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SOME RESULTS IN b-FUZZY METRIC SPACE

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ABSTRACT. The Banach Principle offers us a general condition for a function to obtain a fixed point in a classical manner. Many researchers have provided a variety of fixed point results using different kinds of mappings and metric spaces in accordance with this well-known idea. The primary goal of this work is to present the connection between b-Fuzzy and non-fuzzy metric spaces. This study uses b-contractive fuzzy mappings to present some fixed point results as well as some theoretical findings in b-fuzzy metric space ($b\mathcal{FMS}$). The b-distance function that is produced between fuzzy points is the foundation for these findings.

1. INTRODUCTION

The idea of fuzzy sets was initially introduced by Zadeh [27] in 1965 that is most frequently used for uncertainty management. Chang [5] developed and explored fuzzy sets' topological features for the first time in 1968. Using the concept of fuzzy sets, Kramosil and Michalek [22] introduced $\mathcal{FMS}$ in 1975 with the involvement of some specific operations on membership function in a metric. Erceg [10] investigated $\mathcal{FMS}$ in the theory of fuzzy sets in 1979 and their interaction with statistical metric spaces. As a result, several definitions of this topic are provided, each employing a different methodology. A number of these definitions suffer from structural imprecision; for example, some define the distance function in terms of how the α-level sets are defined, while others treat it as a fuzzy map with a specific function of membership that can be defined using the widely accepted fuzzy set theory extension principle ([20, 23, 24]). In 1982, Zike [7] studied fuzzy points and defined specific metrics between the two fuzzy points. Butnariu [4] studied the fixed-point theorem in fuzzy map and explained some fixed-point theorems with algorithms for fuzzy map to estimate certain fixed points when fuzzy map have certain defined properties.

After that, the idea of Kramosil and Michalek [22] was modified by George and Veeramani [15] in 1994 with some altered properties. A study of fuzzy fixed-point theorem using the Hausdorff distance function [18] between the α-level sets corresponding to fuzzy sets was introduced and presented by Fadhel [11] in 1998. Since then, a great deal of writing has been done to apply this idea in topology and analysis through the study of fuzzy sets theory and applications. When studying fuzzy sets, fuzzy metric space $\mathcal{FMS}$ may be thought of as a straightforward function. As a result, many definitions of this subject are offered using different approaches. Many of these formulations have compositional problems since certain distance functions depend on the description of α-level sets. According to some, the distance function

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may be understood as a fuzzy mapping of a specific membership function that can be defined by using the well-known fuzzy set theory extension concept. George and Sapena [14] gave the fixed-point theorem according to the sense of George and Veeramani [15] for complete $\mathcal{FMS}$. The property of completeness and fixed-point theorems for some particular fuzzy metric spaces were also examined by Fadhel and Majeed ([12, 13]) in 2012. In 2015, Rana [24] *et al.* demonstrated the Banach fixed point theorem in fuzzy domain normed spaces of formal balls and investigated the contraction mapping principle in these spaces. In 2018, Patir *et al.* [23] developed and demonstrated a number of fixed-point theorems for self-mappings in fuzzy metric spaces with various contractive conditions. Bollenbacher and Hicks' generalized Carista fixed point theorem for p-orbitally complete fuzzy metric spaces was extended by Karayilan and Telci [19]. In 2021, Khan *et al.* [21] used a contraction map in complete metric space to demonstrate sequential characterization features of Lebesgue metric space. In recent years, several authors have taken alternative approaches and have taken into consideration different kinds of mappings and metric spaces (fuzzy or nonfuzzy)([1, 2, 8, 9, 16, 17, 25, 26]).

Motivated by the $\mathcal{FMS}$ version introduced by Zike Deng [7] using support set, α-level sets and fuzzy uniformity, the main goal of this study is to provide some new notions and findings in $b\mathcal{FMS}$ and introduce some analytical results pertaining to this framework. We also examine their connection to non-$\mathcal{FMS}$s. The distance function created between b-fuzzy points serves as the foundation for our fixed-point theorems.

2. Preliminaries

The idea of b-metric was introduced and used by Bakhtin [3] and Czerwik [6] as an extension of metric spaces by involving a b-metric constant (≥ 1) in triangle inequality of metric axioms.

Definition 2.1 ([6])**.** For a b-metric on a set $\mathcal{W}$ is a map $\mathfrak{B} : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ if the following three axioms hold true for all $w, h, l \in \mathcal{W}$:

(b_1) $\mathfrak{B}(w, h) = 0 \Leftrightarrow w = h$;

(b_2) $\mathfrak{B}(w, h) = \mathfrak{B}(h, w)$;

(b_3) $\mathfrak{B}(w, h) \leq b(\mathfrak{B}(w, l) + \mathfrak{B}(l, h))$ *where* $b \geq 1$.

The set $\mathcal{W}$ equipped with a metric $\mathfrak{B}$ defined on it, is called a b-metric space and is symbolized by $(\mathcal{W}, \mathfrak{B})$.

