$\begin{aligned}w(z)&=\frac{(\psi(z)-\psi(0))^{\delta-1}}{B_1}\Big[w_0+\Sigma\frac{c_j}{\Gamma(k)}\int_0^{\tau_j}\wedge_\psi^k(\tau_j,s)Aw(s)ds\\&\quad+\Sigma\frac{1}{\Gamma(k)}c_j\int_0^{\tau_j}\wedge_\psi^k(\tau_j,s)F(s)ds+\Sigma\frac{c_j}{\Gamma(k)}\int_0^{\tau_j}\wedge_\psi^k(\tau_j,s)Bv(s)ds\Big]\\&\quad+\frac{1}{\Gamma(k)}\Big(\int_0^z\wedge_\psi^k(z,s)(Aw(s)+F(s)+Bv(s))ds\Big),\end{aligned} \tag{3.6}$$

Conversly, we assume that $w\in C_{1-\delta,\psi}^{\delta}[0,d]$ satisfies the above integral equation. We prove $w(z)$ also satisfies (1.1)-(1.2) .

Multiply c_k and put $z=\tau_k$ to the above equation

$$\begin{aligned}\Sigma c_k w(\tau_k)&=\frac{\Sigma c_j(\psi(\tau_j)-\psi(0))^{\delta-1}}{B_1}\Big[w_0+\Sigma\frac{c_j}{\Gamma(k)}\int_0^{\tau_j}\wedge_\psi^k(\tau_j,s)\big(Aw(s)+F(s)\\&\quad+Bv(s)\big)ds\Big]+\frac{\Sigma c_j}{\Gamma(k)}\Big(\int_0^{\tau_j}\wedge_\psi^k(\tau_j,s)(Aw(s)+F(s)+Bv(s))ds\Big),\end{aligned}$$

$$\Sigma c_k w(\tau_k) = \Big[1 + \frac{\Sigma c_j(\psi(\tau_j) - \psi(0))^{\delta-1}}{B_1}\Big]\Sigma\frac{c_j}{\Gamma(k)}\int_0^{\tau_j} \wedge_\psi^k(\tau_j, s)\big(Aw(s) + F(s)$$
$$+ Bv(s)\big)ds + \frac{(\Sigma c_j\psi(\tau_j) - \psi(0))^{\delta-1}}{B_1}w_0$$

which implies

$$\Sigma c_k w(\tau_k) = \frac{\Gamma(\delta)}{B_1}\Big[\Sigma\frac{c_j}{\Gamma(k)}\int_0^{\tau_j} \wedge_\psi^k(\tau_j, s)\big(Aw(s) + F(s) + Bv(s)\big)ds\Big]$$
$$+ \left(\frac{\Gamma(\delta)}{B_1} - 1\right)w_0,$$
$$\Sigma c_k w(\tau_k) + w_0 = \frac{\Gamma(\delta)}{B_1}\Big[\Sigma\frac{c_j}{\Gamma(k)}\int_0^{\tau_j} \wedge_\psi^k(\tau_j, s)\big(Aw(s) + F(s) + Bv(s)\big)ds + w_0\Big]$$
$$= I_{0^+}^{1-\delta;\psi} w(0).$$

Hence we obtain equation (3.2), with this we validate the problem. □

4. Existence and controllability result

In this section, we establish controllability result for given system.

Definition 4.1. The mild solution of the equation (1.1)-(1.2) is a function, that satisfies

$$w(z) = P_{k,\epsilon,\psi}(z)[w_0 + \Sigma_{j=1}^m c_j w(\tau_j)] + \int_0^z \mathcal{K}_{k,\psi}(z, u)\psi'(u)h(u, w(u), \chi w(s))du$$
$$+ \int_0^z \mathcal{K}_{k,\psi}(z, u)\psi'(u)Bv(u)du$$

where $P_{k,\epsilon,\psi}(z) = I_0^{\epsilon(1-k);\psi}\mathcal{K}_{k,\psi}(z)$, $Q_{k,\psi}(z) = \int_0^\infty k\zeta M_k(\zeta)\mathcal{F}((\psi(z) - \psi(u))^k\zeta)d(\zeta)$, $\mathcal{K}_{k,\psi}(z, u) = (\psi(z) - \psi(u))^{k-1}Q_{k,\psi}(z)$, i.e.

