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GAP FUNCTIONS AND EXISTENCE RESULTS FOR APPROXIMATE VECTOR VARIATIONAL INEQUALITY PROBLEMS ON HADAMARD MANIFOLDS

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Abstract. This paper is concerned with the study of a class of weak approximate vector variational inequality problems on Hadamard manifolds. We formulate a gap function and a regularized gap function to solve the considered weak approximate vector variational inequality problems on Hadamard manifolds. We derive the conditions for the existence of solutions for the considered problem without monotonicity and relaxed compactness assumptions on Hadamard manifolds. Moreover, we also establish the existence results for the solutions of the considered problem using geodesic α-monotonicity assumption on Hadamard manifolds. Some nontrivial numerical examples have been given to demonstrate the significance of these results. The results presented in this paper extend and generalize some existing results in the literature.

1. Introduction

In linear topological spaces, the concept of convex set relies on connecting two given points of the space by line segments. However, in many real life problems, it is not always possible to connect the points through line segments. This led to the idea of generalizations of the notion of convex sets. Rapcsák [47] and Udrişte [53] generalized the concept of a line segment between two points by a geodesic in the framework of Riemannian manifolds. Furthermore, Udrişte [53] introduced the notion of geodesic convex functions in the Riemannian manifold setting. A complete, simply connected Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold. Following Udrişte [53], several other generalizations of convex sets and functions have been proposed on Riemannian and Hadamard manifolds, for instance, Ansari et al. [1] have widely discussed the various notions of geodesic convexity of a real-valued function defined on a geodesic convex subset of a Riemannian manifold in terms of a bifunction. Further, they established several relationships between different forms of monotonicity of a bifunction and introduced

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geodesic convexity for real-valued functions. For more exposition, we refer the readers to [9, 10, 40, 46, 54, 58] and references cited therein.

In many optimization problems, the nonsmooth phenomena occur naturally and frequently. The nonsmooth analysis of vector valued maps and their generalized Jacobian matrices, such as Clarke generalized Jacobian matrices of locally Lipschitz functions and Mordukhovich coderivatives for general vector valued maps on Banach spaces have wider applications, see for example [16, 41] and the references cited therein. Therefore, the nonsmooth vector fields, which are natural generalizations of nonsmooth vector valued maps are important both theoretically as well as computationally. Németh [43] introduced the notion of monotone vector fields. Da Cruz Neto et al. [18] and Li et al. [32] have extended the notion of monotone vector fields to the set-valued mappings.

The notion of convexity has wider applications in optimization theory and related areas. However, convexity is very restrictive notion and does not fit in modeling and solution of several real world problems, for instance, mathematical economics. This led to the introduction of the classes of generalized convex functions, which preserve some of the properties of the convex functions. For recent survey and more exposition about generalized convex functions, we refer to [36, 39]. In an effort to generalize the convexity notion, Luc et al. [35] introduced a new class of generalized convex functions, namely ϵ-convex functions having applications in approximate calculus. Daniilidis and Georgiev [19] established that a locally Lipschitz function is approximately convex if and only if its Clarke subdifferential is a submonotone operator.

In 1980, Giannessi [22, 23] introduced vector valued version of variational inequalities introduced by Minty [37] and Stampacchia [49] in the Euclidean space setting. Németh [43] introduced variational inequalities on Hadamard manifolds and established some existence results for the solution of variational inequalities. Several scholars studied vector variational inequalities and its generalizations, see for instance, [2, 15, 27, 51, 56, 57, 61] and the references cited therein. Barani [8] proposed the notions of strong monotonicity for set-valued mappings and some notions of strong convexity for locally Lipschitz functions on Hadamard manifolds. Li et al. [32] investigated the existence of solution and convexity of the solution set for the variational inequality problems for set-valued mapping on Riemannian manifolds. Recently, Chen and Huang [15] studied the relationship between Stampacchia and Minty vector variational inequalities and nonsmooth convex vector optimization problems using Clarke subdifferential and proved certain existence theorems under relaxed compactness assumption. Huang [25] presented a fixed point theorem for a more general type of KKM mapping in the setting of Hadamard manifolds.

The study of existence results, stability criteria, and solution methods for optimization problems relied heavily on gap functions. Variational inequalities can be converted to optimization problems using gap functions (see, for example, [5,6,21,62] and the references cited therein). Gap functions are also useful to find the error bounds for variational inequalities. For a class of variational inequalities involving convex functions, Auslender [4] developed the concept of a gap function in the setting of the Euclidean space. Fukushima [21] defined the regularized gap functions for variational inequalities in the Euclidean space framework. Charitha and Dutta [12], formulated the regularized gap functions and obtained global error bounds for the solution of vector variational inequality problems on the Euclidean space. Ansari et al. [2] studied gap functions and global error bounds for nonsmooth variational inequalities on Hadamard manifolds in terms of bifunctions. Li and He [33] defined gap functions for generalized vector variational inequalities and obtained existence results for their solutions in the framework of Euclidean space. Further, Islam and Irfan [26] have studied minimum and maximum principle sufficiency properties of nonsmooth variational inequality problems by utilizing the gap functions in the setting of Hadamard manifolds. They have provided some characterizations of obtained sufficiency properties along with the error bounds for considered variational inequalities.

1.1 **The Proposed Work**

The novelty and contributions of our work are of two folds:

In the first fold, motivated by the work of Ansari et al. [2], we have formulated a gap function and a regularized gap function for the considered weak approximate vector variational inequality problem. The gap functions formulated in this work are more general than the gap functions considered by Charitha et al. [13], Charitha and Dutta [12] and Yamashita and Fukushima [62] as it is defined for a more general problem namely, weak approximate vector variational inequality problem as well as to a more general space, namely Hadamard manifold.

In the second fold, motivated by the works of Barani [8], Chen and Fang [14] and Oveisiha and Zafarani [45], we establish the existence results for the solutions of considered weak approximate vector variational inequality problems on Hadamard manifolds. The existence results presented in this paper extend some existing results by Ansari and Rezaei [3], Lee [31], and Upadhyay and Mishra [55] to a more general space as well as to a more general class of functions. Furthermore, the results established in the paper generalize some existence results derived by Chen and Huang [15] from Stampacchia weak vector variational inequality problem to a more general problem, namely (WAVVIP).