Definition 2.2 ([6])**.** Let $(\mathcal{W}, \mathfrak{B})$ be a b-metric space. For any sequence of points $\{w_j\} \in \mathcal{W} \neq \emptyset$, a point $w \in \mathcal{W}$ is said to be the limit of $\{w_j\}$ if for every set $\mathcal{W} \neq \emptyset$, $\varepsilon > 0$, $\exists\ j_\varepsilon \in \mathbb{N}$, for which $\mathfrak{B}(w_j, w) < \varepsilon$ for $j \geq j_\varepsilon$ and we assert that $\{w_j\}$ converges to w.

Definition 2.3 ([6])**.** A sequence $\{w_j\}$ in $\mathcal{W}$ is called Cauchy if for each $\varepsilon > 0$, $\exists$ some $j_\varepsilon \in \mathbb{N}$, for which $\mathfrak{B}(w_j, w_q) < \varepsilon$ for all $j, q \geq j_\varepsilon$.

Definition 2.4 ([6])**.** Every Cauchy sequence $\{w_j\} \in \mathcal{W}$ if converges in $\mathcal{W}$, will constitute a complete b-metric space $(\mathcal{W}, \mathfrak{B})$.

Definition 2.5 ([27])**.** An ordinary fuzzy set $\mathfrak{F}$ in a set $\mathcal{W}$ is characterized by a membership function $\theta_{\mathfrak{F}} : \mathcal{W} \to [0,1]$. The value of $\theta_{\mathfrak{F}}$ at w is represented by $\theta_{\mathfrak{F}}(w)$ represents the grade of membership of w in $\mathfrak{F}$ and is a point in $[0,1]$ and it is a fuzzy number.

Example 2.6 ([27])**.** Let $\mathcal{W} = \mathbb{R}$ and let D be a fuzzy set of numbers which are much greater than 1. Then characterization of D can be given by specifying $\theta_D(w)$ as a function of $\mathbb{R}$, $\theta_D(0) = 0$; $\theta_D(5) = 0.05$; $\theta_D(10) = 0.2$.

The following definitions has been applied to the acronyms and basic results used in this study: $\mathcal{W}$ is the non-empty universal set, $\mathcal{U} = [0,1]$, $\mathcal{U}_{\mathcal{W}}$ is the set of all fuzzy subsets of $\mathcal{W}$, $(\mathcal{W}, d_{\mathcal{W}})$ is employed to represent the nonfuzzy metric space, $(\mathcal{U}_{\mathcal{W}},\ d_{\mathcal{U}_{\mathcal{W}}})$ is the corresponding fuzzy metric space and and p_w^{μ} is the fuzzy point with $w \in \mathcal{W}$ as support and $\mu \in [0,1]$. The fuzzy point $p_w^{1-\mu}$ represents complement of p_w^{μ}. Given that p_w^{μ} is a fuzzy point as well, the set of all fuzzy points in $\mathcal{W}$ will be $\mathcal{U}_{\mathcal{W}}$.

For the purpose of creating $\mathcal{FMS}$, [7] developed and introduced the fuzzy distance function, which is a nonfuzzy function defined between two fuzzy points.

Definition 2.7 ([7])**.** Consider the metric space $(\mathcal{W}, d_{\mathcal{W}})$. For $u \in \mathcal{U}_{\mathcal{W}}$, the map $d_{\mathcal{U}_{\mathcal{W}}} : \{u\} \times \{u\} \to \mathbb{R}^+$ is a distance map between fuzzy points if for fuzzy points $p_w^{\mu}, p_y^{\nu}, p_l^{\omega} \in \mathcal{U}_{\mathcal{W}}$, where $w, h, l \in \mathcal{W}$ and $\mu, \nu, \omega \in [0,1]$; the map $d_{\mathcal{U}_{\mathcal{W}}}$ satisfies the following requirements:

(1) $d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) = 0\ iff\ w = h\ and\ \mu \leq \nu$;

(2) $d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) = d_{\mathcal{U}_{\mathcal{W}}}(p_h^{1-\nu}, p_w^{1-\mu})$;

(3) $d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) \leq d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_l^{\omega}) + d_{\mathcal{U}_{\mathcal{W}}}(p_l^{\omega}, p_h^{\nu})$;

(4) $d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) \leq j$;

where $j \geq 0$ implies that there exists $\hat{\mu} > \mu$ such that

$$d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\hat{\mu}}, p_h^{\nu}) < j.$$

The 2-tuple $(\mathcal{U}_{\mathcal{W}}, d_{\mathcal{U}_{\mathcal{W}}})$ is called a $\mathcal{FMS}$.

3. b-Fuzzy Metric Space

We now present the idea of $b\mathcal{FMS}$ and define the concept of contraction in this setup.