$$w(z) = P_{k,\epsilon,\psi}(z)[w_0 + \Sigma_{j=1}^m c_j w(\tau_j)]$$
$$+ \int_0^z (\psi(z) - \psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z)h(u, w(u), \chi w(s))du$$
$$+ \int_0^z (\psi(z) - \psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z)Bv(u)du.$$

Lemma 4.2. *The system* (1.1) *and* (1.2) *is said to be controllable in* J *if for every continuous initial value function, there exists* $v \in L^2(J, \mathbb{R})$ *and the mild solution* $w(z)$ *of (1.1) with (1.2) satisfies* $w(d) = w_1$.

To prove our main result, we require the following hypotheses :

(H_1)
- (a) *Let* A *be the almost sectorial operator which generates analytic semigroup* $\{\mathcal{F}(z), z > 0\}$ *in* $\mathbb{R}$ *such that* $\| \mathcal{F}(z) \|\leq L'$ *where* $L' \geq 0$ *is a constant.*
- (b) *For any fixed* $z > 0$, *the linear operator* $P_{k,\epsilon,\psi}(z)$ *is such that,* $\|P_{k,\epsilon,\psi}(z)\| \leq K_2$.

(H_2) *The function* $h : J\times\mathcal{C}\times\mathbb{R} \to \mathbb{R}$ *such that* $h(., w(.), (\chi w)(.)) \in C_{1-\delta,\psi}[J, \mathcal{C}, \mathbb{R}]$ *satisfies Caratheodory condition for all* $w(z) \in \mathcal{C}$ *:*

(a) *$h(.,w(.),(\chi w)(.))$ is measurable for all $h(.,w(.),(\chi w)(.))$ and is continuous for a.e. $z \in J$,*

(b) *The function $h : C_{1-\delta,\psi}[J,\mathcal{C},\mathbb{R}] \to \mathbb{R}$ satisfies Lipschitz condition :*

$$\|h(z_1,w(z_1),\chi(z_1) - h(z_2,w(z_2),\chi(z_2))\| \leq L_h\|z_1 - z_2\|.$$

(c) *There exists a constant $K_0 > 0$ such that $\| h(.,w(.),(\chi w)(.)) \| \leq K_0$ for every $z \in J$.*

(H_3) *The linear operator $W : L^2(J,\mathbb{R}) \to E$ is defined by*

$$W(v) = \int_0^d (\psi(d) - \psi(u))^{k-1}\psi'(u)Q_{k,\psi}(d,u)Bv(u)du.$$

(a) *W has an invertible operator W^{-1} which takes value in $L^2(J,\mathbb{R})/KerW$ and there exists K_3 such that $\|W^{-1}\| \leq K_3$.*

(b) *There exists a constant K_W such that for any bounded set $\mathfrak{D} \subset E$, $\varsigma((W^{-1}\mathfrak{D})(z)) \leq K_W(z)\varsigma(\mathfrak{D})$.*

(H_4) *There is a constant $\hat{K} > 0$ satisfying $\varsigma(\mathfrak{D}) \leq \hat{K}\varsigma(\mathfrak{D})$ for every bounded subset $\mathfrak{D} \subset E$ and following estimation holds true:*

$$\hat{K} = \Big[K_1 + K_0 + K_2 + K_1K_3K_W(w_d - (K_2 + K_0))\Big] < 1.$$

Theorem 4.3. *Suppose assumptions $(H_1)-(H_4)$ hold, then ψ-Hilfer fractional differential equation* (1.1)-(1.2) *has a solution with $\hat{K} = (K_1+K_0+K_2+K_1K_3K_W(w_d-(K_2+K_0))) < 1$.*

Proof. Now we consider the operator $S : \Pi \to \Pi$ which is well defined and the fixed point of this operator is the solution of our problem.

