Journal of Nonlinear and Convex Analysis
Volume 26, Number 11, 2025, 3123–3143

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

NEW PROOFS OF FIXED POINT THEOREMS ON QUASI-METRIC SPACES

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Abstract. Based on our 2023 Metatheorem in Ordered Fixed Point Theory, we give very simple proofs of known fixed point theorems for various multimap classes on quasi-metric spaces. Such classes are represented by the Banach contractions, the Rus-Hicks-Rhoades maps, the Nadler multimaps, Covitz-Nadler multimaps, and others. Consequently, we obtain simple proofs of a large number of known theorems on extremal elements, fixed points, stationary points for several classes of maps or multimaps. Finally, we add some known theorems for which our Metatheorem does not work.

1. Introduction

Let (X, d) be a metric space. A Banach contraction $T : X \to X$ is a map such that, for some $r \in (0, 1)$,

$$d(Tx, Ty) \leq r\, d(x, y) \quad \text{for all } x, y \in X.$$

There have been appeared thousands of articles related to generalizations of the Banach contraction.

Recently, we introduced the Rus-Hicks-Rhoades (RHR) map $T : X \to X$ for some $r \in (0, 1)$ satisfying

$$d(Tx, T^2x) \leq r\, d(x, Tx) \text{ for all } x \in X.$$

We found that the class of the Rus-Hicks-Rhoades maps contains a large number of maps. See our recent works [31, 34, 36, 39]. Moreover, we found that the RHR maps characterize the metric completeness. See [36, 37].

In this paper, based on our 2023 Metatheorem in Ordered Fixed Point Theory [30, 32, 35], we give very simple proofs of fixed point theorems for various multimap classes in quasi-metric spaces. Such classes are represented by the Banach contractions, the Rus-Hicks-Rhoades maps, the Nadler multimaps, Covitz-Nadler multimaps, and others. Consequently, we obtain another large number of new theorems on extremal elements, fixed points, stationary points for several classes of maps or multimaps.

The present paper is organized as follows: In Section 2, we recall our 2023 Metatheorem as the basis of this paper. Section 3 devotes basic facts and theorems on quasi-metric spaces as preliminaries. In Section 4, we introduce equivalent formulations of the Covitz-Nadler theorem as an application of Metatheorem. Sections 5-8 devote to list examples of theorems satisfying each items of Metatheorem. Such theorems can be trivially proved by applying Metatheorem. In Section 9, we

2020 *Mathematics Subject Classification.* 06A75, 47H10, 54E35, 54H25, 58E30, 65K10.

Key words and phrases. Fixed point, quasi-metric space, Banach contraction, Rus-Hicks-Rhoades map, Nadler multimap, Covitz-Nadler multimap, stationary point, extremal element.

give a few example of fixed point theorems which can not applicable our Metatheorem. Finally, Section 10 is an epilogue.

2. Our 2023 Metatheorem

We obtained the following statement called the new 2023 Metatheorem in [30,32, 35]:

Metatheorem. *Let X be a set, A its nonempty subset, and $G(x,y)$ a sentence formula for $x,y \in X$. Then the following are equivalent:*

- (α) *There exists an element $v \in A$ such that the negation of $G(v,w)$ holds for any $w \in X\backslash\{v\}$.*
- ($\beta1$) *If $f : A \to X$ is a map such that for any $x \in A$ with $x \neq fx$, there exists a $y \in X\backslash\{x\}$ satisfying $G(x,y)$, then f has a fixed element $v \in A$, that is, $v = fv$.*
- ($\beta2$) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ such that for any $x \in A$ with $x \neq fx$, there exists a $y \in X\backslash\{x\}$ satisfying $G(x,y)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*
- ($\gamma1$) *If $f : A \to X$ is a map such that $G(x,fx)$ for any $x \in A$, then f has a fixed element $v \in A$, that is, $v = fv$.*
- ($\gamma2$) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $G(x,fx)$ for all $x \in A$ with $x \neq fx$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*
- ($\delta1$) *If $F : A \multimap X$ is a multimap such that, for any $x \in A\backslash Fx$ there exists $y \in X\backslash\{x\}$ satisfying $G(x,y)$, then F has a fixed element $v \in A$, that is, $v \in Fv$.*
- ($\delta2$) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for any $x \in A\backslash Fx$ there exists $y \in X\backslash\{x\}$ satisfying $G(x,y)$. Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in Fv$ for all $F \in \mathfrak{F}$.*
- ($\epsilon1$) *If $F : A \multimap X$ is a multimap satisfying $G(x,y)$ for any $x \in A$ and any $y \in Fx\backslash\{x\}$, then F has a stationary element $v \in A$, that is, $\{v\} = Fv$.*
- ($\epsilon2$) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that $G(x,y)$ holds for any $x \in A$ and any $y \in Fx\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = Fv$ for all $F \in \mathfrak{F}$.*
- (η) *If Y is a subset of X such that for each $x \in A\backslash Y$ there exists a $z \in X\backslash\{x\}$ satisfying $G(x,z)$, then there exists a $v \in A \cap Y$.*

For the proof, see [30,32,35].

This Metatheorem guarantees the truth of all items when one of them is true. Since 1985, we have shown nearly one hundred cases of such situations.

As an application of Metatheorem, we can apply it to preordered sets with $G(x,y)$ means $x \preceq y$ (resp. $y \preceq x$). A maximal or minimal element will be called an extremal element. Note that (α) can be applied to the existence of extremal points and that ($\beta2$) $-$ ($\epsilon2$) are equivalent to ($\beta1$) $-$ ($\epsilon1$), respectively.

3. Quasi-metric spaces

It is well-known that many key-results in Metric Fixed Point Theory hold for quasi-metric spaces. For example, Banach contraction principle, Nadler or Covitz-Nadler fixed point theorem, Ekeland variational principle, Caristi fixed point theorem, Takahashi minimization principle, and many others; see [31, 38].

We recall the following:

Definition 3.1. A *quasi-metric* on a nonempty set X is a function $q : X \times X \to \mathbb{R}^+ = [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$:

(a) (self-distance) $q(x, y) = q(y, x) = 0 \Longleftrightarrow x = y$;
(b) (triangle inequality) $q(x, z) \leq q(x, y) + q(y, z)$.

A *metric* on a set X is a quasi-metric q satisfying
(c) (symmetry) $q(x, y) = q(y, x)$ for all $x, y \in X$.

Remark 3.2.