Definition 3.1. For $k \geq 1$, the map $d_{\mathcal{U}_{\mathcal{W}}} : \mathcal{U}_{\mathcal{W}} \times \mathcal{U}_{\mathcal{W}} \to \mathbb{R}^+$ is a b-distance function if for fuzzy points $p_w^{\mu}, p_y^{\nu}, p_l^{\omega} \in \mathcal{U}_{\mathcal{W}}$, where $w, h, l \in \mathcal{W}$ and $\mu, \nu, \omega \in [0,1]$; the map $d_{\mathcal{U}_{\mathcal{W}}}$ satisfies the following requirements:

(1) $d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) = 0\ iff\ w = h\ and\ \mu = \nu$;

(2) $d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) = d_{\mathcal{U}_{\mathcal{W}}}(p_h^{1-\nu}, p_w^{1-\mu})$;

(3) $d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) \leq k\{d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_l^{\omega}) + d_{\mathcal{U}_{\mathcal{W}}}(p_l^{\omega}, p_h^{\nu})\}$;

(4) $d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) \leq j$;

where $j \geq 0$ implies that there exists $\hat{\mu} > \mu$ such that

$$d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\hat{\mu}}, p_h^{\nu}) < j.$$

The pair $(\mathcal{U}_{\mathcal{W}}, d_{\mathcal{U}_{\mathcal{W}}})$ is called a $b\mathcal{FMS}$.

Example 3.2. Let $\mathcal{W} = \mathbb{Z}^+$ and $d_{\mathcal{U}_{\mathcal{W}}}$ be defined as

$$d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) = \begin{cases} \max\{\mu - \nu, 0\} & for\ \mu, \nu \in [0,1] \\ \frac{(w-h)^2}{2} & for\ w, h \in \mathcal{W}. \end{cases}$$

(1) If $\mu = \nu$ and $w = h \Rightarrow d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) = 0$ and vice-versa.

(2) It is evident that $d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) = d_{\mathcal{U}_{\mathcal{W}}}(p_h^{1-\nu}, p_w^{1-\mu})$.

(3) Let $d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) > 0$ where $\mu > \nu$ and $w > h$.

Case 1: If $d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_l^{\omega}) = 0$, then $\mu = \omega$ and $w = l$. Thus, $\omega = \mu > \nu$ which implies $\omega - \nu = \mu - \nu > 0$. Thus,

$$2d_{\mathcal{U}_{\mathcal{W}}}(p_l^{\omega}, p_h^{\nu}) = \begin{cases} 2\max\{\omega - \nu, 0\} = 2(\omega - \nu) = 2(\mu - \nu) > \mu - \nu & for\ \mu > \nu \\ \frac{(l-h)^2}{2} = \frac{(w-h)^2}{2} & for\ w, h, l \in \mathcal{W}. \end{cases}$$

which shows $2d_{\mathcal{U}_{\mathcal{W}}}(p_l^{\omega}, p_h^{\nu}) \geq d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu})$. The proof is same in the case that $d_{\mathcal{U}_{\mathcal{W}}}(p_l^{\omega}, p_h^{\nu}) = 0$.

Case 2: If both $d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_l^{\omega}) = d_{\mathcal{U}_{\mathcal{W}}}(p_l^{\omega}, p_h^{\nu}) = 0$, then $\mu = \omega = \nu$ which leads to contradiction.

Case 3: If both $d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_l^{\omega}) > 0$ and $d_{\mathcal{U}_{\mathcal{W}}}(p_l^{\omega}, p_h^{\nu}) > 0$, then $\mu > \omega > \nu$ and $w > l > h$. This implies

$$\mu - \nu = \mu - \omega + \omega - \nu \leq 2\{d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_l^{\omega}) + d_{\mathcal{U}_{\mathcal{W}}}(p_l^{\omega}, p_h^{\nu})\}.$$

Also,

$$\begin{aligned} d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) =& \frac{(w-h)^2}{2} \\ \leq& (w-l)^2 + (l-h)^2 \\ =& 2\{d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_l^{\omega}) + d_{\mathcal{U}_{\mathcal{W}}}(p_l^{\omega}, p_h^{\nu})\}. \end{aligned}$$

(4) $d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) \leq j \Rightarrow \exists\ \hat{\mu} > \mu$ such that

$$d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\hat{\mu}}, p_h^{\nu}) < j.$$

Now, $0 \leq d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) \Rightarrow \mu - \nu \leq d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) \leq j$ and $w - h \leq d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) \leq j$. Choose $\hat{\mu}$ for which $\mu < \hat{\mu} < \min\{2, j + \nu\}$. Then, $\hat{\mu} - \nu < j$ which implies $d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\hat{\mu}}, p_h^{\nu}) < j$.

The next remark contains the definition of the fuzzy distance function that will be utilized in the work that follows. Its proof is simple and relies on definition 3.1.