The operator $S : \Pi \to \Pi$ is defined as

$$Sw(z) = P_{k,\epsilon,\psi}(z)[w_0 + \Sigma_{j=1}^m c_j w(\tau_j)] + \int_0^z \mathcal{K}_{k,\psi}(z,u)\psi'(u)h(u,w(u),\chi w(s))du + \int_0^z \mathcal{K}_{k,\psi}(z,u)\psi'(u)Bv(u)du,$$

where

$$v(z) = W^{-1}\big[w(d) - P_{k,\epsilon,\psi}(d)[w_0 + \Sigma_{j=1}^m c_j w(\tau_j)] - \int_0^d \mathcal{K}_{k,\psi}(d,u)\psi'(u)h(u,w(u),\chi w(s))du\big].$$

Define a bounded, closed and convex set $B_\epsilon = \{w \in \Pi : \|w\| \leq \epsilon\}$.

By using Mönch fixed point theorem, we prove the existence of the solution in following steps:

Step 1 : The operator S maps the set B_ϵ into itself. i.e. $SB_\epsilon \subset B_\epsilon$.

$$S(w(z)) = P_{k,\epsilon,\psi}(z)[w_0 + \Sigma_{j=1}^m c_j w(\tau_j)] + \int_0^z \mathcal{K}_{k,\psi}(z,u)\psi'(u)h(u,w(u),\chi w(s))du + \int_0^z \mathcal{K}_{k,\psi}(z,u)\psi'(u)Bv(u)du,$$

$$\|Sw(z)\| \leq \Big\|z^{-1+\epsilon-k\epsilon+k}\Big[P_{k,\epsilon,\psi}(z)[w_0 + \Sigma_{k=1}^m c_k w(\tau_k)]$$

$$
\begin{aligned}
&+\int_0^z \mathcal{K}_{k,\psi}(z,u)\psi'(u)h(u,w(u),\chi w(s))du \\
&+\int_0^z \mathcal{K}_{k,\psi}(z,u)\psi'(u)Bv(u)du\Big]\Big\| \\
\leq\ & \|d^{-1+\epsilon-k\epsilon+k}\|\Big(\|P_{k,\epsilon,\psi}(z)\|\|w_0+\Sigma_{k=1}^{m}c_k w(\tau_k)\| \\
&+\Big\|\int_0^z \mathcal{K}_{k,\psi}(z,u)\psi'(u)h(u,w(u),\chi w(s))du\Big\| \\
&+\Big\|\int_0^z \|\mathcal{K}_{k,\psi}(z,u)\psi'(u)\|\|B\|\Big\|W^{-1}\Big\|\Big[w_d-\|P_{k,\epsilon,\psi}(d)[w_0+\Sigma_{j=1}^{m}c_j w(\tau_j)]\| \\
&-\Big\|\int_0^d \mathcal{K}_{k,\psi}(d,u)\psi'(u)h(u,w(u),\chi w(s))du\Big\|\Big]\Big) \\
\leq\ & \|d^{\delta-1}\|\ \big(K_1+K_0+K_2+K_1K_3(w_d-(K_2+K_0))\big) \\
\leq\ & \epsilon\hat{K}.
\end{aligned}
$$

It follows that $\|Sw\|_\Pi<\epsilon$. Thus $SB_\epsilon\subset B_\epsilon$.

Step 2 : S is continuous on B_ϵ.

Let $\{w_n\}_{n=1}^{\infty}$ be a sequence such that $w_n\to w$ in B_ϵ as $n\to\infty$ then for each $z\in J$, we have

$$
\begin{aligned}
\|(Sw_n(z)-Sw(z))\| &\leq \|z^{-1+\epsilon-k\epsilon+k}\|\ \Big[\Big\|\int_0^z \mathcal{K}_{k,\psi}(z,u)\psi'(u)(h(u,w_n(u),\chi w_n(s)) \\
&\qquad -h(u,w(u),\chi w(s)))du\Big\|\Big] \\
&\leq \|d^{-1+\epsilon-k\epsilon+k}\|\ \big(\|hw_n-hw\|\big) \\
&\longrightarrow 0\ as\ n\to\infty.
\end{aligned}
$$

where $hw_n=h(u,w_n(u),\chi w_n(s))$ and $hw=h(u,w(u),\chi w(s))$.

By Lebesgue convergence theorem, we conclude that $\|Sw_n-Sw\|\to 0$ as $n\to\infty$. Hence operator ς is continuous on B_ϵ.