(1) For quasi-metric spaces, the convergence of a sequence, Cauchy sequences, completeness, orbits, and orbital continuity are routinely defined; see Jleli-Samet [16].
(2) Such definitions also work for a topological space X and a function $q : X \times X \to \mathbb{R}^+ = [0, \infty)$ such that $q(x, y) = 0$ implies $x = y$ for $x, y \in X$
(3) Every quasi-metric induces a metric, that is, if (X, q) is a quasi-metric space, then the function $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = \max\{q(x, y), q(y, x)\}$$

is a metric on X; see Jleli-Samet [16].

Definition 3.3. Let (X, q) be a quasi-metric space and $T : X \to X$ a selfmap. The *orbit* of T at $x \in X$ is the set

$$O_T(x) = \{x, Tx, \dots, T^n x, \dots\}.$$

The space X is said to be *T-orbitally complete* if every right-Cauchy sequence in $O_T(x)$ is convergent in X. A selfmap T of X is said to be *orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\to\infty} T^n x = x_0 \Longrightarrow \lim_{n\to\infty} T^{n+1} x = Tx_0$$

for any $x \in X$.

In our previous works, we obtained the following Rus-Hicks-Rhoades (RHR) Contraction Principle:

Theorem P ([38, 39]). *Let (X, q) be a quasi-metric space and let $f : X \to X$ be an RHR map for $0 \leq \alpha < 1$; that is,*

$$q(fx, f^2x) \leq \alpha\, q(x, fx) \text{ for every } x \in X,$$

such that X is f-orbitally complete. Then

(i) *for each* $x \in X$*, there exists a point* $x_0 \in X$ *such that*

$$\lim_{n\to\infty} f^n x = x_0,$$

$$q(f^n x, x_0) \le \frac{\alpha^n}{1-\alpha} q(x, fx), \quad n = 1, 2, \ldots,$$

$$q(f^n x, x_0) \le \frac{\alpha}{1-\alpha} q(f^{n-1}x, f^n x), \quad n = 1, 2, \ldots,$$

and

(ii) x_0 *is a fixed point of* f *if and only if* f *is orbitally continuous at* x_0*.*

Corollary P.1. *Let* f *be a continuous selfmap of a complete quasi-metric space* (M, q) *satisfying*

$$q(fx, f^2x) \le \alpha\, q(x, fx) \quad \textit{for every } x \in M,$$

where $0 < \alpha < 1$*. Then, for each* $x \in X$*,* f *has a fixed point* $x_0 \in X$ *satisfying* (i) *of Theorem P.*

Later, we found that, for metric spaces, some refined versions of Theorem P were given by I.A. Rus [48, 49] in 2016, where an RHR map was called a *graphic contraction*.

The following in [38] extends the standard Banach contraction principle:

Theorem Q. *Let* (X, q) *be a quasi-metric space and let* $T : X \to X$ *be a contraction, that is,*

$$q(Tx, Ty) \le r\, q(x, y) \textit{ for every } x, y \in X,$$

with $0 < r < 1$*. If* (X, q) *is T-orbitally complete, then* T *has a unique fixed point* $x_0 \in X$*.*

Remark 3.5. Definition 3.2 also works for a topological space X and a function $q : X \times X \to \mathbb{R}^+ = [0, \infty)$ such that $q(x, y) = 0$ implies $x = y$ for $x, y \in X$

Every quasi-metric induces a metric, that is, if (X, q) is a quasi-metric space, then the function $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = \max\{q(x, y), q(y, x)\}$$

is a metric on X; see Jleli-Samet [16].

The following is given in [31, 38]:

Theorem 3.6. *A selfmap* $T : X \to X$ *of a quasi-metric space* (X, q) *has a fixed point* $z \in X$ *if and only if* z *is a fixed point of the selfmap* T *of the induced metric space* (X, d)*.*

4. A form of our new 2023 Metatheorem

Let (X,q) be a quasi-metric space and $\mathrm{Cl}(X)$ denote the family of all nonempty closed subsets of X (not necessarily bounded). For $A, B \in \mathrm{Cl}(X)$, set

$$H(A,B) = \max\{\sup\{q(a,B) : a \in A\},\ \sup\{q(b,A) : b \in B\}\},$$

where $q(a,B) = \inf\{q(a,b) : b \in B\}$. Then H is called a generalized Hausdorff quasi-distance on $\mathrm{Cl}(X)$ induced by q and it may have infinite values. Then the continuity of a map $T : X \to \mathrm{Cl}(X)$ is obviously defined.

Recently, as a basis of Ordered Fixed Point Theory [30], we obtained the 2023 Metatheorem and Theorem H including Nadler's fixed point theorem [28] in 1969 and its extended version by Covitz-Nadler [9] in 1970.

The following is the new version of Theorem H in [39] which assumes the continuity of all involved maps and multimaps:

Theorem H. *Let (X,q) be a quasi-metric space and $0 < r < 1$. All maps $f : X \to X$ and multimaps $T : X \to \mathrm{Cl}(X)$ are supposed to be continuous.*

Then the following statements are equivalent:

- (0) *(X,q) is complete.*
- (α) *For a multimap $T : X \to \mathrm{Cl}(X)$, there exists an element $v \in X$ such that $H(Tv,Tw) > r\,q(v,w)$ for any $w \in X\backslash\{v\}$.*
- (β) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ such that, for any $x \in X\backslash\{fx\}$, there exists a $y \in X\backslash\{x\}$ satisfying $q(fx,fy) \le r\,q(x,y)$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*
- (γ) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ satisfying $q(fx, f^2x) \le r\,q(x,fx)$ for all $x \in X\backslash\{fx\}$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*
- (δ) *Let $\mathfrak{F}$ be a family of multimaps $T : X \to \mathrm{Cl}(X)$ such that, for any $x \in X\backslash Tx$, there exists $y \in X\backslash\{x\}$ satisfying $H(Tx,Ty) \le r\,q(x,y)$. Then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v \in Tv$ for all $T \in \mathfrak{F}$.*
- (ϵ) *If $\mathfrak{F}$ is a family of multimaps $T : X \to \mathrm{Cl}(X)$ satisfying $H(Tx,Ty) \le r\,q(x,y)$ for all $x \in X$ and any $y \in Tx\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in X$, that is, $\{v\} = Tv$ for all $T \in \mathfrak{F}$.*
- (η) *If Y is a subset of X such that for each $x \in X\backslash Y$ there exists a $z \in X\backslash\{x\}$ satisfying $H(Tx,Tz) \le r\,q(x,z)$ for a $T : X \to \mathrm{Cl}(X)$, then there exists a $v \in X \cap Y = Y$.*

Proof. The equivalency $(\alpha) - (\eta)$ follows from our 2023 Metatheorem in [30,32,35]. When $\mathfrak{F}$ is a singleton, (β)-(ϵ) are denoted by ($\beta1$)-($\epsilon1$), respectively, They are also logically equivalent to (α)-(η). Note that ($\gamma1$) follows from Theorem P for quasi-metric spaces. The equivalency of (0) and ($\gamma1$) is given in [36]. Then Theorem H holds. $\square$

Remark 4.1

1. ($\beta1$) implies the Banach contraction principle, which does not characterize the metric completeness.

