Journal of Nonlinear and Convex Analysis
Volume 26, Number 4, 2025, 1003–1016

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

A FIXED POINT ITERATIVE APPROACH TO THE SOLUTION OF AN ELASTIC BEAM EQUATION

FAEEM ALI AND JAVID ALI

Abstract. This paper applies a fixed point iteration method to approximate the solution of a non-linear elastic beam equation in Banach space. The existence and uniqueness of the solution of the elastic beam equation are proved. Moreover, some convergence results are demonstrated for almost ϕ-contraction by F^*-fixed point iteration method. Some illuminative numerical examples are formulated to validate our results. The results presented in this paper are new and extend, improve, and unify several relevant results in the literature.

1. Introduction and preliminaries

In recent years, much attention has been paid to "nonlinear fourth−order two point boundary value problems (BVPs)" for elastic beam equations. Such type BVPs have the importance in science, engineering and applied in the field of physics, aircraft designes, chemical sensors, medical diagnostics, electromechanical systems, etc. In recent past, many authors have been studied various forms of the elastic beam equations and proved existence and uniqueness of the solutions of such BVPs under certain assumptions, for instance, see [14]. In the present paper, we consider the following nonlinear equation of the elastic beam which is a special case of the fourth−order BVPs.

$$\begin{cases} z^{''''}(y) = g(y, z(y)), \ 0 \leq y \leq 1; \\ z(0) = z^{'}(0) = z^{''}(1) = z^{'''}(1) = 0, \end{cases} \tag{1.1}$$

where $g : [0,1] \times [0,\infty) \longrightarrow [0,\infty)$ is a continuous function. Equation (1.1) is known as an elastic beam equation of length 1 whose both ends are fixed. Such a beam model to be also known as "cantilever beam model" in material mechanics. The existence of the solution of problem (1.1) has been proved by several authors, e.g., see [1, 15]. In this paper, we also give the simplest proof of the existence and uniqueness of the solution of problem (1.1) with mild conditions in Banach space. The exact solution of the equation of elastic beams is not known in the various circumstances. Therefore, the approximate solution is a best choice for the researchers and which is very useful in the various fields beyond the mathematics. There are several methods available in the literature to approximate the solution of the BVPs like variational method [12, 17], monotone variational method [13], etc. One of the most important and applicable methods is the fixed point iteration method for the solution of BVPs [3,6,16,19,20]. Several researchers have introduced and studied different fixed point iteration methods to find the solutions of nonlinear

2020 *Mathematics Subject Classification.* 47H09, 47H10, 54H25.

Key words and phrases. Nonlinear elastic beam equation, F^*-fixed point iteration method, almost ϕ-contraction, Banach space, fixed point.

problems, for instance, see [5, 16]. Recently, Ali and Ali [4] introduced a two-step fixed point iteration method known to be F^*-iteration method which is defined as follows:

Let S be a self mapping defined on a Banach space Y and $y_0 \in Y$ be an initial point. Then a sequence $\{y_n\}$ generated by y_0 defined as:

$$\begin{cases} y_0 \in Y, \\ w_n = S((1-\wp_n)y_n + \wp_n S y_n), \\ y_{n+1} = S w_n, \ n = 0, 1, 2, 3, \dots, \end{cases} \tag{1.2}$$

where $\{\wp_n\}$ is a sequence in $(0,1)$. They also claimed the F^*-iteration method is almost stable for weak contractions and more efficient than the some leading iteration methods given in [2, 18, 21, 22, 24, 25]. In this paper, we consider F^*-iteration method and study its convergence behaviour for almost ϕ-contractions.

Before the concept of almost ϕ-contractions, Berinde [8] introduced a noted concept of almost contractions, sometimes also referred as weak contractions which is defined as follows.

A mapping $\mathcal{S} : Y \to Y$ is said to be almost contraction if there is a constant $\delta \in (0,1)$ and for some $L \geq 0$ such that for all $y, z \in Y$,

$$\|\mathcal{S}y - \mathcal{S}z\| \leq \delta\|y - z\| + L\|y - \mathcal{S}z\|, \tag{1.3}$$

and

$$\|\mathcal{S}y - \mathcal{S}z\| \leq \delta\|y - z\| + L\|z - \mathcal{S}y\|. \tag{1.4}$$

He proved the following result for the existence and uniqueness of a fixed point.

Theorem 1.1 ([8]). *Let Y be a Banach space and $S : Y \to Y$ be an almost contraction satisfying with $\delta \in (0,1)$ and for some $L \geq 0$,*

$$\|Sy - Sz\| \leq \delta\|y - z\| + L\|y - Sy\|, \ \forall \ y, z \in Y. \tag{1.5}$$

Then S has a unique fixed point in Y.

On the other hand, in 1968, Browder [11] coined the concept of ϕ-contraction involving a function ϕ on $\mathbb{R}^+ = [0, \infty)$.

Definition 1.2. A mapping S from a normed space Y into itself is called a ϕ-contraction if there is a function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$\|Sy - Sz\| \leq \phi(\|y - z\|), \ \forall \ y, z \in Y. \tag{1.6}$$

Afterward, Rus [23] and Berinde [7] improved the conditions of a comparison function ϕ.