Remark 3.3. Presume a nonfuzzy metric space $(\mathcal{W}, d_{\mathcal{W}})$ and the related fuzzy metric space $(\mathcal{U}_{\mathcal{W}}, d_{\mathcal{U}_{\mathcal{W}}})$. The following definition may be used to determine the distance between two fuzzy points:

$$d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) = |\mu - \nu| + d_{\mathcal{W}}(w,h), \ \textit{for all } w, h \in \mathcal{W} \textit{ and } \mu, \nu \in [0,1]. \tag{3.1}$$

It should be noted that several fuzzy distance function examples might be obtained based on how the nonfuzzy distance function $d_{\mathcal{W}}(w,h)$ is defined in equation (3.1).

4. Convergence and Completeness

The concept of convergence and Cauchy sequence for the aforementioned $b\mathcal{FMS}$ is defined as follows:

Definition 4.1. A fuzzy point sequence $\{p_{w_j}^{\mu_j}\}, \ j \in \mathbb{N}$ is said to converge to a fuzzy point $p_w^{\mu} \in \mathcal{U}_{\mathcal{W}}$ in $\mathcal{U}_{\mathcal{W}}$ if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that:

$$d_{\mathcal{U}_{\mathcal{W}}}(p_{w_j}^{\mu_j}, p_w^{\mu}) < \varepsilon, \forall \ j \geq N$$
$$\textit{where } p_{w_j}^{\mu_j}, p_w^{\mu} \in \mathcal{U}_{\mathcal{W}}, \ w_j, w \in \mathcal{W} \textit{ and } \mu_j, \mu \in [0,1].$$

Definition 4.2. A fuzzy point sequence $\{p_{w_j}^{\mu_j}\}, \ j \in \mathbb{N}$ is said to be Cauchy if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that:

$$d_{\mathcal{U}_{\mathcal{W}}}(p_{w_j}^{\mu_j}, p_{w_q}^{\mu_q}) < \varepsilon, \forall \ j, q \geq N$$
$$\textit{where } p_{w_j}^{\mu_j}, p_{w_q}^{\mu_q} \in \mathcal{W} \textit{ and } \mu_j, \ \mu_q \in [0,1].$$

Remark 4.3. A sequence of fuzzy points $\{p_{w_j}^{\mu_j}\}, (\mu_j \in [0,1], \ j \in \mathbb{N})$ is said to converge to a fuzzy point p_w^{μ} if and only if there is a crisp monotonic sequence of real numbers $\{\mu_j\} \in [0,1]$ converging to $\mu \in \mathbb{R}$ and a series of support points $\{w_j\}, \ w_j \in \mathcal{W}, \ j \in \mathbb{N}$, which converge to $w \in \mathcal{W}$, as j tends to ∞.

Lemma 4.4. *If $\{p_{w_j}^{\mu_j}\}$ ($w_j \in \mathcal{W}, \ \mu_j \in [0,1], \ j \in \mathbb{N}$) is a convergent sequence of fuzzy points in $b\mathcal{FMS}$ $(\mathcal{U}_{\mathcal{W}}, d_{\mathcal{U}_{\mathcal{W}}})$, then $p_{w_j}^{\mu_j}$ is a Cauchy sequence in $(\mathcal{U}_{\mathcal{W}}, d_{\mathcal{U}_{\mathcal{W}}})$.*

Definition 4.5. A $b\mathcal{FMS}$ $(\mathcal{U}_{\mathcal{W}}, d_{\mathcal{U}_{\mathcal{W}}})$ is said to be complete, if every Cauchy sequence in $\mathcal{U}_{\mathcal{W}}$ converges to a fuzzy point in $\mathcal{U}_{\mathcal{W}}$.

Lemma 4.6. *Suppose that $(\mathcal{U}_{\mathcal{W}}, d_{\mathcal{U}_{\mathcal{W}}})$ is the fuzzy metric space induced from non-$\mathcal{FMS}$ $(\mathcal{W}, d_{\mathcal{W}})$, then $(\mathcal{U}_{\mathcal{W}}, d_{\mathcal{U}_{\mathcal{W}}})$ is complete if and only if $(\mathcal{W}, d_{\mathcal{W}})$ is a complete.*

5. New Findings

We define the contraction in $b\mathcal{FMS}$ stated as follows:

Definition 5.1. Let $(\mathcal{U}_{\mathcal{W}}, d_{\mathcal{U}_{\mathcal{W}}})$ be a $b\mathcal{FMS}$ and $g : \mathcal{U}_{\mathcal{W}} \to \mathcal{U}_{\mathcal{W}}$ be a fuzzy mapping, then g is a b-contractive fuzzy mapping in $\mathcal{U}_{\mathcal{W}}$ if there exists $\varsigma \in (0,1)$, such that:

$$d_{\mathcal{U}_{\mathcal{W}}}(g(p_w^{\mu}), g(p_h^{\nu})) \leq \varsigma d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, p_h^{\nu}) \tag{5.1}$$

for fuzzy points $p_w^{\mu}, p_h^{\nu} \in \mathcal{U}_{\mathcal{W}}$ and $\mu, \nu \in [0,1]$.

Following result is the Banach version b-Fuzzy fixed point theorem.