(2) Moreover, ($\delta 1$) and ($\epsilon 1$) extend the well-known theorems of Nadler [28] and Covitz-Nadler [9].

The following ($\gamma 1$) of Theorem H is the basis in the present article and will be called the RHR theorem, which characterizes the metric completeness.

Theorem H($\gamma 1$)**.** *Let (X, q) be a complete quasi-metric space and $0 < r < 1$.*

($\gamma 1$) *If a continuous map $f : X \to X$ satisfies $q(fx, f^2x) \leq r\, q(x, fx)$ for all $x \in X \backslash \{fx\}$, then f has a fixed element $v \in X$, that is, $v = fv$.*

Consequently, this is a close relative of Theorems of Rus [47] and Hicks-Rhoades [14]. They are all close relatives of the Banach contraction principle in 1922; see [4].

5. The α-type theorem

We have a single-valued version of Theorem H(α) as follows:

Theorem H($\alpha 1$)**.** *Let (X, q) be a complete quasi-metric space and $0 < r < 1$. For a continuous map $f : X \to X$, there exists an element $v \in X$ such that $q(fv, fw) > r\, q(v, w)$ for any $w \in X \backslash \{v\}$.*

This is also equivalent to all items in Theorem H. In some sense, this shows that the Banach contraction principle does not characterize the metric completeness.

6. The β-class of the Banach type

In this section, we want to collect certain extensions of the Banach contraction satisfying Theorem H(β). There have known huge number of extensions of the Banach contraction principle, however, we have known only one example of the β-class.

Banach [4] in 1922

Let $f : X \to X$ be a contraction of a metric space (X, d), that is, there exists $r \in (0, 1)$ such that

$$d(Tx, Ty) \leq r\, d(x, y) \quad \text{for all } x,\ y \in X.$$

Recall that the Banach contraction principle follows from Theorems P, Q and others in Sections 3 and 4.

From now on, the numbers attached to Definitions, Theorems, Corollaries and others are the same ones for the original sources.

7. The γ-class of the Rus-Hicks-Rhoades type

Many generalizations of the Banach principle are of the Rus-Hicks-Rhoades type; see [36]. Rhoades [45] in 1978 noted that the analogues of most of the conditions in his well-known list [44] could be extended to the RHR type of contractive definitions; see also [41].

In this section, we list some typical old and new examples of Theorem H(γ). Most of examples hold for quasi-metric spaces, but we state their original forms.

Kannan [17] in 1969

Recall the following:

Theorem 1.1. (Kannan) *Let (X, d) be a complete metric spaces and $T : X \to X$ be a Kannan contraction mapping, i.e.,*

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X,$$

where $\lambda \in [0, 1/2)$. Then T has a unique fixed point.

Comment: This is a simple consequence of Theorem P for quasi-metric spaces. In fact, for $y = Tx$, we have

$$d(Tx, T^2x) \leq \frac{\lambda}{1-\lambda} d(x, Tx) \text{ and } 0 \leq \frac{\lambda}{1-\lambda} < 1.$$

Kannan's example does not require the continuity of the map at every point, although maps satisfying his condition are continuous at fixed points. Hence Kannan's example follows from Theorem P.

Reich [43] in 1971

The following theorem proved by Reich generalizes Banach's fixed point theorem and Kannan's fixed point theorem.

Theorem. (Reich) *Let f be a selfmap on a complete metric space (X, d). If there exist constants $a, b, c \in [0, 1)$ with $a + b + c < 1$ such that*

$$d(f(x), f(y)) \leq ad(x, f(x)) + bd(y, f(y)) + cd(x, y) \quad for \ all \ x, \ y \in X,$$

then f has a unique fixed point.

Comment: Note that f is an RHR map and that Theorems P can be applicable to f. The uniqueness follows from the contractive condition.

Rus, Reich, Ćirić in 1971-2001

This is from the following Rus-Reich-Ćirić theorem in Karapinar-Agarwal [22] in 2019:

Theorem 1.3. *Let (X, d) be a complete metric spaces and $T : X \to X$ be a Rus-Reich-Ćirić contraction mapping, i.e.,*

$$d(Tx, Ty) \leq \lambda[d(x, y) + d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$, where $\lambda \in [0, 1/3)$. Then T has a unique fixed point.

Comment: This is a simple consequence of the RHR theorem for quasi-metric spaces. In fact, for $y = Tx$, we have

$$d(Tx, T^2x) \leq \frac{2\lambda}{1-\lambda} d(x, Tx) \text{ and } 0 \leq \frac{2\lambda}{1-\lambda} < 1.$$

Therefore, Theorem P can be applicable to T.

Ćirić [7] in 1974

Theorem 1.1. (Ćirić) *Let (M, d) be a quasi-metric space, and let $T : M \to M$ be a given mapping. Suppose that the following conditions are satisfied:*

(i) *T is orbitally continuous on M;*

(ii) (M, d) *is T-orbitally complete;*
(iii) *There exists a constant* $q \in (0, 1)$ *such that*

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(y, Tx)\} \leq q\, d(x, y), \quad \text{for all } x, y \in M.$$

Then, for every $x \in M$*, the Picard sequence* $\{T^n x\}$ *converges to a fixed point of* T*.*

Comment: For $y = Tx$, we can consider the following:

$$\min\{d(Tx, T^2x), d(x, Tx), d(Tx, T^2x)\} - \min\{d(x, T^2x), d(Tx, Tx)\} \leq q\, d(x, Tx),$$

for all $x, y = Tx \in M$.