The organization of this paper is given as follows: In Section 2, we recall some basic definitions and preliminaries related to Hadamard manifolds. We formulate a gap function and a regularized gap function to solve weak approximate vector variational inequality problems on Hadamard manifolds in Section 3. In Section 4, an analogous to the KKM lemma have been employed to derive some existence theorems for considered weak approximate vector variational inequality problems under relaxed compactness and without monotonicity assumptions as well as using geodesic α-monotonicity assumptions.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $\mathbb{R}^n_+$ be the nonnegative orthant of $\mathbb{R}^n$ and 0 be the origin of the nonnegative orthant of $\mathbb{R}^n$. The symbol $\operatorname{int}\mathbb{R}^n_+$ is used to represent the positive orthant of $\mathbb{R}^n$.

We recall the following definitions and preliminaries about the Riemannian manifolds from [14, 47, 53].

Let M be an n-dimensional connected Riemannian manifold endowed with a Riemannian metric $\mathcal{G}$. The tangent space at $p \in M$ is denoted by T_pM, which is a vector space of dimension n over the field of real numbers. The symbols $\langle \cdot, \cdot \rangle_p$ and $\| \cdot \|_p$ are used to represent the inner product and its associated norm on the tangent space T_pM, respectively. The dual space of T_pM is represented by T_pM^*. The symbol $\mathbb{B}_p^*(v, \epsilon)$ represents the ball centered at $v \in T_pM$ of radius $\epsilon > 0$. Notably, the tangent space T_pM, being a finite-dimensional complete normed space, is a Banach space. The symbol T_pM^{**} represents the dual of T_pM^* and the canonical embedding $\mathcal{C} : T_pM \to T_pM^{**}$ is defined as follows:

$$\mathcal{C}(v) = \mathcal{J}_v, \ \mathcal{J}_v \in T_pM^{**},$$

where $\mathcal{J}_v(f) = f(v), f \in T_pM^*$ (see, [30]). The tangent bundle of M is denoted by $TM = \cup_{p\in M} T_pM$, which is naturally a manifold. Moreover, the symbol TM^* is employed to denote the cotangent bundle of M, which is defined as $TM^* := \bigcup_{p\in M} T_pM^*$. Let us recall that the length of a piecewise differentiable curve $\Omega : [a, b] \to M$ joining p and q in M, such that $\Omega(a) = p$ and $\Omega(b) = q$, is defined by

$$L(\Omega) := \int_a^b \left\|\Omega'(t)\right\| dt.$$

Minimizing this length functional on the set of all piecewise differentiable curves joining p and q in M, we get a distance function $d(p, q)$. This distance function d induces the original topology on M. Let $\chi(M)$ denote the space of all vector fields on M. The Riemannian metric induces a map $f \mapsto \operatorname{grad} f \in \chi(M)$, which associates to each f its gradient via the rule $\langle df, X \rangle = df(X)$, for each $X \in \chi(M)$. On every Riemannian manifold there exists exactly one covariant derivation called Levi-Civita connection denoted by $\nabla_X Y$ for any vector fields $X, Y \in \chi(M)$. We also recall that a geodesic is a C^∞ smooth path Ω whose tangent is parallel along the path Ω, that is, Ω satisfies the equation

$$\nabla_{\Omega'} \Omega^{'} = 0.$$

Any path Ω joining p and q in M such that $L(\Omega) = d(p, q)$ is a geodesic, and it is called a minimal geodesic. It is known that a Levi-Civita connection ∇ induces an isometry $P_{t_1,\Omega}^{t_2} : T_{\Omega(t_1)}M \to T_{\Omega(t_2)}M$ so called parallel translation along Ω from $\Omega(t_1)$ to $\Omega(t_2)$. The symbol $P_{x,y}$ represents the parallel translation along a minimal geodesic connecting points x and y. The parallel translation $P_{x,y}$ induces another linear isometry $P_{x,y}^* : T_xM^* \to T_yM^*$, such that for every $\xi \in T_xM^*$ and $\nu \in T_yM$, the following condition holds:

$$\langle P_{x,y}^*(\xi), \nu \rangle = \langle \xi, P_{y,x}(\nu) \rangle.$$

Here, $\langle P_{x,y}^*(\xi), \nu \rangle = P_{x,y}^*(\xi)(\nu)$, and $\langle \xi, P_{y,x}(\nu) \rangle = \xi(P_{y,x}(\nu))$, since $P_{x,y}^*(\xi)$ and ξ are linear functionals on T_yM^* and T_xM^*, respectively. We will use the same symbol $P_{x,y}$ to denote the isometry $P_{x,y}^*$.

The following remark is from Azagra et al. [7], which is a consequence of parallel translation along a curve as a solution to an ordinary linear differential equation.

Remark 2.1. Let $D : TM^* \to T_yM^*, \quad D(x, \xi) = P_{x,y}(\xi)$, then the defined map D is continuous at (y, ζ), that is, if $(x_n, \xi_n) \to (y, \zeta)$ in TM^* then $P_{x_n,y}(\xi_n) \to \zeta$, for every $(y, \zeta) \in TM^*$.

A Riemannian manifold is said to be complete if, for any $x \in M$, all geodesic emanating from x are defined for all $-\infty < t < \infty$. If M is complete, then any points in M can be joined by a minimal geodesic. Suppose that M is complete. The exponential map $\exp_x : T_xM \to M$ at x in M is defined by $\exp_x(v) := \Omega_v(1, x)$, for every $v \in T_xM$, where $\Omega(.) := \Omega_v(., x)$ is the geodesic starting at x with velocity v, that is $\Omega(0) = x$ and $\Omega'(0) = v$. It is easy to see that $\exp_x(tv) = \Omega_v(t, x)$, for each real number t. We note that the map $\exp_x$ is differentiable on T_xM, for every $x \in M$.