Definition 1.3 ([9]). A mapping $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is called a (c)-comparison function if the following conditions are satisfied.

(1) ϕ is monotonically increasing, *i.e.* if $s_1 < s_2 \implies \phi(s_1) \leq \phi(s_2)$;
(2) $\sum_{n=0}^{\infty} \phi^n(s) < \infty, \ \forall \ s \in \mathbb{R}^+$.

Remark 1.4 ([10]). If $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a (c)-comparison function, then

(1) $\phi(s) < s, \ \forall s > 0$;
(2) $\phi(0) = 0$;

(3) ϕ is right continuous at 0;
(4) the sequence $\{\phi^n(s)\}_{n=0}^{\infty}$ of nth iterate of ϕ converges to zero, for all $s \in \mathbb{R}^+$.

In 2003, Berinde [9] combined the concept of almost contractions with (c)-comparison function ϕ which is known as almost ϕ-contraction and defined as follows:

Definition 1.5 ([9])**.** Let $(Y, \|\cdot\|)$ be a Banach space. A self mapping $S : Y \longrightarrow Y$ is called almost ϕ-contraction if there is a (c)-comparison function ϕ and for some $L \geq 0$ such that for all $y, z \in Y$,

$$\|\mathcal{S}y - \mathcal{S}z\| \leq \phi(\|y - z\|) + L\|y - \mathcal{S}z\|, \tag{1.7}$$

and

$$\|\mathcal{S}y - \mathcal{S}z\| \leq \phi(\|y - z\|) + L\|z - \mathcal{S}y\|. \tag{1.8}$$

To demonstrate that an operator S is an almost ϕ-contraction, both inequalities (1.7) and (1.8) must be verified.

Remark 1.6. (1) Any almost contraction is an almost ϕ-contraction with $\phi(s) = \beta s$, $s \in \mathbb{R}^+$, $\beta \in [0, 1)$;
(2) Any ϕ-contraction is an almost ϕ-contraction with $L = 0$. This means that the class of almost ϕ-contractions properly contains the classes of almost contractions and ϕ-contractions.

He proved the following existence and uniqueness result.

Theorem 1.7 ([9])**.** *Let Y be a Banach space and $S : Y \to Y$ be an almost ϕ-contraction satisfying* (1.7) *and the following inequality for some $L \geq 0$*

$$\|Sy - Sz\| \leq \phi(\|y - z\|) + L\|y - Sy\|, \ \forall \ y, z \in Y. \tag{1.9}$$

Then S has a unique fixed point in Y. Moreover, the sequence of Picard iterates converges to the unique fixed point of the mapping S.

In this paper, the existence and uniqueness of the solution of elastic beam equation are proved by using an almost ϕ-contraction. We also proved some convergence results in a Banach space. We propose a novel F^*-Green's iteration method to solve the elastic beam equation. This method is based on a F^* iteration method that utilizes a Green's function, $G(s, \tau)$ to enhance the convergence of the iteration process and handle a wide range of boundary conditions more effectively. The proposed approach offers improvements in both accuracy and computational efficiency over some existing methods.

2. Convergence analysis for almost ϕ-contractions

In this section, we prove our main results for almost ϕ-contractions via F^* iteration method (1.2) in a Banach space.

Theorem 2.1. *Let Y be a Banach space and $S : Y \longrightarrow Y$ be an almost ϕ-contraction satisfying* (1.9)*. Then the iterative method* (1.2) *converges to a fixed point of S.*

Proof. Since S is an almost ϕ-contraction satisfying (1.9), therefore $F(S)$ is nonempty. For $p \in F(S)$ and $y \in Y$, we have

$$\begin{aligned} \|Sp - Sy\| &\leq \phi(\|p-y\|) + L\|p - Sp\| \\ &= \phi(\|p-y\|). \end{aligned}$$

Now by using the properties of (c)-comparison function ϕ and F^* iteration method (1.2), we get

$$\begin{aligned} \|w_n - p\| &= \|S((1-\vartheta_n)y_n + \vartheta_n Sy_n) - p\| \\ &\leq \phi(\|(1-\vartheta_n)y_n + \vartheta_n Sy_n - p\|) \\ &< \|(1-\vartheta_n)y_n + \vartheta_n Sy_n - p\| \\ &= \|(1-\vartheta_n)(y_n - p) + \vartheta_n(Sy_n - p)\| \\ &\leq (1-\vartheta_n)\|y_n - p\| + \vartheta_n\|Sy_n - p\| \\ &\leq (1-\vartheta_n)\|y_n - p\| + \vartheta_n\phi(\|y_n - p\|) \\ &< (1-\vartheta_n)\|y_n - p\| + \vartheta_n\|y_n - p\| \\ &= \|y_n - p\|. \end{aligned} \tag{2.1}$$

By (2.1) and since ϕ is monotonically increasing, it follows that

$$\begin{aligned} \|y_{n+1} - p\| &= \|Sw_n - p\| \\ &\leq \phi(\|w_n - p\|) \leq \phi(\|y_n - p\|) \\ &\leq \phi^2(\|y_{n-1} - p\|) \leq \cdots \leq \phi^n(\|y_0 - p\|). \end{aligned} \tag{2.2}$$

In view of Remark 1.4, the nth iterate $\phi^n(s)$, $s \in \mathbb{R}^+$ of ϕ converges to zero as $n \longrightarrow \infty$. This implies that the sequence $\{y_n\}$ converges to p, the fixed point of S. □

Theorem 2.1 extends, unifies, refines, and enhances a range of existing results in the following ways:

Remark 2.2.
- One can deduce the results of Ali and Ali [4] by setting $\phi(s) = \beta s$, $0 \leq \beta < 1$.
- One can obtain the results for ϕ-contractions by setting $L = 0$.
- One can obtain the results for contractions by setting $\phi(s) = \beta s$, $0 \leq \beta < 1$ and $L = 0$.