Case 1: If $d(Tx, T^2x) \leq d(x, Tx)$, then $d(Tx, T^2x) \leq q\, d(x, Tx)$, possible.
Case 2: If $d(Tx, T^2x) \geq d(x, Tx)$, then $d(x, Tx) \leq q\, d(x, Tx)$ implies $x = Tx$.

In any case, Theorem P is applicable to T.

Dass and Gupta [10] in 1975

Theorem 1. (Dass-Gupta) *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping. If there exist* $k_1, k_2 \in [0, 1)$*, with* $k_1 + k_2 < 1$ *such that*

$$d(Tx, Ty) \leq k_1 \cdot d(y, Ty)\frac{1 + d(x, Tx)}{1 + d(x, y)} + k_2 \cdot d(x, y),$$

for all $x, y \in X$*, then* T *has a unique fixed point* $u \in X$ *and the sequence* $\{T^n x\}$ *converges to the fixed point* u *for all* $x \in X$*.*

Comment: Note that T is an RHR map and can be applied Theorem P. The uniqueness follows from the contractive condition.

Jaggi [15] in 1977

Consider the following:

Theorem 2. (Jaggi) *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a continuous mapping. If there exist* $k_1, k_2 \in [0, 1)$*, with* $k_1 + k_2 < 1$ *such that*

$$d(Tx, Ty) \leq k_1 \cdot \frac{d(x, Tx)}{d(y, Ty)} d(x, y) + k_2 \cdot d(x, y)$$

for all distinct $x, y \in X$*, then* T *possesses a unique fixed point in* X*.*

Comment: For $y = Tx$, we have the following

$$d(Tx, T^2x) \leq k_1 \cdot \frac{d(x, Tx)}{d(Tx, T^2x)} d(x, Tx) + k_2 \cdot d(x, Tx),$$

which implies $d(Tx, T^2x) \leq (k_1 + k_2) d(x, Tx)$.
Hence T is an RHR map and can be applied Theorems P and H(γ1).

Ran and Reurings [42] in 2004

In this paper, the following fixed point theorem in a partially ordered metric space is proved:

Theorem 1.2. (Ran-Reurings) *Let $(M, \leq)$ be an ordered set and d be a metric on M such that (M, d) is a complete metric space. Let $U : M \to M$ be a nondecreasing mapping, i.e. $Ux \leq Uy$, for every x, $y \in M$ with $x \leq y$. Suppose that there exists $x_0 \in M$ with $x_0 \leq Ux_0$ and $L \in [0, 1)$ such that*

$$d(Ux, Uy) \leq L\, d(x, y), \text{ for every } x, \ y \in M \text{ with } x \leq y.$$

If U is continuous, then it has a fixed point in M.

Comment: Note that U is an RHR map and can be applied Theorems P and H(γ1).

Suzuki [50] in 2008

Suzuki generalized the Banach contraction principle as follows:

Theorem 2. (Suzuki) *Let (X, d) be a complete metric space and T be a mapping on X. Define a nonincreasing function θ from $[0, 1)$ onto $(1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq (\sqrt{5} - 1)/2, \\ (1 - r)r^{-2} & \text{if } (\sqrt{5} - 1)/2 \leq r \leq 2^{-1/2}, \\ (1 + r)^{-1} & \text{if } 2^{-1/2} \leq r < 1. \end{cases} \tag{7.1}$$

Assume there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq r\, d(x, y)$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in X$.

Comment: Note that T is an RHR map and Theorems P or H(γ1) can be applied. Now it suffices to show the uniqueness of the fixed point z. If $w \in X$ is a fixed point, then

$$\theta(r)d(z, Tz) \leq d(z, w) \text{ implies } d(Tz, Tw) \leq r\, d(z, w).$$

Hence $d(z, w) = d(Tz, Tw) \leq r\, d(z, w)$ and consequently $d(z, w) = 0$. This is a simple proof of Theorem 2.

There have been appeared a large number of variants of Suzuki's theorem. Many of them can be improved by easy proofs as shown as above.

In order to show uselessness of Theorems 2 and 3 in [50] and other works of Suzuki, let us consider any function $\theta' : [0, \infty) \to [0, 1]$:

Theorem 2.′ *Replace the function θ in Theorem 2 by θ'. Then the conclusion of Theorem 2 holds.*

Proof. Note that, by putting $y = Tx$, T becomes an RHR map. Then by Theorems P or H(γ1), T has a fixed point and its uniqueness follows as in our proof of Theorem 2. □

Altun and Erduran [1] in 2011

The authors present a fixed-point theorem for a single-valued map in a complete metric space using implicit relation, which is a generalization of several previously stated results including that of Suzuki [50].

The aim of this paper is to generalize the above results using the implicit relation technique in such a way that

$$F(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\le 0,$$

for $x,\ y\in X$, where $F:[0,\infty)^6\to\mathbb{R}$ is a function as given in Section 2 with 5 examples.

Theorem 3.1. *Let (X,d) be a complete metric space, and let T be a mapping on X. Define a nonincreasing function $\theta:[0,1)\to(1/2,1]$ as in Suzuki* [50]*. Assume that there exists an F as above, such that $\theta(r)d(x,Tx)\le d(x,y)$ implies*

$$F(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\le 0,$$

for all $x,\ y\in X$, then T has a unique fixed-point z and $\lim_n T^n x=z$ holds for every $x\in X$.

Comment: In the proof, the authors showed that $d(Tx,T^2x)\le r\,d(x,Tx)$ for all $x\in X$, that is, T is an RHR map. Hence T has a fixed point by Theorem P and its uniqueness follows from properties of F.

Khojasteh, Abbas, Costache [25] in 2014

Theorem 1. *Let (X,d) be a complete metric space and let T be a mapping from X into itself. Suppose that T satisfies the following condition:*

$$d(Tx,Ty)\le\frac{d(x,Ty)+d(y,Tx)}{d(x,Tx)+d(y,Ty)+1}d(x,y)$$

for all $x,\ y\in X$. Then

(a) *T has at least one fixed point $\dot{x}\in X$,*
(b) *$\{T^n x\}$ converges to a fixed point, for all $x\in X$;*
(c) *if $\dot{x}$ and $\dot{y}$ are distinct fixed points of T, then $d(\dot{x},\ \dot{y})\le 1/2$.*

Comment: Note that T is an RHR map and can be applicable Theorem P. The (c) follows from the contractive condition.