A simply connected complete Riemannian manifold with nonpositive sectional curvature is called a Hadamard manifold. If M is a Hadamard manifold, then $\exp_x : T_xM \to M$ is a diffeomorphism for every $x \in M$ and its inverse $\exp_x^{-1} : M \to T_xM$ satisfies $\exp_x^{-1}(x) = 0_x$, where $0_x \in T_xM$ is the zero tangent vector. Furthermore, for any $x, y \in M$, there always exists a unique minimal geodesic joining x and y.

The definitions of injectivity radius for a complete $n-$dimensional Riemannian manifold is from Sakai [50].

Definition 2.2. The injectivity radius related to an element $x \in M$ is denoted by η_x, and is defined as follows:

$$\eta_x := \sup\{r > 0 \mid \exp_x : \mathbb{B}_x^*(0_x, r) \subset T_xM \to \exp_x(\mathbb{B}_x^*(0_x, r)) \text{ is a diffeomorphism}\}.$$

The injectivity radius of M is $\eta(M) = \inf\{\eta_x \mid x \in M\}$.

Remark 2.3. If M is an $n-$dimensional Hadamard manifold, then $\eta_x = +\infty$, for every $x \in M$.

In the following definition from Jost [28], we recall the notion of a totally geodesic submanifold.

Definition 2.4. A Riemannian submanifold H of a Riemannian manifold M is called totally geodesic submanifold if all geodesics in H are also geodesics in M.

For example, each closed geodesic in a Riemannian manifold defines a 1-dimensional compact totally geodesic submanifold.

Remark 2.5. In general, the Riemannian manifold may not have any totally geodesic submanifolds of dimension greater than 1.

The following lemmas are from Sakai [50].

Lemma 2.6. *Let M be a Riemannian manifold of constant sectional curvature, then the following condition holds: Let V be any $k-$dimensional $(k \leq n)$ subspace of $T_xM, x \in M$. Then $S := \exp_x(V \cap B_x^*(0_x, \epsilon))$ is a $k-$dimensional totally geodesic submanifold of M, where $0 < \epsilon < \eta_x$.*

The aforementioned condition is known as the axiom of plane.

However, in view of the Remark 2.3, if M is a Hadamard manifold with constant sectional curvature, the following lemma holds.

Lemma 2.7. *Let M be a Hadamard manifold having constant sectional curvature and V be any $k-$dimensional $(k \leq n)$ subspace of T_xM for any $x \in M$. Then $S = \exp_x(V)$ is a complete and simply connected $k-$dimensional totally geodesic* *submanifold of M and called a $k-$dimensional subspace of M. Moreover, S is isometric to a $k-$dimensional Hadamard manifold.*

Throughout the paper, we assume that M is an $n-$dimensional Hadamard manifold, unless otherwise specified.

The notion of approximate monotonicity in finite dimensional spaces have been studied by Spingarn [48] and in infinite dimensional spaces by Daniilidis and Georgiev [19] and Ngai and Penot [42].

Let $A : M \to 2^M$ be a multivalued vector field such that $Ax \subseteq T_xM$, for each $x \in M$, and the domain $D(A)$ of A is defined by

$$D(A) := \{x \in M : A(x) \neq \emptyset\}.$$

Now, we recall the notion of geodesic α-monotone multivalued vector fields on Hadamard manifolds from Upadhyay et al. [60].

Definition 2.8. A is said to be geodesic α-monotone on M, if for given $\alpha > 0$, for every $x, y \in D(A)$, and for every $\xi \in A(x), \eta \in A(y)$, one has

$$\left\langle P^0_{1,\Omega}\xi - \eta, \exp_y^{-1}(x)\right\rangle_y \geq -\alpha \left\|\exp_y^{-1}(x)\right\|_y,$$

where $\Omega(t) := \exp_y\left(t \exp_y^{-1}(x)\right), t \in [0, 1]$.

In the following definition, we recall the notion of a geodesic convex set in the framework of Hadamard manifolds from Udrişte [53].

Definition 2.9. A nonempty subset K of M is said to be a geodesic convex set, if for any points $x, y \in K$, the unique minimal geodesic joining x and y is contained in K. That is,

$$\exp_x(t \exp_x^{-1}(y)) \in K, \text{ for all } t \in [0, 1].$$

In the following definitions, we recall some notions of nonsmooth analysis (see, for instance, [8, 24]).

Definition 2.10. Let function $g : M \to]-\infty, \infty]$ be a proper function. The function g is said to be locally Lipschitz on M, if for all $z \in M$, there exists a positive constant L_z and $\delta_z > 0$, such that

$$\|g(x) - g(y)\| \leq L_z d(x, y), \forall x, y \in B\left(z, \delta_z\right),$$

where L_z is called Lipschitz rank of g in the neighborhood of z and $B\left(z, \delta_z\right) := \{x \in M : d(z, x) < \delta_z\}$.

Definition 2.11. Let $K \subseteq M$ be an open geodesic convex subset of M and let $g : M \to] - \infty, \infty]$ be a locally Lipschitz function on K. The Clarke generalized directional derivative of g at $x \in K$, in the direction of a vector $\nu \in T_xM$, denoted by $g^\circ(x; \nu)$, is defined as

$$g^\circ(x; \nu) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{g\left(\exp_y t \left(d \exp_x\right)_{\exp_x^{-1}(y)} \nu\right) - g(y)}{t},$$

where $(d \exp_x)_{\exp_x^{-1}(y)} : T_{\exp_x^{-1}(y)}\left(T_xM\right) \to T_yM$ is the differential of exponential mapping at $\exp_x^{-1}(y)$.

Definition 2.12. Let $K \subseteq M$ be an open geodesic convex subset of M and let $g : M \to]-\infty, \infty]$ be a locally Lipschitz function on K. The Clarke generalized subdifferential of g at $x \in K$, denoted by $\partial_c g(x)$, is the subset of T_xM^* defined by

$$\partial_c g(x) := \{\xi \in T_xM^* : g^\circ(x;\nu) \geq \langle \xi, \nu \rangle, \forall \nu \in T_xM\},$$

where $\langle \xi, \nu \rangle = \xi(\nu)$.