Now, we construct the following example to compare the rate of convergence numerically.

Example 2.3. Let $Y = \mathbb{R}^2$ be a Banach space with respect to the norm $\|(y,z)\| = |y| + |z|$ and $\mathcal{A} = \{(y,z) \in [0,1] \times [0,1]\}$ be a subset of Y. Let $S : \mathcal{A} \to \mathcal{A}$ be defined by

$$S(y,z) = \begin{cases} \left(\frac{1}{3}\sin(y), \frac{1}{6}\sin(z)\right), & \text{if } (y,z) \in [0,\frac{1}{3}] \times [0,\frac{1}{3}], \\ \\ \left(\frac{1}{3}y, \frac{1}{6}z\right), & \text{if } (y,z) \in (\frac{1}{3},1] \times (\frac{1}{3},1]. \end{cases}$$

Then S is an almost ϕ-contraction satisfying (1.5) for $L = \frac{1}{3}$.

Using MATLAB, we demonstrate that the iteration method F^* converges more rapidly to a fixed point $p = (0, 0)$ of the mapping S compared to the Picard, Mann, Ishikawa, S, Normal-S and Varat iteration methods. This convergence is observed with an initial point $y_0 = (0.15, 0.45)$ and control sequences $\vartheta_n = 0.95$, $\mu_n = 0.25$ and $\nu_n = 0.55$ for $n \in \mathbb{Z}^+$. These results are illustrated in Tables 1-2 and Fig. 1.

Table 1. Iteration Results

Iter. No.	F^*	Picard	Mann	Ishikawa
1	(0.150000,0.450000)	(0.150000,0.450000)	(0.150000,0.450000)	(0.150000,0.450000)
2	(0.006088,0.002534)	(0.049813,0.072494)	(0.054822,0.091370)	(0.046966,0.077628)
3	(0.000248,0.000015)	(0.016597,0.012072)	(0.020093,0.019015)	(0.014738,0.013605)
4	(0.000010,0.000000)	(0.005532,0.002012)	(0.007367,0.003961)	(0.004626,0.002386)
5	(0.000000,0.000000)	(0.001844,0.000335)	(0.002701,0.000825)	(0.001452,0.000418)
6	(0.000000,0.000000)	(0.000615,0.000056)	(0.000990,0.000172)	(0.000456,0.000073)
7	(0.000000,0.000000)	(0.000205,0.000009)	(0.000363,0.000036)	(0.000143,0.000013)
8	(0.000000,0.000000)	(0.000068,0.000002)	(0.000133,0.000007)	(0.000045,0.000002)
9	(0.000000,0.000000)	(0.000023,0.000000)	(0.000049,0.000002)	(0.000014,0.000000)
10	(0.000000,0.000000)	(0.000008,0.000000)	(0.000018,0.000000)	(0.000004,0.000000)
11	(0.000000,0.000000)	(0.000003,0.000000)	(0.000007,0.000000)	(0.000001,0.000000)
12	(0.000000,0.000000)	(0.000001,0.000000)	(0.000002,0.000000)	(0.000000,0.000000)
13	(0.000000,0.000000)	(0.000000,0.000000)	(0.000001,0.000000)	(0.000000,0.000000)
14	(0.000000,0.000000)	(0.000000,0.000000)	(0.000000,0.000000)	(0.000000,0.000000)

Table 2. Iteration Results

Iter. No.	S	Normal-S	Varat
1	(0.150000,0.450000)	(0.150000,0.450000)	(0.150000,0.450000)
2	(0.041956,0.058752)	(0.018265,0.015207)	(0.041729,0.058358)
3	(0.011768,0.007851)	(0.002232,0.000528)	(0.011641,0.007743)
4	(0.003302,0.001050)	(0.000273,0.000018)	(0.003248,0.001028)
5	(0.000926,0.000140)	(0.000033,0.000001)	(0.000906,0.000136)
6	(0.000260,0.000019)	(0.000004,0.000000)	(0.000253,0.000018)
7	(0.000073,0.000003)	(0.000000,0.000000)	(0.000071,0.000002)
8	(0.000020,0.000000)	(0.000000,0.000000)	(0.000020,0.000000)
9	(0.000006,0.000000)	(0.000000,0.000000)	(0.000005,0.000000)
10	(0.000002,0.000000)	(0.000000,0.000000)	(0.000002,0.000000)
11	(0.000000,0.000000)	(0.000000,0.000000)	(0.000000,0.000000)

Remark 2.4. In the above numerical computation, one can observe that the F^*-iteration method has a better rate of convergence than the some existing methods for an almost ϕ-contraction in Banach space.