Karapinar [18] in 2018

The author started his results by the generalization of the definition of Kannan type contraction via interpolation notion, as follows:

Definition 2.1. Let (X,d) be a metric space. We say that the self-mapping $T:X\to X$ is an interpolative Kannan type contraction, if there exist a constant $\lambda\in[0,1)$ and $\alpha\in(0,1)$ such that

$$d(Tx,Ty)\le\lambda[d(x,Tx)]^{\alpha}\cdot[d(y,Ty)]^{1-\alpha}$$

for all $x,\ y\in X$ with $x\neq Tx$.

Theorem 2.2. *Let (X,d) be a complete metric space and T be an interpolative Kannan type contraction. Then T has a unique fixed point in X.*

Comment: Later, the author withdraw the uniqueness of fixed point. For $y = Tx$, we have

$$d(Tx, T^2x) \leq \lambda[d(x, Tx)]^\alpha \cdot [d(Tx, T^2x)]^{1-\alpha} \Longrightarrow [d(Tx, T^2x)]^\alpha \leq \lambda[d(x, Tx)]^\alpha$$
$$\Longrightarrow d(Tx, T^2x) \leq \lambda^{1/\alpha} d(x, Tx) \text{ with } 0 < \lambda^{1/\alpha} < 1.$$

Therefore, T is an RHR map for which Theorems P and H(γ1) can be applicable. Hence Theorem 2.2 can be stated for T-orbitally complete quasi-metric spaces.

Karapinar, Agarwal, and Aydi [23] in 2018

As a correction of Theorem 1 (Theorem 2.2 of the previous paper), the authors should state:

Theorem 2. *Let (X, ρ) be a complete metric space. A self-mapping $T : X \to X$ possesses a fixed point in X, if there exist constants $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$\rho(T\zeta, T\eta) \leq \lambda[\rho(\zeta, T\zeta)]^\alpha \cdot [\rho(\eta, T\eta)]^{1-\alpha}$$

for all $\zeta,\ \eta \in X \backslash Fix(T)$.

The following theorem was proved by Reich, Rus and Ćirić independently to combine and improve both Banach and Kannan fixed point theorems.

Theorem 3. *In the framework of a complete metric space (X, ρ), if $T : X \to X$ forms a Reich-Rus-Ćirić contraction mapping, i.e.,*

$$\rho(T\zeta, T\eta) \leq \lambda[\rho(\zeta, \eta) + \rho(\zeta, T\zeta) + \rho(\eta, T\eta)],$$

for all $\zeta,\ \eta \in X$, where $\lambda \in [0, 1/3)$, then T possesses a unique fixed point.

Notice that several variations of Reich contractions (in Theorem 3) can be stated. We may state the following:

$$\rho(T\zeta, T\eta) \leq a\rho(\zeta, \eta) + b\rho(\zeta, T\zeta) + c\rho(\eta, T\eta),$$

where $a, b, c \in (0, \infty)$ such that $0 \leq a + b + c < 1$.

In this paper, the authors shall investigate the validity of the interpolation approach for Reich contractions in the context of partial metric spaces that was introduced by Matthews.

Comment: We have two examples of RHR maps in [23] as follows:

For the Reich-Rus-Ćirić contraction, by putting $\eta = T\zeta$, we have

$$\rho(T\zeta, T^2\zeta) \leq 2\lambda\, \rho(\zeta, T\zeta) + \lambda\, \rho(T\zeta, T^2\zeta)$$
$$\Longrightarrow \rho(T\zeta, T^2\zeta) \leq \frac{2\lambda}{1-\lambda}\, \rho(\zeta, T\zeta),\ \ \lambda \in [0, 1/2).$$

For the variation of the Reich contraction, by putting $\eta = T\zeta$, we have

$$\rho(T\zeta, T^2\zeta) \leq (a + b)\rho(\zeta, T\zeta) + c\, \rho(T\zeta, T^2\zeta)$$
$$\Longrightarrow \rho(T\zeta, T^2\zeta) \leq \frac{a+b}{1-c}\, \rho(\zeta, T\zeta),\ \ 0 < \frac{a+b}{1-c} < 1.$$

Therefore, Theorem P can be applied to such two examples for T-orbitally complete quasi-metric spaces.

Karapinar, Alqahtani, Aydi [24] in 2018

By using an interpolative approach, we recognize the Hardy-Rogers fixed point theorem in the class of metric spaces. The obtained result is supported by some examples. We also give the partial metric case, according to our result.

One of generalizations of the Banach Contraction Principle is due to Hardy-Rogers as follows; see [13].

Theorem 3. *Let (X,d) be a complete metric space. Let $T: X \to X$ be a given mapping such that*

$$d(T\theta,T\vartheta) \leq \alpha d(\theta,\vartheta) + \beta d(\theta,T\theta) + \gamma d(\vartheta,T\vartheta) + \frac{\delta}{2}[d(\theta,T\vartheta) + d(\vartheta,T\theta)],$$

for all $\theta,\vartheta \in X$, where $\alpha,\beta,\gamma,\delta$ are non-negative reals such that $\alpha+\beta+\gamma+\delta<1$. Then T has a unique fixed point in X.

In this paper, we introduce the concept of interpolative Hardy-Rogers type contractions, and provide some examples illustrating the obtained result. We also extend our obtained result to partial metric spaces.

Definition 2. Let (X,d) be a metric space. We say that the self-mapping $T: X \to X$ is an *interpolative Hardy-Rogers type contraction* if there exists $\lambda \in [0,1)$ and $\alpha,\beta,\gamma \in (0,1)$ with $\alpha+\beta+\gamma<1$, such that

$$d(T\theta,T\vartheta) \leq \lambda[d(\theta,\vartheta)]^{\beta} \cdot [d(\theta,T\theta)]^{\alpha} \cdot [d(\vartheta,T\vartheta)]^{\gamma} \cdot [\frac{1}{2}(d(\theta,T\vartheta)+d(\vartheta,T\theta))]^{1-\alpha-\beta-\gamma}$$

for all $\theta,\vartheta \in X\backslash Fix(T)$.

Theorem 4. *Let (X,d) be a complete metric space and T be an interpolative Hardy-Rogers type contraction. Then, T has a fixed point in X.*

Comment: For the map in Theorem 3 is an RHR map by putting $\vartheta = T\theta$. In fact, we have

$$\begin{aligned} d(T\theta,T^2\theta) &\leq (\alpha+\beta)d(\theta,T\theta) + \frac{\delta}{2}[d(\theta,T\theta)+d(T\theta,T^2\theta)] \\ &\Longrightarrow (1-\gamma-\delta/2)d(T\theta,T^2\theta) \leq (\alpha+\beta+\delta/2)d(\theta,T\theta), \end{aligned}$$

where $\alpha+\beta+\delta/2 < 1-\gamma-\delta/2$.