Remark 2.13. It is worthwhile to note that T_xM is isomorphic to T_xM^* for every $x \in M$. Therefore, the Clarke subdifferential of g at x can also be defined as follows:

$$\partial_c g(x) := \{\xi \in T_xM : g^\circ(x;\nu) \geq \langle \xi, \nu \rangle, \forall \nu \in T_xM\}.$$

For more details, we refer to [52].

The following definitions of weak topology and weak* topology are from [29, 30].

Definition 2.14. The weak topology on T_xM for some $x \in M$, is the coarsest topology such that every element of T_xM^* is continuous.

Definition 2.15. The weak* topology on T_xM^* for some $x \in M$, is the coarsest topology such that the functionals $\{\mathcal{J}_v \mid v \in T_xM\}$ are all continuous, where $\mathcal{C} : v \mapsto \mathcal{J}_v$ is the canonical embedding of T_xM into T_xM^{**}.

Remark 2.16. It is worthwhile to note that T_xM is a finite-dimensional complete normed space, therefore, the weak topology and weak* topology coincide.

Definition 2.17. Let $V \subseteq T_xM$ for some $x \in M$. Then V is said to be weakly compact, if it is a compact set with respect to the weak topology on T_xM.

In the following definition, we recall the notion of a weak*-compact set from [11].

Definition 2.18. Let $V^* \subseteq T_xM^*$ for some $x \in M$. Then V^* is said to be weak*-compact, if it is a compact set with respect to the weak* topology on T_xM^*.

In order to define the weak*-cluster point of a sequence T_xM^*, we recall the weak* convergence in the following definitions given by Kesavan [29].

Definition 2.19. Let $\{f_n\}$ be a sequence in T_xM^* and $f \in T_xM^*$. Then $\{f_n\}$ weakly* converges to f if $\{f_n(v)\}$ converges to $f(v)$ for every $v \in T_xM$.

In view of the fact that T_xM is a Banach space, one can define the weak*-cluster point of a sequence in T_xM^* as follows:

Definition 2.20. A linear functional $f \in T_xM^*$ is said to be a weak*-cluster point of sequence $\{f_n\} \in T_xM^*$ if and only if there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ weakly* converges to f.

Now onwards, unless otherwise specified, we suppose that K is a non-empty geodesic convex subset of M.

The following lemma is from Hosseini and Pouryayevali [24].

Lemma 2.21. *Let $g : M \to \mathbb{R}$ be a locally Lipschitz function on K with Lipschitz rank L. Then the following hold.*

(i) *For each* $x \in K$, *the function* $g^\circ(x;\nu)$ *is finite, positively homogeneous and subadditive on* T_xM, *and satisfies*

$$\|g^\circ(x;\nu)\| \leq L\|\nu\|, \ \forall \nu \in T_xM.$$

(ii) $g^\circ(\cdot,\cdot)$ *is upper semicontinuous on* $K \times T_xM$ *and* $g^\circ(x,\cdot)$ *is locally Lipschitz with Lipschitz rank* L *on* T_xM, *for all* $x \in K$.

(iii) $g^\circ(x;-\nu) = (-g^\circ)(x;\nu)$, *for all* $x \in K$ *and* $\nu \in T_xM$.

The following lemma is from Hosseini and Pouryayevali [24]. We provide the proof of the following lemma for the sake of the convenience of readers.

Lemma 2.22. *Let* M *be a Riemannian manifold, and* $g : M \to \mathbb{R}$ *be a Lipschitz function near* $x \in M$. *Then*

(i) $\partial_c g(x)$ *is a nonempty, convex and weak*-compact subset of* T_xM^*, *and* $\|\xi\|_{T_xM^*} \leq L_x, \forall \xi \in \partial_c g(x)$, *where* L_x *is the Lipschitz rank of* g *in the neighborhood of* x.

(ii) *Let* $\{x_n\}$ *and* $\{\xi_n\}$ *be sequences in* M *and* TM^*, *respectively, such that* $\xi_n \in \partial_c g(x_n)$, *for each* n. *Moreover, if we assume that* $\{x_n\}$ *converges to* x, *and* ξ *is a weak*-cluster point of the sequence* $\{P_{x_n,x}(\xi_n)\}$, *then, we have* $\xi \in \partial_c g(x)$.

Proof. (i) In view of Lemma 2.21 and Hahn-Banach Theorem (see, [30]) on T_xM, there exist at least one linear functional $\xi : T_xM \to \mathbb{R}$, that is, $\xi \in T_xM^*$, such that

$$g^\circ(x;\nu) \geq \langle \xi, \nu\rangle, \forall \nu \in T_xM.$$

Therefore, $\xi \in \partial_c g(x)$, which implies that $\partial_c g(x)$ is nonempty.

Let ξ_1, ξ_2 be any two arbitrary elements of $\partial_c g(x) \subseteq T_xM^*$. In view of the fact that T_xM^* is a linear space, it follows that $t\xi_1 + (1-t)\xi_2 \in T_xM^*, \forall t \in [0,1]$. Moreover,

$$\begin{aligned}\langle t\xi_1 + (1-t)\xi_2, \nu\rangle =& t\langle \xi_1, \nu\rangle + (1-t)\langle \xi_2, \nu\rangle,\\ \leq& t g^\circ(x;\nu) + (1-t) g^\circ(x;\nu) = g^\circ(x;\nu).\end{aligned}$$

Therefore, $\partial_c g(x)$ is a convex subset of T_xM^*. From the definition of $\partial_c g(x)$, we have

$$\langle \xi, \nu\rangle \leq g^\circ(x;\nu), \forall \nu \in T_xM.$$

Therefore, by using Lemma 2.21, we have

$$||\xi||_{T_xM^*} \leq ||g^\circ(x;\nu)|| \leq L_x||\nu|| \leq L_x,$$

where, $||\xi||_{T_xM^*} := \sup\{\langle \xi, \nu\rangle \mid \nu \in T_xM, ||\nu|| \leq 1\}$. By Banach-Alaoglu Theorem in [29], $\partial_c g(x)$ is a weak*-compact subset of T_xM^*.