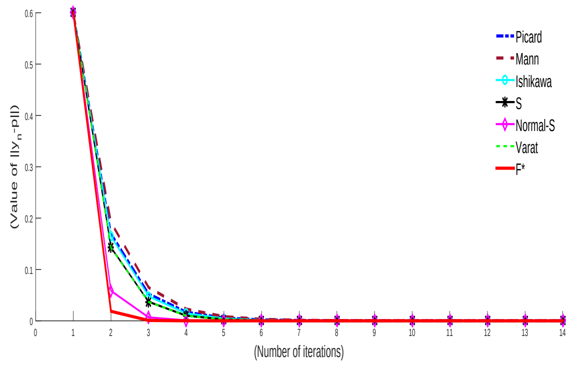

FIGURE 1. Comparison of errors examined by various iteration methods.

3. EXISTENCE AND UNIQUENESS OF THE SOLUTION OF ELASTIC BEAM EQUATIONS

In the current section, we use $C[0,1]$ to denote the class of continuous real-valued functions on the interval $[0,1]$. We will proceed to prove the existence and uniqueness theorem for finding a solution to the boundary value problem (BVP) (1.1).

Theorem 3.1. *Along with the problem* (1.1)*, suppose there exists a* (c)*-comparison function* ϕ *satisfying*

$$0 \leq g(s;\beta) - g(s;\alpha) \leq \phi(\beta - \alpha), \quad \forall s \in [0,1], \quad \forall \beta, \alpha \in \mathbb{R} \text{ with } \beta \geq \alpha. \tag{3.1}$$

Then the BVP (1.1) *has a unique solution.*

Proof. The boundary value problem (1.1) can be transformed into the integral equation:

$$u(s) = \int_0^1 G(s,\tau) g(\tau; u(\tau)) d\tau, \quad \forall s \in [0,1], \tag{3.2}$$

where the Green's function $G(s,\tau)$ is defined as

$$G(s,\tau) = \begin{cases} \frac{1}{6}\tau^2(3s - \tau), & \text{for } 0 \leq s \leq \tau \leq 1, \\ \frac{1}{6}s^2(3\tau - s), & \text{for } 0 \leq \tau \leq s \leq 1. \end{cases} \tag{3.3}$$

It can be verified that

$$0 \leq G(s,\tau) \leq \frac{1}{2}s^2\tau, \quad \forall s,\tau \in [0,1]. \tag{3.4}$$

We define a norm $\|\cdot\|$ on the space $P := C[0,1]$ as:

$$\|\psi\| = \max_{s\in[0,1]} |\psi(s)|, \quad \forall \psi \in P. \tag{3.5}$$

Then P is a Banach space. The function $J : P \to P$ can be defined as follows:

$$J(\theta)(s) = \int_0^1 G(s,\tau) g(\tau; \theta(\tau))\, d\tau, \quad \forall s \in [0,1], \quad \forall \theta \in P. \tag{3.6}$$

Utilizing Eq. (3.1), for any $s \in [0,1]$ and any $\theta, \psi \in P$, $\theta(s) \geq \psi(s)$, $\forall s \in [0,1]$, one has:

$$\begin{aligned}
\|J\theta - J\psi\| &= |J(\theta)(s) - J(\psi)(s)| \\
&= \left| \int_0^1 G(s,\tau)(g(\tau;\theta(\tau)) - g(\tau;\psi(\tau)))\, d\tau \right| \\
&\leq \int_0^1 G(s,\tau)\phi(|\theta(\tau) - \psi(\tau)|)\, d\tau \\
&\leq \left(\int_0^1 G(s,\tau)\, d\tau \right) \phi(\|\theta - \psi\|) \quad \text{(using the increasing property of } \phi).
\end{aligned}$$

From Eq. (3.4), we know that:

$$\int_0^1 G(s,\tau)\, d\tau \leq \frac{s^2}{4} \leq \frac{1}{4}. \tag{3.7}$$

Hence, we have:

$$\frac{1}{4}\phi(\|\theta - \psi\|) \leq \phi(\|\theta - \psi\|) + L\|\theta - J\phi\|, \tag{3.8}$$

where $L \geq 0$ is an arbitrary constant. This implies:

$$\|J\theta - J\psi\| \leq \phi(\|\theta - \psi\|) + L\|\theta - J\theta\|. \tag{3.9}$$

All the conditions of Theorem 1.7 have thus been verified, implying the existence of a unique function $\theta \in C([0,1])$ that satisfies $J(\theta) = \theta$. This function is the unique solution of the integral equation (3.2), and hence a solution of the boundary value problem (1.1). □

4. Approximation of the solution of elastic beam equations

The purpose of this section is to approximate the solution of the BVP (1.1) under some mild conditions by developing an efficient iteration method embedding Green's function.