Theorem 5.2. *If $g : \mathcal{U}_{\mathcal{W}} \longrightarrow \mathcal{U}_{\mathcal{W}}$ is a b-contractive fuzzy mapping and $(\mathcal{U}_{\mathcal{W}}, d_{\mathcal{U}_{\mathcal{W}}})$ is a complete $b\mathcal{FMS}$, then g has a fuzzy fixed point in $\mathcal{U}_{\mathcal{W}}$.*

Proof. If $p_{w_0}^{\mu_0} \in \mathcal{U}_{\mathcal{W}}$, and suppose that $p_{w_1}^{\mu_1} = g(p_{w_0}^{\mu_0})$ where g^j refers to j^{th} composition of the fuzzy mapping g.

Now, let $p_{w_j}^{\mu_j}, (w_j \in \mathcal{W},\ \mu_j \in [0,1],\ j \in \mathbb{N})$ be a sequence of fuzzy points in $\mathcal{U}_{\mathcal{W}}$ and from the contractility of g, we obtain,

$$\begin{aligned}
d_{\mathcal{U}_{\mathcal{W}}}(p_{w_2}^{\mu_2}, p_{w_1}^{\mu_1}) &= d_{\mathcal{U}_{\mathcal{W}}}(g(p_{w_1}^{\mu_1}, g(p_{w_0}^{\mu_0})) \\
&\leq \varsigma d_{\mathcal{U}_{\mathcal{W}}}(p_{w_1}^{\mu_1}, p_{w_0}^{\mu_0}) \\
d_{\mathcal{U}_{\mathcal{W}}}(p_{w_3}^{\mu_3}, p_{w_2}^{\mu_2}) &= d_{\mathcal{U}_{\mathcal{W}}}(g(p_{w_2}^{\mu_2}), g(p_{w_1}^{\mu_1})) \\
&\leq \varsigma d_{\mathcal{U}_{\mathcal{W}}}(p_{w_2}^{\mu_2}, p_{w_1}^{\mu_1}) \\
&\leq \varsigma^2 d_{\mathcal{U}_{\mathcal{W}}}(p_{w_1}^{\mu_1}, p_{w_0}^{\mu_0}) \\
&\ \vdots \\
d_{\mathcal{U}_{\mathcal{W}}}(p_{w_j}^{\mu_j}, p_{w_{j-1}}^{\mu_{j-1}}) &\leq \varsigma^j d_{\mathcal{U}_{\mathcal{W}}}(p_{w_1}^{\mu_1}, p_{w_0}^{\mu_0}).
\end{aligned}$$

Also, the triangle inequality is used to get:

$$d_{\mathcal{U}_{\mathcal{W}}}(p_{w_j}^{\mu_j}, p_{w_q}^{\mu_q}) \leq k[d_{\mathcal{U}_{\mathcal{W}}}(p_{w_j}^{\mu_j}, p_{w_k}^{\mu_k}) + d_{\mathcal{U}_{\mathcal{W}}}(p_{w_k}^{\mu_k}, p_{w_q}^{\mu_q})].$$

Therefore, application of equation (3.1) will give:

$$d_{\mathcal{U}_{\mathcal{W}}}(p_{w_j}^{\mu_j}, p_{w_q}^{\mu_q}) = |\mu_j - \mu_q| + d(w_j, w_q),$$

Consequently, using the triangle inequality $d(w_j, w_{j+2}) \leq d(w_j, w_{j+1}) + d(w_{j+1}, w_{j+2})$ for each sequence $\{w_j\}$ and $\{\mu_j\}$ as well as considering the fact that $|\mu_j - \mu_{j-2}| \leq |\mu_j - \mu_{j+1}| + |\mu_{j+1} - \mu_{j+2}|$, we get the following results:

$$\begin{aligned}
d_{\mathcal{U}_{\mathcal{W}}}(p_{w_j}^{\mu_j}, p_{w_q}^{\mu_q}) &= |\mu_j - \mu_q| + d(w_j, w_q) \\
&\leq |\mu_j - \mu_{j+1}| + |\mu_{j+1} - \mu_q| + d(w_j, w_{j+1}) + d(w_{j+1}, w_q) \\
\Rightarrow d_{\mathcal{U}_{\mathcal{W}}}(p_{w_j}^{\mu_j}, p_{w_q}^{\mu_q}) &\leq |\mu_j - \mu_{j+1}| + d(w_j, w_{j+1}) + |\mu_{j+1} - \mu_q| + d(w_{j+1}, w_q) \\
d_{\mathcal{U}_{\mathcal{W}}}(p_{w_j}^{\mu_j}, p_{w_q}^{\mu_q}) &\leq d_{\mathcal{U}_{\mathcal{W}}}(p_{w_j}^{\mu_j}, p_{w_{j+1}}^{\mu_{j+1}}) + d_{\mathcal{U}_{\mathcal{W}}}(p_{w_{j+1}}^{\mu_{j+1}}, p_{w_q}^{\mu_q}) \\
&\leq k[d_{\mathcal{U}_{\mathcal{W}}}(p_{w_j}^{\mu_j}, p_{w_{j+1}}^{\mu_{j+1}}) + d_{\mathcal{U}_{\mathcal{W}}}(p_{w_{j+1}}^{\mu_{j+1}}, p_{w_q}^{\mu_q})] \\
&\leq k d_{\mathcal{U}_{\mathcal{W}}}(p_{w_j}^{\mu_j}, p_{w_{j+1}}^{\mu_{j+1}}) + k^2 d_{\mathcal{U}_{\mathcal{W}}}(p_{w_{j+1}}^{\mu_{j+1}}, p_{w_{j+2}}^{\mu_{j+2}}) + \cdots \\
&\leq \varsigma^j k d_{\mathcal{U}_{\mathcal{W}}}(p_{w_0}^{\mu_0}, p_{w_1}^{\mu_1}) + \varsigma^{j+1} k^2 d_{\mathcal{U}_{\mathcal{W}}}(p_{w_0}^{\mu_0}, p_{w_1}^{\mu_1}) + \cdots \\
&= k\varsigma^j d_{\mathcal{U}_{\mathcal{W}}}(p_{w_0}^{\mu_0}, p_{w_1}^{\mu_1})[1 + k\varsigma + (k\varsigma)^2 + \cdots] \\
&= \frac{k\varsigma^j}{1 - k\varsigma} d_{\mathcal{U}_{\mathcal{W}}}(p_{w_0}^{\mu_0}, p_{w_1}^{\mu_1}).
\end{aligned}$$

Taking limit $j \to \infty$, we get for $\varepsilon > 0$,

$$\lim_{j \to \infty} d_{\mathcal{U}_{\mathcal{W}}}(p_{w_0}^{\mu_0}, p_{w_1}^{\mu_1}) = 0.$$

Therefore,

$$d_{\mathcal{U}_{\mathcal{W}}}(p_{w_j}^{\mu_j}, p_{w_q}^{\mu_q}) \leq \varepsilon$$

Therefore, $p_{w_j}^{\mu_j},\ j \in \mathbb{N}$ is a Cauchy sequence.

As $(\mathcal{U}_{\mathcal{W}}, d_{\mathcal{U}_{\mathcal{W}}})$ is a complete $b\mathcal{FMS}$, therefore, $\{p_{w_j}^{\mu_j}\}$ converges to $p_w^{\mu} \in \mathcal{U}_{\mathcal{W}}$.

Now, we demonstrate that p_w^{μ} is in fact g's fixed point. Given the continuous nature of $p_{w_j}^{\mu_j} \longrightarrow p_w^{\mu}$ and g, we have,

$$g(p_{w_j}^{\mu_j}) \longrightarrow g(p_w^{\mu}),\ \ i.e.\ p_{w_{j+1}}^{\mu_{j+1}} \to g(p_w^{\mu}).$$

Also, since $\{p_{w_j}^{\mu_j}\} \longrightarrow p_w^{\mu}$, so $g(p_{w_j}^{\mu_j}) = p_w^{\mu}$ and thus p_w^{μ} is a fixed point of this mapping.

□

Definition 5.3. Let $(\mathcal{U}_{\mathcal{W}}, d_{\mathcal{U}_{\mathcal{W}}})$ be a complete $b\mathcal{FMS}$. A fuzzy map $g : \mathcal{U}_{\mathcal{W}} \to \mathcal{U}_{\mathcal{W}}$ is described as b-contractive on $\mathcal{U}_{\mathcal{W}}$ if and only if for $\varsigma \in (0,1)$, $p_w^{\mu}, p_h^{\nu} \in \mathcal{U}_{\mathcal{W}}$, where $w, h \in \mathcal{W}$ and $\mu, \nu \in [0,1]$ implies:

$$d_{\mathcal{U}_{\mathcal{W}}}(g(p_w^{\mu}), g(p_h^{\nu})) \leq \varsigma \max\{d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, g(p_w^{\mu})), d_{\mathcal{U}_{\mathcal{W}}}(p_h^{\nu}, g(p_h^{\nu}))\}. \tag{5.2}$$

Theorem 5.4. *If a contractive fuzzy mapping $g : \mathcal{U}_{\mathcal{W}} \to \mathcal{U}_{\mathcal{W}}$ fulfilling inequality (5.2) and a complete $b\mathcal{FMS}$ $(\mathcal{U}_{\mathcal{W}}, d_{\mathcal{U}_{\mathcal{W}}})$ are given, then g has a fuzzy fixed point in $\mathcal{U}_{\mathcal{W}}$.*

Proof. By (5.2), we have

$$d_{\mathcal{U}_{\mathcal{W}}}(g(p_w^{\mu}), g(p_h^{\nu})) \leq \varsigma \max\{d_{\mathcal{U}_{\mathcal{W}}}(p_w^{\mu}, g(p_w^{\mu})), d_{\mathcal{U}_{\mathcal{W}}}(p_h^{\nu}, g(p_h^{\nu}))\}.$$