(ii) Fix $\nu \in T_xM$. Since $\{\xi_n\} \in \partial_c g(x_n)$, it follows that for each n, we have

$$g^\circ\left(x_n; P_{x,x_n}(\nu)\right) \geq \left\langle \xi_n, P_{x,x_n}(\nu)\right\rangle.$$

The sequence $\{\langle \xi_n, P_{x,x_n}(\nu)\rangle\} = \{\langle P_{x_n,x}(\xi_n), \nu\rangle\}$ is bounded in $\mathbb{R}$, and contains terms that are arbitrarily close to $\langle \xi, \nu\rangle$. In view of the fact that ξ is a weak*-cluster point of sequence $\{P_{x_n,x}(\xi_n)\}$, this implies that there exists a subsequence $\{P_{x_{n_k},x}(\xi_n^k)\}$ of sequence $\{P_{x_n,x}(\xi_n)\}$, such that

$$\left\{\left\langle P_{x_{n_k},x}\left(\xi_n^k\right), \nu\right\rangle\right\} = \left\{\left\langle \xi_n^k, P_{x,x_{n_k}}(\nu)\right\rangle\right\} \to \langle \xi, \nu\rangle.$$

By Remark 2.1, we have that $P_{x,x_{n_k}}(\nu) \to \nu$. Since g° is upper semicontinuous in (x,ν), it follows that

$$g^\circ(x;\nu) \geq \langle \xi, \nu \rangle.$$

Since ν was an arbitrary element in T_xM, we conclude that $\xi \in \partial_c g(x)$. □

Now, we have the following notion of geodesic α-convex functions on Hadamard manifolds from Upadhyay et al. [60].

Definition 2.23. Let $\alpha > 0$ be given and $g : K \to \mathbb{R}$ be a locally Lipschitz function. Then g is said to be geodesic α-convex on K, if for all $x, y \in K$, one has

$$g(y) - g(x) \geq \left\langle \xi, \exp_x^{-1}(y) \right\rangle_x - \alpha \left\| \exp_x^{-1}(y) \right\|_x, \forall \xi \in \partial_c g(x).$$

The following theorem from [60] establishes the relationship between a geodesic α-convex function and its Clarke subdifferential on Hadamard manifolds.

Theorem 2.24. *Let $g : K \to \mathbb{R}$ be a locally Lipschitz function on K. Then g is geodesic α-convex on K if and only if $\partial_c g$ is geodesic α-submonotone on K.*

These definitions and properties can be extended to a locally Lipschitz vector valued function $f : K \to \mathbb{R}^p$. Denote by $f_i, i \in I := \{1, 2, \ldots, p\}$, the components of f, the Clarke generalized subdifferential of f at x is the set

$$\partial_c f(x) := \partial_c f_1(x) \times \partial_c f_2(x) \times \cdots \times \partial_c f_p(x).$$

Let $C := -\mathbb{R}^p_+ \backslash \{0\}$ and $\operatorname{int} C := -\operatorname{int} \mathbb{R}^p_+$ be its interior. For any $x, y \in \mathbb{R}^p$, we use the following ordering relation:

$$y \leq_C x \iff y - x \in C; \quad y \nleq_C x \iff y - x \notin C.$$

We formulate the following weak approximate vector variational inequality problem in terms of Clarke subdifferential on Hadamard manifolds:

(WAVVIP) Find a point $\bar{x} \in K$, such that for any $y \in K$, there exist $\beta > 0$, and $\xi \in \partial_c f(\bar{x})$, satisfying:

$$\left\langle \xi, \exp_{\bar{x}}^{-1}(y) \right\rangle_{\bar{x}} + \beta \left\| \exp_{\bar{x}}^{-1}(y) \right\|_{\bar{x}} e \notin \operatorname{int} C,$$

or equivalently,

$$\Big(\left\langle \xi_1, \exp_{\bar{x}}^{-1}(x) \right\rangle_{\bar{x}} + \beta \left\| \exp_{\bar{x}}^{-1}(x) \right\|_{\bar{x}}, \ldots, \left\langle \xi_p, \exp_{\bar{x}}^{-1}(x) \right\rangle_{\bar{x}} + \beta \left\| \exp_{\bar{x}}^{-1}(x) \right\|_{\bar{x}} \Big) \notin \operatorname{int} C,$$

where $e = \underbrace{(1, 1, \ldots, 1)}_{p \text{ times}}$.

Remark 2.25.
1. If $\beta = 0$, then (WAVVIP) reduces to the weak Stampacchia vector variational inequality problem, namely (WSVVIP) considered by Chen and Huang [15].
2. If $M = \mathbb{R}^n$, then $\exp_x^{-1}(y) = y - x$. Moreover, if $\beta = 0$, then (WAVVIP) reduces to the weak Stampacchia vector variational inequality problems (WVVIP) considered by Mishra and Upadhyay [38] and Upadhyay et al. [59].

3. Gap functions

In this section, we define the gap function and regularized gap function to solve the weak approximate vector variational inequality problems (WAVVIP).

Definition 3.1. A real-valued function $\Pi : K \to \mathbb{R}$ is said to be a gap function for (WAVVIP), if the following hold:

(a) $\Pi(x) \geq 0, \forall x \in K$.
(b) $\bar{x} \in K$ is such that $\Pi(\bar{x}) = 0$ if and only if $\bar{x}$ is a solution of (WAVVIP).

Let $x, y \in K$ and $\xi \in \partial_c f(y)$. Consider the function $\Theta : K \to \mathbb{R}$ defined as

$$\Theta(y) := \inf_{\xi \in \partial_c f(y)} \sup_{x \in K} \min_{i \in I} \left\{ \left\langle P^1_{0,\Omega}\xi_i, \exp^{-1}_x(y) \right\rangle_x - \beta \left\| \exp^{-1}_x(y) \right\|_x \right\}, \tag{3.1}$$

where $\Omega(t) = \exp_y\left(t \exp^{-1}_y(x)\right), \forall t \in [0,1]$.