4.1. **Overview of Green's function and methodology.**

Consider the following equation of elastic beam:

$$\begin{cases} z''''(y) = g(y, z(y)), \ 0 \leq y \leq 1; \\ z(0) = z'(0) = z''(1) = z'''(1) = 0. \end{cases} \tag{4.1}$$

The existence and uniqueness of the solution of the problem (4.1) is proved in the Section 3. The Green's function $G(s,\tau)$ corresponding to $z'''' = 0$ is given by

$$G(s,\tau) = \begin{cases} c_1z_1 + c_2z_2 + c_3z_3 + c_4z_4, \ 0 \leq \tau < s, \\ d_1y_1 + d_2y_2 + d_3z_3 + d_4z_4, \ s \leq \tau \leq 1, \end{cases} \tag{4.2}$$

where z_1, z_2, z_3 and z_4 are linearly independent solutions of $z'''' = 0$. The constants c_i, d_i, $i = 1,2,3,4$ can be determined using the properties of Green's function. The particular solution of the problem (4.1) can be expressed in terms of Green's function as follows:

$$z_p = \int_0^1 G(s,\tau)g(\tau, z(\tau))d\tau. \tag{4.3}$$

It can be easily seen that the problem (4.1) has a general solution $z = z_h + z_p$, where z_h and z_p denote the homogeneous and particular solutions of the problem (4.1).

The core concept behind the proposed method involves defining an integral operator using Green's function and subsequently applying the F^*-iteration method. We start by considering the linear integral operator:

$$J(z) = z_h + \int_0^1 G(s,\tau) z''''(\tau)\, d\tau. \tag{4.4}$$

By adding and subtracting $G(s,\tau)g(\tau,z)$ within the integrand, we derive the expression:

$$J(z) = z_h + \int_0^1 G(s,\tau)\left(z'''' - g(\tau,z)\right)\, d\tau + \int_0^1 G(s,\tau) g(\tau,z)\, d\tau. \tag{4.5}$$

Based on Eq. (4.3), we can express:

$$J(z) = z_h + \int_0^1 G(s,\tau)\left(z'''' - g(\tau,z)\right)\, d\tau + z_p, \tag{4.6}$$

or equivalently,

$$J(z) = z + \int_0^1 G(s,\tau)\left(z'''' - g(\tau,z)\right)\, d\tau, \tag{4.7}$$

after expressing z as $z_h + z_p$. The later Eq. (4.7) represents our integral operator, which is essential for constructing the iteration method.

Next, we apply the F^*-iteration method (1.2) to the operator J in (4.7). This yields:

$$\begin{cases} w_n = J((1-\vartheta_n)z_n + \vartheta_n J z_n), \\ z_{n+1} = J w_n, \ n = 0,1,2,3,\ldots, \end{cases} \tag{4.8}$$

or equivalently:

$$\begin{cases} w_n = z_n + \vartheta_n \int_0^1 G(s,\tau)(z_n'''' - g(\tau,z_n))\, d\tau, \\ q_n = w_n + \int_0^1 G(s,\tau)(w_n'''' - g(\tau,w_n))\, d\tau, \\ z_{n+1} = q_n + \int_0^1 G(s,\tau)(q_n'''' - g(\tau,q_n))\, d\tau, \ n = 0,1,2,3,\ldots, \end{cases} \tag{4.9}$$

which is a required iteration method.

4.2. **Convergence analysis.** In this section, we explore the convergence of the proposed iterative method. Consider the following elastic beam equation

$$L[z] = z^{''''}(y) = g(y, z(y)), \ 0 \le y \le 1; \tag{4.10}$$

with the boundary conditions:

$$z(0) = z^{'}(0) = z^{''}(1) = z^{'''}(1) = 0. \tag{4.11}$$

The Green's function for the problem (4.10)-(4.11) is given in (3.3). To implement the iteration method, we interchange the variables s and τ within the integrand. This requires using the adjoint Green's function $G^*(s,\tau)$, which satisfies the equation $L^*G^*(t,s) = \delta(t-s)$, where δ denotes the Dirac delta function. This is different from $G(s,\tau)$, which satisfies $LG(s,\tau) = \delta(s-\tau)$. The reason for using the adjoint function is due to the inner product relationship $\langle LG, G\rangle = \langle G, L^*G^*\rangle$, where L^* is the adjoint of the linear operator L. In our scenario, $G^*(s,\tau) = -G(s,\tau)$, and thus it takes on the following form:

$$G^*(s,\tau) = \begin{cases} \frac{1}{6}\tau^2(\tau - 3s), & \text{for } 0 \leq \tau \leq s, \\ \frac{1}{6}s^2(s - 3\tau), & \text{for } s \leq \tau \leq 1. \end{cases} \tag{4.12}$$

Now, let's define the operator $U_G : P \to P$ as follows:

$$U_{G^*}z = z + \int_0^1 G^*(s,\tau)(z'''' - g(\tau,z))\,d\tau. \tag{4.13}$$

Hence, (4.8) takes the following form:

$$\begin{cases} w_n = U_{G^*}((1-\vartheta_n)z_n + \vartheta_n U_{G^*}z_n), \\ z_{n+1} = U_{G^*}w_n, \ n = 0,1,2,3,\ldots, \end{cases} \tag{4.14}$$

The iteration method (4.14) is the desired F^*-Green's iteration method. Now we prove its convergence theoretically as follows.