Let $p_{w_0}^{\mu_0} \in \mathcal{U}_{\mathcal{W}}$ and suppose that

$$\begin{aligned}
p_{w_1}^{\mu_1} &= g(p_{w_0}^{\mu_0}),\\
p_{w_2}^{\mu_2} &= g(p_{w_1}^{\mu_1}) = g(g(p_{w_0}^{\mu_0})) = g^2(p_{w_0}^{\mu_0}),\\
p_{w_3}^{\mu_3} &= g(p_{w_2}^{\mu_2}) = g(g^2(p_{w_0}^{\mu_0})) = g^3(p_{w_0}^{\mu_0}),\\
&\vdots\\
p_{w_j}^{\mu_j} &= g^j(p_{w_0}^{\mu_0}).
\end{aligned}$$

Now,

$$\begin{aligned}
d_{\mathcal{U}_{\mathcal{W}}}(p_{w_2}^{\mu_2}, p_{w_3}^{\mu_3}) &\leq \varsigma \max\{d_{\mathcal{U}_{\mathcal{W}}}(p_{w_1}^{\mu_1}, g(p_{w_1}^{\mu_1})), d_{\mathcal{U}_{\mathcal{W}}}(p_{w_2}^{\mu_2}, g(p_{w_2}^{\mu_2}))\}\\
&= \varsigma \max\{d_{\mathcal{U}_{\mathcal{W}}}(g(p_{w_0}^{\mu_0}), g(p_{w_1}^{\mu_1})), d_{\mathcal{U}_{\mathcal{W}}}(g(p_{w_1}^{\mu_1})), g(p_{w_2}^{\mu_2}))\}\\
&\leq \varsigma \max\{\varsigma \max\{d_{\mathcal{U}_{\mathcal{W}}}(p_{w_0}^{\mu_0}, g(p_{w_0}^{\mu_0})), d_{\mathcal{U}_{\mathcal{W}}}(p_{w_1}^{\mu_1}, g(p_{w_1}^{\mu_1})\},\\
&\qquad \varsigma \max\{d_{\mathcal{U}_{\mathcal{W}}}(p_{w_1}^{\mu_1}, g(p_{w_1}^{\mu_1})), d_{\mathcal{U}_{\mathcal{W}}}(p_{w_2}^{\mu_2}, g(p_{w_2}^{\mu_2}))\}\}\\
&= \varsigma^2 \max\{d_{\mathcal{U}_{\mathcal{W}}}(p_{w_0}^{\mu_0}, g(p_{w_0}^{\mu_0})), d_{\mathcal{U}_{\mathcal{W}}}(p_{w_1}^{\mu_1}, g(p_{w_1}^{\mu_1})), d_{\mathcal{U}_{\mathcal{W}}}(p_{w_2}^{\mu_2}, g(p_{w_2}^{\mu_2}))\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
d_{\mathcal{U}_{\mathcal{W}}}(p_{w_3}^{\mu_3}, p_{w_4}^{\mu_4}) \leq \varsigma^3 \max\{&d_{\mathcal{U}_{\mathcal{W}}}(p_{w_0}^{\mu_0}, p_{w_0}^{\mu_0}), d_{\mathcal{U}_{\mathcal{W}}}(p_{w_1}^{\mu_1}, p_{w_1}^{\mu_1}),\\
&d_{\mathcal{U}_{\mathcal{W}}}(p_{w_2}^{\mu_2}, p_{w_2}^{\mu_2}), d_{\mathcal{U}_{\mathcal{W}}}(p_{w_3}^{\mu_3}, p_{w_3}^{\mu_3})\}
\end{aligned}$$

$$\vdots$$

$$\Rightarrow d_{\mathcal{U_W}}(p_{w_j}^{\mu_j}, p_{w_{j+1}}^{\mu_{j+1}}) \leq \varsigma^j \max\{d_{\mathcal{U_W}}(p_{w_0}^{\mu_0}, p_{w_0}^{\mu_0}), d_{\mathcal{U_W}}(p_{w_1}^{\mu_1}, p_{w_1}^{\mu_1}), \ldots, d_{\mathcal{U_W}}(p_{w_j}^{\mu_j}, p_{w_j}^{\mu_j})\}.$$