For an interpolative Hardy-Roger contraction, by putting $\vartheta = T\theta$, we have

$$\begin{aligned} &d(T\theta,T^2\theta) \leq \lambda[d(\theta,T\theta)]^{\alpha+\beta} \cdot [d(T\theta,T^2\theta)]^{\gamma} \cdot [\max\{d(\theta,T\theta),d(T\theta,T^2\theta)\}]^{1-\alpha-\beta-\gamma} \\ &\Longrightarrow d(T\theta,T^2\theta)^{1-\gamma} \leq \lambda[d(\theta,T\theta)]^{\alpha+\beta+1-\alpha-\beta-\gamma} \text{ or} \\ &\qquad [d(T\theta,T^2\theta)]^{1-\gamma} \leq \lambda[d(\theta,T\theta)]^{\alpha+\beta} \cdot [d(T\theta,T^2\theta)]^{1-\alpha-\beta-\gamma} \\ &\Longrightarrow d(T\theta,T^2\theta)^{1-\gamma} \leq \lambda[d(\theta,T\theta)]^{1-\gamma} \text{ or } d(T\theta,T^2\theta)^{\alpha+\beta} \leq \lambda[d(\theta,T\theta)]^{\alpha+\beta} \\ &\Longrightarrow d(T\theta,T^2\theta) \leq \lambda^p d(\theta,T\theta) \text{ or } d(T\theta,T^2\theta) \leq \lambda^q d(\theta,T\theta). \end{aligned}$$

Note that $0<\lambda^p := \lambda^{\frac{1}{1-\gamma}} < 1$ and $0<\lambda^q := \lambda^{\frac{1}{\alpha+\beta}} < 1$.

Consequently, Theorem P can be applied to such two examples on T-orbitally complete quasi-metric spaces.

Karapinar [19] in 2019

The author collects and combine several non-unique fixed point results in the context of several distinct abstract spaces. The main goal is to give a brief background on the topic as well as the to underline the importance of the non-unique fixed points. By using the auxiliary functions, some of the given results are reformulated in a more general form to cover the existing results on the topic in the literature.

Given nonunique fixed point theorems are due to Ćirić (1974), Achari (1976), Pachpatte (1979), Ćirić-Jotić (1998), and Karapinar [19] in 2019. For example,

Theorem 5. (Karapinar) *Let $T : X \to X$ be an orbitally continuous self-map on the T-orbitally complete metric space (X, d). Suppose there exist real numbers a_1, a_2, a_3, a_4, a_5 and a self mapping $T : X \to X$ which satisfies $E(x, y) \leq a_4 d(x, y) + a_5 d(x, T^2x)$, where*

$$E(x, y) := a_1 d(Tx, Ty) + a_2[d(x, Tx) + d(y, Ty)] + a_3[d(y, Tx) + d(x, Ty)],$$

for all $x, y \in X$. Then, T has at least one fixed point.

Comment: Here $\{a_i\}_{i=1}^5$ are chosen to hold the RHR condition $d(Tx, T^2x) \leq r\, d(x, Tx)$ with $r \in [0, 1)$ for all $x \in X$. Note that Theorem 5 holds for quasi-metric spaces, and Theorems P and H(γ1) can be applied.

Miñana and Valero [27] in 2019

Many G-metric fixed point results can be retrieved from classical ones given in the (quasi-)metric framework. Indeed, many G-contractive conditions can be reduced to a quasi-metric counterpart assumed in the statement of celebrated fixed point results. In this paper, we show that the existence of fixed points for the most part in the aforesaid G-metric fixed point results is guaranteed by a very general celebrated result by Park, even when the G-contractive condition is reduced to a quasi-metric one which is not considered as a contractive condition in any celebrated fixed point result. Moreover, in all those cases in which a quasi-metric contractivity can be raised, we show that the uniqueness of the fixed point is also derived from it. ...

We are able to show that most fixed point results obtained in G-metric spaces can be deduced from a fixed point result stated in quasi-metric spaces obtained by Park in [29]. To this end, let us recall such a result.

Theorem 1. *Let (X, τ) be a topological space, let $d : X \times X \to [0, \infty[$ be a continuous map, such that $d(x, y) = 0 \iff x = y$, and let $f : X \to X$ be a map. Suppose that there exist $x,\ x_0 \in X$, such that the following conditions hold:*

1. $\lim_{n\to\infty} d(f^n(x_0), f^{n+1}(x_0)) = 0$;
2. $(f^n(x_0))_{n\in\mathbb{N}}$ *converges to x with respect to τ;*
3. *f is orbitally continuous at x with respect to τ.*

Then $x \in \mathrm{Fix}(f) = \{y \in X : f(y) = y\}$.

It must be stressed that Park's original version of the preceding result was stated for lower semicontinuous mappings d. However, we have focused our attention on continuous ones, because it is enough for our announced purpose.

Corollary 1. *Let (X, G) be a G-metric space and let $f : X \to X$ be a mapping. Suppose that there exist $x,\ x_0 \in X$, such that the following conditions hold:*

1. $\lim_{n\to\infty} d_G(f^n(x_0), f^{n+1}(x_0)) = 0$;
2. $(f^n(x_0))_{n\in\mathbb{N}}$ *converges to x with respect to τ_{d_G};*
3. f *is orbitally continuous at x with respect to τ_{d_G}.*

Then $x = f(x)$.

Moreover, the authors have shown that the existence of fixed points in many of the aforesaid G-metric fixed point results is a consequence of Park's celebrated result, even when the G-contractive condition is reduced to a quasi-metric one that is not considered as a contractive condition in the statement of any known fixed point result.

Aouine and Aliouche [3] in 2021

Abstract: We prove unique fixed point theorems for a self-mapping in complete metric spaces and that the fixed point problem is well-posed. Examples are provided to illustrate the validity of our results and we give some remarks about three papers.... Afterwards, we apply our result to study the possibility of optimally controlling the solution of an ordinary differential equation via dynamic programming.

Definition 1. (Reich-Zaslavski) Let (X, d) be a metric space and $T : X \to X$ a mapping. The fixed point problem of T is said to be well-posed if

i) T has a unique fixed point z in X,
ii) for any sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} d(Ty_n, y_n) = 0$, we have $\lim_{n\to\infty} d(y_n, z) = 0$.