Theorem 4.1. *Let $U_{G^*} : P \to P$ be the operator defined by*

$$U_{G^*}z = z + \int_0^1 G^*(s,\tau)(z'''' - g(\tau,z))\,d\tau. \tag{4.15}$$

where P is a Banach space, $G^*(s,\tau)$ is a Green's function associated with the problem (4.10)-(4.11)*. Suppose that the (c)-comparison function ϕ satisfies the following condition*

$$0 \leq g(s;\theta(s)) - g(s;\psi(s)) \leq \phi(\theta(s) - \psi(s)), \quad \forall \theta,\psi \in P \text{ with } \theta(s) \geq \psi(s),\ \forall s \in [0,1]. \tag{4.16}$$

If $\{z_n\}$ is the sequence of iterates obtained using the F^*-Green's iteration method (4.14)*, then $\{z_n\}$ converges strongly to the unique fixed point of U_{G^*}, which is the solution of the elastic beam equation* (4.10)-(4.11)*.*

Proof. The operator defined in (4.15) can be written as:

$$\begin{aligned} U_{G^*}z &= z + \int_0^1 G^*(s,\tau)z''''\,d\tau - \int_0^1 G^*(s,\tau)g(\tau,z))\,d\tau \\ &= z + \int_0^s G^*(s,\tau)z''''\,d\tau + \int_s^1 G^*(s,\tau)z''''\,d\tau - \int_0^1 G^*(s,\tau)g(\tau,z))\,d\tau \\ &= z + \frac{1}{6}\int_0^s \tau^2(\tau - 3s)z''''\,d\tau + \frac{1}{6}\int_s^1 s^2(s-3\tau)z''''\,d\tau - \int_0^1 G^*(s,\tau)g(\tau,z))\,d\tau. \end{aligned}$$

Using integration by parts, we get

$$U_{G^*}z = -\int_0^1 G^*(s,\tau)g(\tau,z)\,d\tau. \tag{4.17}$$

Utilizing Eq. (4.16), for any $s \in [0,1]$ and any $\theta,\psi \in P$, $\theta(s) \geq \psi(s)$, $\forall s \in [0,1]$, one has:

$$\|U_{G^*}\theta - U_{G^*}\psi\| = |U_{G^*}(\theta(s)) - U_{G^*}(\psi(s))|$$

$$
\begin{aligned}
&= \left|\int_0^1 G^*(s,\tau)(g(\tau;\theta(\tau)) - g(\tau;\psi(\tau)))\,d\tau\right| \\
&\leq \int_0^1 G^*(s,\tau)\phi(|\theta(\tau)-\psi(\tau)|)\,d\tau \\
&\leq \left(\int_0^1 G^*(s,\tau)\,d\tau\right)\phi(\|\theta-\psi\|)
\end{aligned}
$$

(4.18) (using the increasing property of ϕ).

From Eq. (4.12), we know that:

$$\int_0^1 G^*(s,\tau)\,d\tau \leq \frac{s^2}{4} \leq \frac{1}{4}. \tag{4.19}$$

Hence, we have:

$$\frac{1}{4}\phi(\|\theta-\psi\|) \leq \phi(\|\theta-\psi\|) + L\|\theta - U_{G^*}\phi\|, \tag{4.20}$$

where $L \geq 0$ is an arbitrary constant. This implies:

$$\|U_{G^*}\theta - U_{G^*}\psi\| \leq \phi(\|\theta-\psi\|) + L\|\theta - U_{G^*}\theta\|. \tag{4.21}$$

This means that U_{G^*} is an almost ϕ-contraction on a Banach space $P = C[0,1]$. Therefore, by applying Theorem 2.1, one can conclude that the sequence $\{z_n\}$ defined in (4.14) converges to the solution of the elastic beam equation (4.10)-(4.11). □

4.3. **Numerical computation.** In this section, we evaluate the performance of our proposed F^*-Green's iteration approach alongside existing methods. Tables presented here demonstrate the accuracy of our new approach across different parameter sets. The results validate our theoretical findings.

Example 4.2. Consider the following equation of elastic beam

$$z^{''''}(s) = \left(z(s) + \frac{s^4}{24} - \frac{s^3}{6} + \frac{s^2}{4}\right),\ 0 \leq s \leq 1; \tag{4.22}$$

with the boundary conditions:

$$z(0) = z^{'}(0) = z^{''}(1) = z^{'''}(1) = 0. \tag{4.23}$$

In this case, the F^*-Green's iteration method becomes.

(4.24)

$$
\begin{cases}
w_n = z_n + \vartheta_n \int_0^1 G^*(s,\tau)\left(z_n^{''''} - \left(z_n + \frac{\tau^4}{24} - \frac{\tau^3}{6} + \frac{\tau^2}{4}\right)\right)d\tau, \\
q_n = w_n + \int_0^1 G^*(s,\tau)\left(w_n^{''''} - \left(w_n + \frac{\tau^4}{24} - \frac{\tau^3}{6} + \frac{\tau^2}{4}\right)\right)d\tau, \\
z_{n+1} = q_n + \int_0^1 G^*(s,\tau)\left(q_n^{''''} - \left(q_n + \frac{\tau^4}{24} - \frac{\tau^3}{6} + \frac{\tau^2}{4}\right)\right)d\tau,\ n = 0,1,2,3,\ldots,
\end{cases}
$$

where $G^*(s,\tau)$ is the Green's function defined in the Eq. (4.12).