Step 3 : S is equicontinuous on B_ϵ.

For any $z_1,z_2\in J$ such that $0<z_1<z_2<d, w\in B_\epsilon$, we have

$$
\begin{aligned}
\|Sw(z_2)-Sw(z_1)\| \leq \Big\| & z_2^{1-\epsilon+k\epsilon-k}(P_{k,\epsilon,\psi}(z_2)[w_0+\Sigma c_k w(\tau_k)] \\
&+\int_0^{z_2}(\psi(z_2)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_2)h(u,w(u),\chi w(s))du \\
&+\int_0^{z_2}(\psi(z_2)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_2)Bv(u)du) \\
&-\Big[z_1^{1-\epsilon+k\epsilon-k}(P_{k,\epsilon,\psi}(z_1)[w_0+\Sigma c_k w(\tau_k)] \\
&+\int_0^{z_1}(\psi(z_2)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_1)h(u,w(u),\chi w(s))du
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{z_1}(\psi(z_1)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_1)Bv(u)du)\Big]\Big\|.\\
&\leq \|(z_2^{1-\epsilon+k\epsilon-k}P_{k,\epsilon,\psi}(z_2)-z_1^{1-\epsilon+k\epsilon-k}P_{k,\epsilon,\psi}(z_1))(w_0+\Sigma c_k w(\tau_k))\|\\
&\quad+\Big\|z_2^{1-\epsilon+k\epsilon-k}\int_0^{z_2}(\psi(z_2)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_2)h(u,w(u),\chi w(s))du\\
&\quad-z_1^{1-\epsilon+k\epsilon-k}\int_0^{z_1}(\psi(z_1)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_1)h(u,w(u),\chi w(s))du\Big\|\\
&\quad+\Big\|z_2^{1-\epsilon+k\epsilon-k}\int_0^{z_2}(\psi(z_2)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_2)Bv(u)du\\
&\quad-z_1^{1-\epsilon+k\epsilon-k}\int_0^{z_1}(\psi(z_1)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_1)Bv(u)du\Big\|\\
&\leq \Big\|z_2^{1-\epsilon+k\epsilon-k}P_{k,\epsilon,\psi}(z_2)-z_1^{1-\epsilon+k\epsilon-k}P_{k,\epsilon,\psi}(z_1)\Big\|(w_0+\Sigma c_k w(\tau_k))\\
&\quad+\Big\|z_2^{1-\epsilon+k\epsilon-k}\int_0^{z_1}(\psi(z_2)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_1)h(u,w(u),\chi w(s))du\\
&\quad-z_1^{1-\epsilon+k\epsilon-k}\int_0^{z_1}(\psi(z_1)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_1)h(u,w(u),\chi w(s))du\Big\|\\
&\quad+\Big\|z_2^{1-\epsilon+k\epsilon-k}\int_{z_1}^{z_2}(\psi(z_2)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_2)h(u,w(u),\chi w(s))du\Big\|\\
&\quad+\Big\|z_2^{1-\epsilon+k\epsilon-k}\int_0^{z_1}(\psi(z_2)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_1)Bv(u)du\\
&\quad-z_1^{1-\epsilon+k\epsilon-k}\int_0^{z_1}(\psi(z_1)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_1)Bv(u)du\Big\|\\
&\quad+\Big\|z_2^{1-\epsilon+k\epsilon-k}\int_{z_1}^{z_2}(\psi(z_2)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_2)Bv(u)du\Big\|\\
&\leq \Big\|z_2^{1-\epsilon+k\epsilon-k}P_{k,\epsilon,\psi}(z_2)-z_1^{1-\epsilon+k\epsilon-k}P_{k,\epsilon,\psi}(z_1)\|(w_0+\Sigma c_k w(\tau_k))\\
&\quad+d^{1-\epsilon+k\epsilon-kv}\Big\|\int_0^{z_1}((\psi(z_2)-\psi(u))^{k-1}-(\psi(z_1)-\psi(u))^{k-1})\\
&\times\psi'(u)Q_{k,\psi}(z_1)h(u,w(u),\chi w(s))du\|\\
&\quad+d^{1-\epsilon+k\epsilon-k}\|\int_{z_1}^{z_2}(\psi(z_2)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_2)h(u,w(u),\chi w(s))du\|\\
&\quad+d^{1-\epsilon+k\epsilon-k}\Big\|\int_0^{z_1}((\psi(z_2)-\psi(u))^{k-1}\\
&\qquad\qquad-(\psi(z_1)-\psi(u))^{k-1})\psi'(u)Q_{k,\psi}(z_1)Bv(u)du\Big\|\\
&\quad+d^{1-\epsilon+k\epsilon-k}\Big\|\int_{z_1}^{z_2}(\psi(z_2)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_2)Bv(u)du\Big\|.\\
&\leq \Sigma_{i=1}^5 I_i
\end{aligned}
$$