Theorem 4. *Let $(X; d)$ be a complete metric space and T a mapping from X into itself satisfying the following condition*

$$d(Tx, Ty) \le \frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1} \max\{d(x, Tx), d(y, Ty)\}$$

for all $x,\ y \in X$.

Then

a) *T has a unique fixed point $z \in X$,*
b) *The fixed point problem of T is well-posed, and*
c) *T is continuous at z.*

Comment: For $y = Tx$,

$$\frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1} < 1$$

and hence $\max\{d(x, Tx), d(y, Ty)\} = d(x, Tx)$. Therefore T is an RHR map and applicable Theorems P and H(γ1).

Aouine [2] in 2022

In this paper, we prove a fixed point theorem for p-contraction mappings in partially ordered metric spaces. As an application, we investigate the possibility of optimally controlling the solution of the ordinary differential equations.

Definition 4. Let (X, d) be a metric space. A mapping $T : Y \subset X \to X$ is said to be a *metric p-contraction* (or simply *p-contraction* mapping if Y is T-invariant and it satisfies the following inequality:

$$d(T(x), T^2(x)) \leq p(x)d(x, T(x)) \ for \ all \ x \in Y,$$

where $p : Y \to [0, 1]$ is a function such that $p(x) < 1$ for all $x \in Y$ and $\sup_{x \in Y} p(Tx) = \alpha < 1$.

Further, if $\bigcap_{n=0}^{\infty} T^n(Y)$ is a singleton set, where $T^n(Y) = T(T^{n-1}(Y))$ for each $n \in \mathbb{N}$ and $T^0(Y) = Y$, then T is said to be a *strong p-contraction.*

Comment: Note that T is an RHR map.

Romaguera [46] in 2022

The above theorem suggests the following natural question (see Section 2 for notation and concepts).

Question. Let F be a self map of a bicomplete (or at least, Smyth complete) quasi-metric space (X, d) and let $c \in (0, 1)$ be a constant, such that for every $x, y \in X$, the following contraction condition holds:

$$d(x, Fx) \leq 2d(x, y) \Longrightarrow d(Fx, Fy) \leq c\, d(x, y).$$

Under the above assumptions, does F admit a fixed point?

In Section 3, we will give an example showing that this question has a negative answer in the general quasi-metric context.

Comment: Note that F is an RHR map.

Petrov [42] in 2023

Petrov defined generalized Kannan type maps and obtained:

Proposition 1.3. *Let (X, d) be a metric space and let $T : X \to X$ be a generalized Kannan type metric with some $\lambda \in [0, 2/3)$. If x is an accumulation point of X and T is continuous at x, then the inequality*

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)/2]$$

holds for all points $y \in X$.

Theorem 2.1. *Let $(X, d), |X| > 3,$ be a complete metric space and let the mapping $T : X \to X$ satisfy the following two conditions:*

(i) *$T(T(x)) \neq x$ for all $x \in X$ such that $Tx \neq x$.*
(ii) *T is a generalized Kannan-type mapping on X.*

Then T has a fixed point. The number of fixed points is at most two.

Comment: For any point $x \in X$, the inequality in Proposition 1.3 for $y = Tx$ implies

$$\left(1 - \frac{\lambda}{2}\right) d(Tx, T^2x) \leq \lambda\, d(x, Tx) \text{ or } d(Tx, T^2) \leq \alpha\, d(x, Tx) \text{ where } \alpha = \frac{\lambda}{2-\lambda} \in [0,1).$$

Hence, T is an RHR map and has a fixed point by Theorems P and H(γ1).

Suppose that there exists at least three pairwise distinct fixed points x, y and z. Then $Tx = x$, $Ty = y$ and $Tz = z$, which contradicts to the definition of generalized Kannan-type map.

Anyway, the concept of generalized Kannan-type maps can be replaced by RHR maps.

8. The δ-ϵ-class of Covitz-Nadler type

Theorem H((δ)-(ϵ) generalize the celebrated multi-valued versions of the Banach contraction versions due to Nadler and Covitz-Nadler.

Nadler [28] in 1969

Some fixed point theorems for multi-valued contraction mappings (m.v.c.m.) are proved, as well as a theorem on the behaviour of fixed points as the mappings vary.

Theorem 5. (Nadler) *Let (X, d) be a complete metric space. If $F : X \to BC(X)$ is a m.v.c.m., then F has a fixed point.*

Finally in [28], the following is added:

5. Added in proof. In a forthcoming paper with Professor Covitz on multi-valued contraction mappings in generalized metric spaces the author has extended Theorems 5 and 6 of this paper to mappings into $\mathrm{Cl}(X) = \{C : C$ is a nonempty closed subset of $X\}$ with the generalized Hausdorff distance. These results give an affirmative answer to problems posed in this remark and show that even boundedness of point images is not necessary.

Comment: Nadler's and Covitz-Nadler's fixed point theorems are consequences of Theorem H(δ) with simple elementary proofs. In fact, Theorems H(δ)–(ϵ) are new theorems.

Fierro and Pizarro [12] in 2023

From Text: In this note, we prove a fixed point existence theorem for set-valued functions by extending the usual Banach orbital condition concept for single valued mappings. As we show, this result applies to various types of set-valued contractions existing in the literature.

Given a multimap $T : X \to \mathrm{BC}(X)$, $x_0 \in X$, and $k \in [0, 1)$, we say T satisfies the *multivalued Banach orbital* (MBO) condition at x_0 with constant k, whenever for all $x \in O(x_0, T)$, $\inf_{y \in Tx} d(y, Ty) \leq k\, d(x, Tx)$, and that, T satisfies the *strong multivalued Banach orbital* (SMBO) condition at x_0 with constant k, whenever for all $x \in O(x_0, T)$, $\sup_{y \in Tx} d(y, Ty) \leq k\, d(x, Tx)$.

Theorem 3.1. *Let $T : X \to \mathrm{BC}(X)$ be a set-valued mapping satisfying the MBO condition at $x_0 \in X$ with constant k. Then, there exist $x^* \in X$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x^* such that, for all $n \in \mathbb{N}$, $x_{n+1} \in Tx_n$, and the following two conditions hold:*

(i) $d(x_n, Tx_n) \leq d(x_n, x_{n+1}) \leq k^n\, d(x_0, Tx_0)$ *and*
(ii) $d(x^*, Tx_n) \leq \{k^{n+1}/(1-k)\} d(x_0, Tx_0)$*, for all* $n \in \mathbb{N}$.