We utilize the initial guess z_0, which is chosen as the solution of the corresponding homogeneous equation $z'''' = 0$ subject to the specified boundary conditions: $z(0) = z'(0) = z''(1) = z'''(1) = 0$. We set the control parameter $\vartheta_n = 0.95$ within the range

(0, 1), and present the obtained results for $s = 0.2$ and $s = 0.8$ in Tables 3, and 4 respectively. Clearly, it can be seen that the convergence of F^*-Green's method is better than the Picard-Green's and Mann-Green's methods for the above problem. The numerical results presented in Table 5 are obtained with $\vartheta_n = 0.95$ for various values of s in the interval $[0, 1]$, considering the error $|z_{n+1} - z_n|$. Through numerical comparison, it is observed that the F^*-Green's iteration method exhibits superior convergence, with a minimum errors.

TABLE 3. Comparison of errors by different iteration methods for $s = 0.2$.

Iter. No.	Picard-Green	Mann-Green	F^*-Green
1	6.5560e-04	6.2282e-04	7.1261e-04
2	5.2943e-05	7.8922e-05	5.9052e-07
3	4.2824e-06	1.0007e-05	4.9013e-10
4	3.4640e-07	1.2691e-06	4.0680e-13
5	2.8021e-08	1.6097e-07	3.3764e-16
6	2.2666e-09	2.0418e-08	2.8024e-19
7	1.8335e-10	2.5899e-09	2.3260e-22
8	1.4831e-11	3.2851e-10	1.9305e-25
9	1.1997e-12	4.1671e-11	1.6023e-28
10	9.7045e-14	5.2858e-12	1.3299e-31

TABLE 4. Comparison of errors by different iteration methods for $s = 0.8$.

Iter. No.	Picard-Green	Mann-Green	F^*-Green
1	0.0074	0.0071	0.0081
2	6.0132e-04	8.9578e-04	6.7074e-06
3	4.8641e-05	1.1363e-04	5.5671e-09
4	3.9346e-06	1.4413e-05	4.6207e-12
5	3.1827e-07	1.8283e-06	3.8351e-15
6	2.5745e-08	2.3191e-07	3.1831e-18
7	2.0826e-09	1.6877e-09	2.6420e-21
8	1.6846e-10	3.7314e-09	2.1928e-24
9	1.3627e-11	4.7331e-10	1.8200e-27
10	1.1023e-12	6.0038e-11	1.5106e-30

Example 4.3. Consider the following nonlinear equation of elastic beam

$$z^{''''}(s) = \left(z^2(s) + \frac{s^2}{4} \right), \quad 0 \le s \le 1; \tag{4.25}$$

with the boundary conditions:

$$z(0) = z^{'}(0) = z^{''}(1) = z^{'''}(1) = 0. \tag{4.26}$$

TABLE 5. Comparison of errors at different iteration steps by F^*-Green's iteration method for various values of s.

s	Err. (z_1)	Err. (z_5)	Err. (z_{10})	Err. (z_{20})
0.1	0.0002	0.0089e-14	0.0035e-29	0.0054e-60
0.2	0.0007	0.0338e-14	0.0133e-29	0.0206e-60
0.3	0.0015	0.0721e-14	0.0284e-29	0.0441e-60
0.4	0.0026	0.1215e-14	0.0479e-29	0.0743e-60
0.5	0.0038	0.1795e-14	0.0707e-29	0.1097e-60
0.6	0.0051	0.2438e-14	0.0960e-29	0.1490e-60
0.7	0.0066	0.3124e-14	0.1230e-29	0.1909e-60
0.8	0.0081	0.3835e-14	0.1511e-29	0.2344e-60
0.9	0.0096	0.4559e-14	0.1796e-29	0.2786e-60
0.99	0.0110	0.5213e-14	0.2053e-29	0.3186e-60

We utilize the initial guess z_0, which is chosen as the solution of the corresponding homogeneous equation $z'''' = 0$ subject to the specified boundary conditions: $z(0) = z'(0) = z''(1) = z'''(1) = 0$. We set the control parameter $\vartheta_n = 0.95$ within the range (0, 1), and present the obtained results for different values of s in Table 6. Clearly, it can be seen that the convergence of F^*-Green's method is better than the Picard-Green's and Mann-Green's methods for the nonlinear problem (4.25)-(4.26).

TABLE 6. Comparison of errors for Picard-Green, Mann-Green, and F^*-Green methods at fifth iteration step for different values of s.

Sr. No.	s	Picard-Green, Err. (z_5)	Mann-Green, Err. (z_5)	F^*-Green, Err. (z_5)
1	0.1	0.0032e-12	0.0020e-06	0.0042e-16
2	0.2	0.0121e-12	0.0077e-06	0.0160e-16
3	0.3	0.0260e-12	0.0166e-06	0.0341e-16
4	0.4	0.0441e-12	0.0280e-06	0.0575e-16
5	0.5	0.0655e-12	0.0415e-06	0.0849e-16
6	0.6	0.0895e-12	0.0565e-06	0.1154e-16
7	0.7	0.1153e-12	0.0725e-06	0.1478e-16
8	0.8	0.1422e-12	0.0892e-06	0.1815e-16
9	0.9	0.1697e-12	0.1062e-06	0.2158e-16
10	1.0	0.1973e-12	0.1232e-06	0.2502e-16

5. CONCLUSION

The proposed F^*-Green's iteration method presents a new approach to solving the elastic beam equation, offering greater accuracy and efficiency compared to other leading methods. This method utilizes a Green's function and a customized iteration process to enhance convergence and manage boundary conditions effectively. The existence and uniqueness of the solution to the nonlinear elastic beam equation is established in a straightforward way. This is the first time that fixed points of almost ϕ-contractions are achieved through a general iterative method in the literature. The paper introduces novel findings and improves upon existing results in the field.