Using the triangle inequality $d_{\mathcal{U_W}}(p_{w_j}^{\mu_j}, p_{w_q}^{\mu_q}) \leq k[d_{\mathcal{U_W}}(p_{w_j}^{\mu_j}, p_{w_k}^{\mu_k}) + d_{\mathcal{U_W}}(p_{w_k}^{\mu_k}, p_{w_q}^{\mu_q})]$, $q > j$, we obtain,

$$\begin{aligned} d_{\mathcal{U_W}}(p_{w_j}^{\mu_j}, p_{w_q}^{\mu_q}) \leq\ & k[d_{\mathcal{U_W}}(p_{w_j}^{\mu_j}, p_{w_{j+1}}^{\mu_{j+1}}) + d_{\mathcal{U_W}}(p_{w_{j+1}}^{\mu_{j+1}}, p_{w_{j+2}}^{\mu_{j+2}}) + \cdots \\ & + d_{\mathcal{U_W}}(p_{w_{q-1}}^{\mu_{q-1}}, p_{w_q}^{\mu_q})] \\ \leq\ & k\varsigma^j \max\{d_{\mathcal{U_W}}(p_{w_0}^{\mu_0}, p_{w_0}^{\mu_0}), \ldots, d_{\mathcal{U_W}}(p_{w_j}^{\mu_j}, p_{w_j}^{\mu_j})\} \\ & + k^2\varsigma^{j+1} \max\{d_{\mathcal{U_W}}(p_{w_0}^{\mu_0}, p_{w_0}^{\mu_0}), \ldots, d_{\mathcal{U_W}}(p_{w_{j+1}}^{\mu_{j+1}}, p_{w_{j+1}}^{\mu_{j+1}})\} + \cdots \end{aligned} \tag{5.3}$$

If $d_{\mathcal{U_W}}(p_{w_0}^{\mu_0}, p_{w_0}^{\mu_0})$ is where the inequality is maximized, then:

$$\begin{aligned} d_{\mathcal{U_W}}(p_{w_j}^{\mu_j}, p_{w_q}^{\mu_q}) &\leq k\ \varsigma^j d_{\mathcal{U_W}}(p_{w_0}^{\mu_0}, p_{w_0}^{\mu_0}) + k^2\varsigma^{j+1} d_{\mathcal{U_W}}(p_{w_0}^{\mu_0}, p_{w_0}^{\mu_0}) + \cdots \\ &= k\varsigma^j(1 + k\varsigma + \cdots) d_{\mathcal{U_W}}(p_{w_0}^{\mu_0}, p_{w_0}^{\mu_0}) \\ &= \frac{k\varsigma^j}{1 - k\varsigma} d_{\mathcal{U_W}}(p_{w_0}^{\mu_0}, p_{w_0}^{\mu_0}),\ \ 0 \leq \varsigma < 1. \end{aligned}$$

Taking limit $j \to \infty$, we get

$$\lim_{j\to\infty} d_{\mathcal{U_W}}(p_{w_0}^{\mu_0}, p_{w_0}^{\mu_0}) = 0$$

and therefore,

$$d_{\mathcal{U_W}}(p_{w_j}^{\mu_j}, p_{w_q}^{\mu_q}) \to 0\ as\ j, q \to \infty.$$

Hence, $\{p_{w_j}^{\mu_j}\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{U_W}$.

Since $(\mathcal{U_W}, d_{\mathcal{U_W}})$ is a complete $b\mathcal{FMS}$; $\{p_{w_j}^{\mu_j}\}$ converges to a fuzzy point $p_w^\mu \in \mathcal{U_W}$.

Now, since $(p_{w_j}^{\mu_j}) \to (p_w^\mu)$ and g is continuous; we have $(p_{w_{j+1}}^{\mu_{j+1}}) \to g(p_w^\mu)$ as $j \to \infty$. Thus, the uniqueness of the limit implies $g(p_w^\mu) = p_w^\mu$.

Likewise, in the event that $d_{\mathcal{U_W}}(p_{w_1}^{\mu_1}, p_{w_1}^{\mu_1})$ is the location of the maximum of inequality (5.3), the following will emerge:.

$$d_{\mathcal{U_W}}(p_{w_j}^{\mu_j}, p_{w_q}^{\mu_q}) \leq \frac{k\varsigma^j}{1 - k\varsigma} d_{\mathcal{U_W}}(p_{w_1}^{\mu_1}, p_{w_1}^{\mu_1}),\ \ 0 < \varsigma < 1.$$

Then for $\varepsilon > 0$ and large $N \in \mathbb{N}$ such that $\frac{k\varsigma^j}{1-k\varsigma} d_{\mathcal{U_W}}(p_{w_1}^{\mu_1}, p_{w_1}^{\mu_1}) < \varepsilon,\ \forall\ j \geq N$ and consequently, $d_{\mathcal{U_W}}(p_{w_j}^{\mu_j}, p_{w_q}^{\mu_q}) \leq \varepsilon\ \forall\ q \geq j$ and in the similar manner, we get p_w^μ is a fixed point.

This concludes the theorem's proof. □

6. Conclusion and future scope

In this study, we have presented some new notions and ideas in fuzzy b-metric space and explored some fixed point findings in this setup. As a result, we have established a fruitful environment for the study of certain fixed point theorems in related spaces for future articles. We have developed b-fuzzy versions of the Banach contraction principle and consequently, attempted to expand on the findings found in the literature by using the b-fuzzy distance map and the relationship that is created between fuzzy and non-fuzzy metrics. We anticipate that these existence results will offer a suitable setting for approximating more generalized fixed point solutions.

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