Moreover, the following conditions are equivalent:

(iii) $x^* \in Tx^*$,
(iv) G_T *is* (x_0, T)*-orbitally lower semicontinuous at* x^**, and*
(v) *the function* $h : X \to \mathbb{R}$*, defined by* $h(x) = d(x, Tx)$*, is lower semicontinuous at* x^*.

Comment: The authors only claimed the equivalency of (iii)-(v). In the present article, we showed that (iii)-(v) actually hold when X is T-orbitally complete quasi-metric space.

9. Maps not belonging any of the above classes

There are relatively small number of maps which does not belong to any of the classes in our Metatheorem. We give only a few example of such maps.

Ćirić [7] in 1974

Ćirić proved, in Theorem 1 of [7], a celebrated fixed point theorem which we state as follows:

Theorem 1. (Ćirić) *Let T be a self map of a complete metric space (X, d). If there is a constant $\alpha \in (0, 1)$ such that*

$$d(Tx, Ty) \leq \alpha \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for all $x,\ y \in X$, then T has a unique point $z \in X$ and $d(z, T^n x_0) \to 0$ as $n \to \infty$, for all $x_0 \in X$.

Comments: The map T is not in β-class. However, for $y = Tx$, we have

$$d(Tx, T^2x) \leq \alpha \max\{d(x, Tx), d(x, Tx) + d(Tx, T^2x)\} \leq \frac{\alpha}{1-\alpha} d(x, Tx).$$

Hence, for $\alpha \in [0, 1/2)$, Theorem 1 follows from Theorem P.

Bogin [5] in 1976

Theorem 1. (Bogin) *Let (X, d) be a complete metric space and $F : X \to X$ a mapping satisfying for each $x,\ y \in X$:*

$$d(Fx, Fy) \leq a\, d(x, y) + b[d(x, Fx) + d(y, Fy)] + c[d(x, Fy) + d(y, Fx)]$$

where $a,\ b,\ c > 0$ and $a + 2b + 2c = 1$. Then F has a unique fixed point in X.

Note that F is not an RHR map.

Ćirić [8] in 1993

Theorem 2. (Ćirić) *Let K be a closed convex subset of a complete convex metric space X and $T : K \to K$ a mapping satisfying*

$$d(Tx, Ty) \le a\, d(x, y) + (1 - a) \max\{d(x, Tx), d(y, Ty), b\, [d(x, Ty) + d(y, Tx)]\}$$

for all $x,\ y \in K$, where $0 < a < 1$ and $b \le \frac{1}{2} - \frac{1-a^2}{10+6a^2}$. Then T has a unique fixed point.

Note that T is not an RHR map.

Feng and Liu [11] in 2006

Let (X, d) be a complete metric space. $\mathrm{Cl}(X)$ denotes the collection of all nonempty closed subsets. Let $T : X \to \mathrm{Cl}(X)$ be a multi-valued mapping. Define a function $f : X \to \mathbb{R}$ as $f(x) = d(x, T(x))$. For a positive constant $b \in (0, 1)$, define the set $I_b^x \subset X$ as

$$I_b^x = \{y \in T(x) : b\, d(x, y) \le d(x, T(x))\}.$$

The following theorem is the main result:

Theorem 3.1. (Feng-Liu) *Let (X, d) be a complete metric space, $T : X \to \mathrm{Cl}(X)$ be a multi-valued mapping. If there exists a constant $c \in (0, 1)$ such that for any $x \in X$ there is $y \in I_b^x$ satisfying*

$$d(y, T(y)) \le c\, d(x, y),$$

then T has a fixed point in X provided $c < b$ and f is lower semi-continuous.

Corollary 3.2. (Feng-Liu) *Let (X, d) be a complete metric space, $T : X \to \mathrm{Cl}(X)$ be a multi-valued mapping. If there exists a constant $c \in (0, 1)$ such that for any $x \in X,\ y \in T(x)$,*

$$d(y, T(y)) \le c\, d(x, y),$$

then T has a fixed point in X provided f is lower semi-continuous.

This extends the Covitz-Nadler fixed point theorem [9] or Theorem H(δ1) for metric spaces.

Kumam, Dung, Sitytithakerngkiet [26] in 2015

Kumam et al. obtained in [26] the following improvement of Ćirić's theorem.

Theorem 3.1. (Kumam et al.) *Let T be a self map of a complete metric space (X, d). If there is a constant $\alpha \in (0, 1)$ such that*

$$d(Tx, Ty) \le \alpha \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx),$$
$$d(x, T^2x), d(Tx, T^2x), d(x, T^2y), d(T^2x, Ty)$$

for all $x,\ y \in X$, then T has a unique fixed point $z \in X$ and $d(z, T^n x_0) \to 0$ as $n \to \infty$, for all $x_0 \in X$.

Some corollaries and multi-valued versions of them are added.

Comment: The contractive condition implies the Hegedüs condition

$$d(Tx, Ty) \le q\, \mathrm{diam}\{O_T(x) \cup O_T(y)\}.$$

Hence the main theorem was already known; see Park [29]. The map T is not in β-class.

10. Conclusion

In this paper, we introduce a correct form of Theorem P, which is also called the weak contraction principle or the Rus-Hicks-Rhoades contraction principle. It is a generalization of the Banach contraction principle and a large number of applications in our previous works.

Moreover, we introduced Theorem H based on our previous 2023 Metatheorem. Theorem H claims that the six statements $(\alpha) - (\epsilon)$ and (η) are equivalent and that they characterize the metric completeness (0). We classified multi-valued selfmaps on quasi-metric spaces satisfying each of the statements $(\alpha) - (\epsilon)$. Such classes of multimaps have extremal elements, fixed points, common fixed points, stationary points, common stationary points by the Metatheorem.

For example, γ-class of selfmaps consists of the Rus-Hicks -Rhoades maps and fixed point theorems on them can be easily obtained by the statement $\gamma 1$ in the Metatheorem. The numbers of multimaps in other classes are relatively small. We add some examples of multimaps not belonging to any of α-ϵ classes. Usually such type of theorems have relatively long and difficult proofs.

Consequently, we can destroy possible maps of γ-class in thousands of artificial metric type spaces. This will save the energy of many researchers.

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Manuscript received September 9, 2025

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