6. Future work

The F*-Green's iteration method introduced in this paper shows promise in enhancing the solution of the elastic beam equation. There are several areas for future research that can further extend and optimize the method:

- The current study focuses on one-dimensional elastic beam equations. Extending the F*-Green's iteration method to multidimensional beam problems and nonlinear elastic beam equations would broaden its applicability in engineering and mechanics.
- The principles behind the F*-Green's iteration method can be explored in partial differential equations in solid mechanics and related fields.

Acknowledgments

The authors are grateful to the anonymous referees for their valuable comments and suggestions which improve the paper.

References

[1] R. P. Agarwal and D. O'Regan, *Twin solutions to singular boundary value problems*, Proc. Amer. Math. Soc. **128** (2000), 2085–2094.

[2] R. P. Agarwal, D. O'Regan and D. R. Sahu, *Iterative construction of fixed points of nearly asymptotically non-expansive mappings*, J. Nonlinear Convex Anal. **8** (2007), 61–79.

[3] J. Ali and F. Ali, *A new iterative scheme for approximating fixed points with an application to delay differential equation*, J. Nonlinear Convex Anal. **21** (2020), 2151–2163.

[4] F. Ali and J. Ali, *Convergence, stability and data dependence of a new iterative algorithm with an application*, Comput. Appl. Math. **39** (2020): 667.

[5] F. Ali, J. Ali and J. J. Nieto, *Some observations on generalized non-expansive mappings with an application*, Comput. Appl. Math. **39** (2020): 74.

[6] F. Ali, J. Ali and I. Uddin, *A novel approach for the solution of BVPs via Green's function and fixed point iterative method*, J. Appl. Math. Comput. **66** (2021), 167–181.

[7] V. Berinde, *Generalized Contractions and Applications*, Baia Editura Cub Press, Mare, 1997 (in Romanian).

[8] V. Berinde, *On the approximation of fixed points of weak contractive mappings*, Carpathian J. Math. **19** (2003), 7–22.

[9] V. Berinde, *Approximating fixed points of weak ϕ-contractions*, Fixed Point Theory **4** (2003), 131–142.

[10] V. Berinde, *Iterative Approximation of Fixed Points*, Berlin: Springer, 2007.

[11] F. E. Browder, *On the convergence of successive approximations for nonlinear functional equations*, Indag. Math. **30** (1968), 27–35.

[12] A. Cabada and S. Tersian, *Multiplicity of solutions of a two point boundary value problem for a fourth-order equation*, Appl. Math. Comput. **219** (2013), 5261–5267.

[13] Y. Cui, H. Chen and Y. Zou, *Monotonically iterative method for the cantilever beam equations*, Axioms **12** (2023): 178.

[14] Q. A. Dang and T. K. Q. Ngo, *Existence results and iterative method for solving the cantilever beam equation with fully nonlinear term*, Nonlinear Anal. **36** (2017), 56–68.

[15] P. W. Eloe and J. Henderson, *Singular nonlinear boundary value problems for higher order ordinary differential equations*, Nonlinear Anal. **17** (1991), 1–10.

[16] C. Garodia and I. Uddin, *A new fixed-point algorithm for finding the solution of a delay differential equation*, AIMS Math. **5** (2020), 3182–3200.

[17] S. Heidarkhani, M. Ferrara, A. Salari and M. Azimbagirad, *A variational approach to perturbed elastic beam problems with nonlinear boundary conditions*, Math. Rep. **18** (2016), 573–589.

[18] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150.

[19] S. Khatoon, I. Uddin and D. Baleanu, *Approximation of fixed point and its application to fractional differential equation*, J. Appl. Math. Comput. **66** (2021), 507–525.

[20] S. A. Khuri and A. Sayfy, *An iterative method for boundary value problems*, Nonlinear Sci. Lett. A **8** (2017), 178–186.

[21] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[22] E. Picard, *Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives*, J. Math. Pures Appl. **6** (1890), 145–210.

[23] I. A. Rus, *Generalized contractions*, Seminar Fixed Point Theory **3** (1983), 1–130.

[24] D. R. Sahu, *Applications of the S-iteration process to constrained minimization problems and split feasibility problems*, Fixed Point Theory **12** (2011), 187–204.

[25] W. Sintunavarat and A. Pitea, *On a new iteration scheme for numerical reckoning fixed points of Berinde mappings with convergence analysis*, J. Nonlinear Sci. Appl. **9** (2016), 2553–2562.

Manuscript received May 20, 2024
revised September 17, 2024

F. Ali
Department of Applied Mathematics, Aligarh Muslim University, Aligarh- 202002, India
E-mail address: `fali.am@amu.ac.in`

J. Ali
Department of Mathematics, Aligarh Muslim University, Aligarh- 202002, India
E-mail address: `javid.mm@amu.ac.in`