In the following theorem, we show that $\Theta(y)$ is a gap function for (WAVVIP).

Theorem 3.2. *Let $f_i : K \to \mathbb{R}$ for $i \in I$ be locally Lipschitz functions. Then the function $\Theta(y)$ is a gap function for (WAVVIP).*

Proof. (a) On the contrary, we assume that there exists $y \in K$ such that

$$\Theta(y) < 0.$$

From the definition of $\Theta(y)$, there exists $\xi_i \in \partial_c f_i(y), i \in I$, such that

$$\sup_{x \in K} \min_{i \in I} \left\{ \left\langle P^1_{0,\Omega}\xi_i, \exp^{-1}_x(y) \right\rangle_x - \beta \left\| \exp^{-1}_x(y) \right\|_x \right\} < 0,$$

where $\Omega(t) = \exp_y(t \exp^{-1}_y(x))$, $\forall t \in [0,1]$. It follows that for every $x \in K$ we have

$$\min_{i \in I} \left\{ \left\langle P^1_{0,\Omega}\xi_i, \exp^{-1}_x(y) \right\rangle_x - \beta \left\| \exp^{-1}_x(y) \right\|_x \right\} < 0,$$

which is a contradiction for $x = y$. Therefore, we have

$$\Theta(y) \geq 0, \ \forall y \in K.$$

(b) Let $\bar{x} \in K$ be an element such that $\Theta(\bar{x}) = 0$. Moreover, for $x \in K$, let $\Omega(t) = \exp_{\bar{x}}\left(t \exp^{-1}_{\bar{x}}(x)\right), \forall t \in [0,1]$. Then from the definition of Θ, there exists $\xi_i \in \partial_c f_i(\bar{x})$, $i \in I$ such that

$$\sup_{x \in K} \min_{i \in I} \left\{ \left\langle P^1_{0,\Omega}\xi_i, \exp^{-1}_x(\bar{x}) \right\rangle_x - \beta \left\| \exp^{-1}_x(\bar{x}) \right\|_x \right\} = 0.$$

That is, for every $x \in K$, we have

$$\min_{i \in I} \left\{ \left\langle P^1_{0,\Omega}\xi_i, \exp^{-1}_x(\bar{x}) \right\rangle_x - \beta \left\| \exp^{-1}_x(\bar{x}) \right\|_x \right\} \leq 0. \tag{3.2}$$

From (3.2), it follows that for every $x \in K$, there exist some $k \in I$, such that

$$\left\langle P^1_{0,\Omega}\xi_k, \exp^{-1}_x(\bar{x}) \right\rangle_x - \beta \left\| \exp^{-1}_x(\bar{x}) \right\|_x \leq 0. \tag{3.3}$$

On applying the parallel translation from x to $\bar{x}$ in (3.3) we get

$$\left\langle \xi_k, \exp^{-1}_{\bar{x}}(x) \right\rangle_{\bar{x}} + \beta \left\| \exp^{-1}_{\bar{x}}(x) \right\|_{\bar{x}} \geq 0.$$

Hence, for all $x \in K$, there exist $\xi_i \in \partial_c f_i(\bar{x}),\ i \in I$, such that

$$\Big(\left\langle \xi_1, \exp_{\bar{x}}^{-1}(x)\right\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(x)\right\|_{\bar{x}}, \ldots, \left\langle \xi_p, \exp_{\bar{x}}^{-1}(x)\right\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(x)\right\|_{\bar{x}}\Big) \notin \operatorname{int} C.$$

Thus, $\bar{x}$ is a solution of (WAVVIP).

Conversely, let $\bar{x}$ be a solution of (WAVVIP). Then for all $x \in K$ there exists $\xi_i \in \partial_c f_i(\bar{x}), i \in I$, such that

$$\Big(\left\langle \xi_1, \exp_{\bar{x}}^{-1}(x)\right\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(x)\right\|_{\bar{x}}, \ldots, \left\langle \xi_p, \exp_{\bar{x}}^{-1}(x)\right\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(x)\right\|_{\bar{x}}\Big) \notin \operatorname{int} C.$$

That is, for every $x \in K$, there exist some $k \in I$, such that

$$\left\langle \xi_k, \exp_{\bar{x}}^{-1}(x)\right\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(x)\right\|_{\bar{x}} \geq 0. \tag{3.4}$$

Using parallel translation from $\bar{x}$ to x in (3.4) we have

$$\left\langle P_{0,\Omega}^{1}\xi_k, \exp_{x}^{-1}(\bar{x})\right\rangle_{x} - \beta \left\|\exp_{x}^{-1}(\bar{x})\right\|_{x} \leq 0, \tag{3.5}$$

where $\Omega(t) = \exp_{\bar{x}}\left(t \exp_{\bar{x}}^{-1}(x)\right), \forall t \in [0,1]$. From (3.5) it follows that

$$\sup_{x\in K} \min_{i\in I} \left\{ \left\langle P_{0,\Omega}^{1}\xi_i, \exp_{x}^{-1}(\bar{x})\right\rangle_{x} - \beta \left\|\exp_{x}^{-1}(\bar{x})\right\|_{x} \right\} \leq 0.$$

Therefore, we have

$$\Theta(\bar{x}) = \inf_{\xi\in\partial_c f(\bar{x})} \sup_{x\in K} \min_{i\in I} \left\{ \left\langle P_{0,\Omega}^{1}\xi_i, \exp_{x}^{-1}(\bar{x})\right\rangle_{x} - \beta \left\|\exp_{x}^{-1}(\bar{x})\right\|_{x} \right\} \leq 0. \tag{3.6}$$

Since $\Theta(x) \geq 0$, for all $x \in K$, therefore, from (3.6) we get

$$\Theta(\bar{x}) = 0.$$

Hence, the proof is complete. □

Remark 3.3. Theorem 3.2 generalizes the Theorem 4.1 of Charitha et al. [13] to a more general problem, namely weak approximate vector variational inequality problem (WAVVIP) as well as to a more general space, namely Hadamard manifold.