where

$$I_1=\|z_2^{1-\epsilon+k\epsilon-k}P_{k,\epsilon,\psi}(z_2)-z_1^{1-\epsilon+k\epsilon-k}P_{k,\epsilon,\psi}(z_1)\|(w_0+\Sigma c_k w(\tau_k)). \tag{4.1}$$

$$
\begin{aligned}
I_2 = d^{1-\epsilon+k\epsilon-k}\Big\| \int_0^{z_1} &((\psi(z_2)-\psi(u))^{k-1} \\
&- (\psi(z_1)-\psi(u))^{k-1})\psi'(u)Q_{k,\psi}(z_1)h(u,w(u),\chi w(s))du\Big\|.
\end{aligned}
\tag{4.2}
$$

$$
I_3 = d^{1-\epsilon+k\epsilon-k}\Big\| \int_{z_1}^{z_2} (\psi(z_2)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_2)h(u,w(u),\chi w(s))du\Big\|.
\tag{4.3}
$$

$$
\begin{aligned}
I_4 = d^{1-\epsilon+k\epsilon-k}\Big\| \int_0^{z_1} &((\psi(z_2)-\psi(u))^{k-1} \\
&- (\psi(z_1)-\psi(u))^{k-1})\psi'(u)Q_{k,\psi}(z_1)Bv(u)du\Big\|.
\end{aligned}
\tag{4.4}
$$

$$
I_5 = d^{1-\epsilon+k\epsilon-k}\Big\| \int_{z_1}^{z_2} (\psi(z_2)-\psi(u))^{k-1}\psi'(u)Q_{k,\psi}(z_2)Bv(u)du\Big\|.
\tag{4.5}
$$

By equicontinuity and absolute continuity of Lebesgue integral assumed in our hypothesis, we can see that the right-hand side of (4.1)-(4.5) tends to zero as $z_2 \to z_1$.

Step 4 : Now we show Mönch Condition.

Let $B_0 \subseteq B_\epsilon$ be a countable and $B_0 \subseteq conv(0 \cup \varsigma(B))$. We will show that $\varsigma(B_0) = 0$.

Suppose $\{x_n\}_{n=1}^{\infty} \subseteq S(B_0)$ is a countable set. Then there exists a set $\{w_n\}_{n=1}^{\infty}$ such that $x_n = (Sw_n)(z)$ for all $z \in J, n \geq 1$.

Using Lemmas 2.14 and 2.15, we have

$$
\begin{aligned}
\varsigma(\{x_n(z)\}_{n=1}^{\infty}) &= \varsigma(\{(Sw_n)(z)\}_{n=1}^{\infty}) \\
&= P_{k,\epsilon,\psi}(z)[w_0 + \Sigma_{j=1}^{m} c_j w(\tau_j)] \\
&\quad + \int_0^z \mathcal{K}_{k,,\psi}(z,u)\psi'(u)h(u,w_n(u),\chi w_n(s))du \\
&\quad + \int_0^z \mathcal{K}_{k,,\psi}(z,u)\psi'(u)Bv(u)du. \\
&= J_1 + J_2.
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &= P_{k,\epsilon,\psi}(z)[w_0 + \Sigma_{j=1}^{m} c_j w(\tau_j)] \\
&\leq \big(K_2\big)\varsigma(B_0). \\
J_2 &= \varsigma\Big(\int_0^z \mathcal{K}_{k,\psi}(z,u)\psi'(u)h(u,w_n(u),\chi w_n(s))du + \int_0^z \mathcal{K}_{k,\psi}(z,u)\psi'(u)Bv(u)du\Big) \\
&\leq (K_0 + K_1K_3(w_d - (K_2+K_0)))K_W\ \varsigma(B_0).
\end{aligned}
$$