Motivated by the regularized gap function introduced by Yamashita and Fukushima [62], we propose the following regularized gap function for (WAVVIP).

Let $\gamma > 0$ be a fixed parameter. Let $x, y \in K$ and $\xi \in \partial_c f(y)$. Consider the function $\Theta^\gamma : K \to \mathbb{R}$ be defined as

$$\Theta^\gamma(y) := \inf_{\xi\in\partial_c f(y)} \sup_{x\in K} \min_{i\in I} \left\{ \left\langle P_{0,\Omega}^{1}\xi_i, \exp_{x}^{-1}(y)\right\rangle_{x} - \beta \left\|\exp_{x}^{-1}(y)\right\|_{x} - \frac{1}{2\gamma} \left\|\exp_{x}^{-1}(y)\right\|_{x}^{2} \right\},$$

where $\Omega(t) = \exp_{y}\left(t \exp_{y}^{-1}(x)\right), \forall t \in [0,1]$.

For more exposition to the regularized gap function, we refer to [27, 34].

In the following theorem, we establish that the function $\Theta^\gamma(y)$ is a gap function for (WAVVIP).

Theorem 3.4. *For each $i \in I$, let $f_i : K \to \mathbb{R}$ be locally Lipschitz function and let $\gamma > 0$. Then, the function Θ^γ is a gap function for (WAVVIP).*

Proof. (a) On the contrary, we suppose that there exists $y \in K$ such that

$$\Theta^{\gamma}(y) < 0.$$

From the definition of $\Theta^{\gamma}(y)$, there exists $\xi_i \in \partial_c f_i(y), i \in I$ such that

$$\sup_{x\in K} \min_{i\in I} \left\{ \left\langle P^1_{0,\Omega}\xi_i, \exp_x^{-1}(y)\right\rangle_x - \beta \left\|\exp_x^{-1}(y)\right\|_x - \frac{1}{2\gamma} \left\|\exp_x^{-1}(y)\right\|_x^2 \right\} < 0,$$

where $\Omega(t) = \exp_y(t \exp_y^{-1}(x)),\ \forall t \in [0,1]$. It follows that for every $x \in K$ we have

$$\min_{i\in I} \left\{ \left\langle P^1_{0,\Omega}\xi_i, \exp_x^{-1}(y)\right\rangle_x - \beta \left\|\exp_x^{-1}(y)\right\|_x - \frac{1}{2\gamma} \left\|\exp_x^{-1}(y)\right\|_x^2 \right\} < 0,$$

which is a contradiction for $x = y$. Therefore, we have

$$\Theta^{\gamma}(y) \geq 0,\ \forall y \in K.$$

(b) Let $\bar{x} \in K$ be an element, such that $\Theta^{\gamma}(\bar{x}) = 0$. Then from the definition of Θ^{γ}, there exists $\xi_i \in \partial_c f_i(\bar{x}), i \in I$, such that

$$\sup_{x\in K} \min_{i\in I} \left\{ \left\langle P^1_{0,\Omega}\xi_i, \exp_x^{-1}(\bar{x})\right\rangle_x - \beta \left\|\exp_x^{-1}(\bar{x})\right\|_x - \frac{1}{2\gamma} \left\|\exp_x^{-1}(\bar{x})\right\|_x^2 \right\} = 0,$$

where $\Omega(t) = \exp_{\bar{x}}\left(t \exp_{\bar{x}}^{-1}(x)\right), \forall t \in [0,1]$. That is, for every $x \in K$, we have

$$\min_{i\in I} \left\{ \left\langle P^1_{0,\Omega}\xi_i, \exp_x^{-1}(\bar{x})\right\rangle_x - \beta \left\|\exp_x^{-1}(\bar{x})\right\|_x - \frac{1}{2\gamma} \left\|\exp_x^{-1}(\bar{x})\right\|_x^2 \right\} \leq 0. \tag{3.7}$$

Now, consider an arbitrary but fixed $\hat{x} \in K$. For $t \in (0,1)$, let $\hat{x}_t := \Omega^*(t) := \exp_{\bar{x}}(t \exp_{\bar{x}}^{-1}(\hat{x}))$. Since K is a geodesic convex set, it is evident that $\hat{x}_t \in K$, for all $t \in (0,1)$.

In view of (3.7) and the fact that $\hat{x}_t \in K$ $(t \in (0,1))$, we have

$$\begin{aligned}\min_{i\in I} &\left\{ -\left\langle P^t_{0,\Omega^*}\xi_i, P^t_{0,\Omega^*}\exp_{\bar{x}}^{-1}(\hat{x}_t)\right\rangle_{\bar{x}} - \beta \left\|\exp_{\bar{x}}^{-1}(\hat{x}_t)\right\|_{\bar{x}} \right\} \\ &\leq \frac{1}{2\gamma} \left\|\exp_{\bar{x}}^{-1}(\hat{x}_t)\right\|_{\bar{x}}^2,\end{aligned} \tag{3.8}$$

From (3.8) and $\exp_{\bar{x}}^{-1}(\hat{x}_t) = t\exp_{\bar{x}}^{-1}(\hat{x})$, it follows that

$$\min_{i\in I} \left\{ -\left\langle P^t_{0,\Omega^*}\xi_i, P^t_{0,\Omega^*}\exp_{\bar{x}}^{-1}(\hat{x})\right\rangle_{\bar{x}} - \beta \left\|\exp_{\bar{x}}^{-1}(\hat{x})\right\|_{\bar{x}} \right\} \leq \frac{t}{2\gamma} \left\|\exp_{\bar{x}}^{-1}(\hat{x})\right\|_{\bar{x}}^2. \tag{3.9}$$

In light of the fact that parallel transport preserves the inner product, we yield the following from (3.9):

$$\min_{i\in I} \left\{ -\left\langle \xi_i, \exp_{\bar{x}}^{-1}(\hat{x})\right\rangle_{\bar{x}} - \beta \left\|\exp_{\bar{x}}^{-1}(\hat{x})\right\|_{\bar{x}} \right\} \leq \frac{t}{2\gamma} \left\|\exp_{\bar{x}}^{-1}(\hat{x})\right\|_{\bar{x}}^2. \tag{3.10}$$