$$
\begin{aligned}
J_1 + J_2 &\leq (K_1 + K_0 + K_2 + K_1K_WK_3(w_d - (K_2+K_0)))\|w\|\varsigma(B_0). \\
&\leq \hat{K}\ \varsigma(B_0), \quad \textit{where } \hat{K} = (K_1 + K_0 + K_2 + K_1K_3K_W(w_d - (K_2+K_0)))
\end{aligned}
$$

$$
\varsigma(B_0) \leq \hat{K}\varsigma(B_0).
$$

$$
\varsigma(B_0) = 0.
$$

So, by Lemma 2.16, S has a fixed point w in $\mathcal{B}_\epsilon$. Then w is a mild solution of (1.1)-(1.2) such that $w(d) = w_1$.Hence the system (1.1)-(1.2) is controllable on J. $\square$

5. An illustrative example

In this section, we demonstrate the applicability of the obtained results to the following fractional differential equation:

(5.1)
$$D_{0^+}^{\frac{1}{2},\frac{1}{3},\frac{t}{3}} w(z) = \frac{\partial^2}{\partial z^2} w(z) + \frac{1}{5}\Big(sin(w(z)) + \int_0^z e^{-\frac{1}{2}w(s)} ds\Big) + \rho v(z),\ z \in J = (0,\pi]$$

$$I_{0^+}^{1-\frac{2}{3};\frac{t}{3}} w(z)|_{z=0} = w_0 + \frac{2}{5} w\Big(\frac{2}{3}\Big). \tag{5.2}$$

where $D_{0^+}^{\frac{1}{2},\frac{1}{3},\frac{t}{3}}$ is a ψ-Hilfer fractional derivative operator and $I_{0^+}^{1-\frac{2}{3};\frac{t}{3}}$ is ψ-RL fractional integral operator and $\psi(t) = \frac{t}{3}$.

The operator A is defined as

$$Aw = \frac{\partial^2}{\partial z^2} w(t,z)$$

and

$$D(A) = \{w \in C^2[0,\pi] : w(t,0) = w(t,\pi) = 0\}.$$

Here A is the infinitesimal generator of analytic semigroup $\{\mathcal{F}(t)\}$ on $[0,\pi]$, $\mathcal{F}(t)$ is not a compact semigroup with $\varsigma(\mathcal{F}(t)D) \leq \varsigma(D)$ and there exists $L' \geq 0$ such that $\sup_{t\in(0,\pi]} \|\mathcal{F}(t)\| \leq L'$.

The bounded linear operator $B : L^2(J,\mathbb{R}) \to \mathbb{R}$ is defined as $B(v(z)) = \rho v(z)$, $\rho > 0$.

The above problem (5.1)-(5.2) can be written in abstract form as :

$$D_{0^+}^{\frac{1}{2},\frac{1}{3},\frac{t}{3}}[w(z)] = Aw(z) + h(z, w(z), \chi(z)) + Bv(z),\ z \in J = (0,\pi] \tag{5.3}$$

$$I_{0^+}^{1-\frac{2}{3};\frac{t}{3}} w(z)|_{z=0} = w_0 + \frac{2}{5} w\Big(\frac{2}{3}\Big). \tag{5.4}$$

Hence the existence and controllability of equation (5.1)-(5.2) follow from Theorem 4.3.

6. Conclusion

In this study, we used uniform boundedness, measure of noncompactness and Mönch fixed point theorem, to investigate the abstract fractional evolution control system is controllable in a Banach space. As we considered ψ-Hilfer fractional differential equation which represents general form of differential equations. It is well known that the compactness criterion for the operator can be decreased to equicontinuity. We proved controllabillity of system with generalized differential system along with non-local type conditions. The outcome of this study represents sufficient condition for controllability of the given system.

Acknowledgement

Authors are thankful to the learned refree for the fruitful comments and suggestions.

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Manuscript received May 14, 2024
revised Auguast 20, 2024

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