Letting $t \downarrow 0$ in (3.10), we have

$$\min_{i\in I} \left\{ -\left\langle \xi_i, \exp_{\bar{x}}^{-1}(\hat{x})\right\rangle_{\bar{x}} - \beta \left\|\exp_{\bar{x}}^{-1}(\hat{x})\right\|_{\bar{x}} \right\} \leq 0. \tag{3.11}$$

This implies that there exists $k \in I$ depending on $\hat{x}$, such that

$$\left\langle \xi_i, \exp_{\bar{x}}^{-1}(\hat{x})\right\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(\hat{x})\right\|_{\bar{x}} \geq 0.$$

By following similar steps for any $x \in K$, there exist $\xi_i \in \partial_c f_i(\bar{x}),\ i \in I$, such that

$$\big(\langle \xi_1, \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(x)\right\|_{\bar{x}}, \ldots, \langle \xi_p, \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(x)\right\|_{\bar{x}}\big) \notin \operatorname{int} C,$$

Hence, $\bar{x}$ is a solution of (WAVVIP).

Conversely, let $\bar{x} \in K$ solves (WAVVIP). Then for all $x \in K$, there exists $\xi_i \in \partial_c f_i(\bar{x}), i \in I$, such that

$$\big(\langle \xi_1, \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(x)\right\|_{\bar{x}}, \ldots, \langle \xi_p, \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(x)\right\|_{\bar{x}}\big) \notin \operatorname{int} C.$$

That is, for every $x \in K$ there exists some $k \in I$, such that

$$\langle \xi_k, \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(x)\right\|_{\bar{x}} \geq 0. \tag{3.12}$$

From (3.12), it follows that

$$\langle \xi_k, \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(x)\right\|_{\bar{x}} + \frac{1}{2\gamma} \left\|\exp_{x}^{-1}(\bar{x})\right\|_{\bar{x}}^2 \geq 0. \tag{3.13}$$

Using parallel translation from $\bar{x}$ to x in (3.13) we get

$$\langle P_{0,\Omega}^1 \xi_k, \exp_{x}^{-1}(\bar{x})\rangle_{x} - \beta \left\|\exp_{x}^{-1}(\bar{x})\right\|_{x} - \frac{1}{2\gamma} \left\|\exp_{x}^{-1}(\bar{x})\right\|_{x}^2 \leq 0, \tag{3.14}$$

where $\Omega(t) = \exp_{\bar{x}}\left(t \exp_{\bar{x}}^{-1}(x)\right), \forall t \in [0,1]$. From (3.14) it follows that

$$\sup_{x \in K} \min_{i \in I} \left\{ \langle P_{0,\Omega}^1 \xi_i, \exp_{x}^{-1}(\bar{x})\rangle_{x} - \beta \left\|\exp_{x}^{-1}(\bar{x})\right\|_{x} - \frac{1}{2\gamma} \left\|\exp_{x}^{-1}(\bar{x})\right\|_{x}^2 \right\} \leq 0.$$

$$\begin{aligned}\Theta^{\gamma}(\bar{x}) = \inf_{\xi \in \partial_c f(\bar{x})} \sup_{x \in K} \min_{i \in I} \Big\{ &\langle P_{0,\Omega}^1 \xi_i, \exp_{x}^{-1}(\bar{x})\rangle_{x} - \beta \left\|\exp_{x}^{-1}(\bar{x})\right\|_{x} \\ &- \frac{1}{2\gamma} \left\|\exp_{x}^{-1}(\bar{x})\right\|_{x}^2 \Big\} \\ \leq 0. \end{aligned} \tag{3.15}$$

Since $\Theta^{\gamma}(x) \geq 0$, for all $x \in K$, therefore, from (3.15) we have

$$\Theta^{\gamma}(\bar{x}) = 0.$$

Hence, the proof is complete. □

Remark 3.5. The regularized gap function Θ^{γ} extends the regularized gap function formulated by Charitha et al. [13].

To illustrate the significance of Theorems 3.2 and 3.4 we have the following example.

Example 3.6. Let $M := \{x \in \mathbb{R} : x > 0\}$ be endowed with a Riemannian metric $\mathcal{G}(x) := x^{-2}$. M is a Hadamard manifold (see, [27]) and the tangent space at $x \in M$, denoted by $T_xM = \mathbb{R}$.

The geodesic curve $\Omega : \mathbb{R} \to M$ starting from x with unit tangent vector $w \in T_xM$ is $\Omega(t) := \exp_x(tw) := xe^{\left(\frac{w}{x}\right)t}$. The inverse of the exponential map for any $x \in M$ is $\exp_x^{-1}(y) := x \ln\left(\frac{y}{x}\right),\ \forall y \in M$.

Let $K := \{x : x = e^t, t \in [0,1]\}$ and $f_1, f_2 : K \to \mathbb{R}$ be given as

$$f_1(x) := \begin{cases} x^2, & x \geq 1 \\ 2x-1, & x < 1 \end{cases}, \quad f_2(x) \quad := \begin{cases} x^2 + x, & x \geq 1 \\ x+1, & x < 1 \end{cases}.$$

The Clarke subdifferentials of f_1, f_2 are given by

$$\partial_c f_1(x) = \begin{cases} 2x^3, & x > 1, \\ 2tx^3 + 2(1-t)x^2 : t \in [0,1], & x = 1, \\ 2x^2, & x < 1, \end{cases}$$

$$\partial_c f_2(x) = \begin{cases} 2x^3 + x^2, & x > 1, \\ 2sx^3 + x^2 : s \in [0,1], & x = 1, \\ x^2, & x < 1. \end{cases}$$

Consider the following weak approximate vector variational inequality problem for $\beta = 1$: Find $\bar{x} \in K$ such that for every $y \in K$, there exists $\xi_i \in \partial_c f_i(\bar{x}),\ i \in I$, satisfying:

$$\text{(WAVVIP1)} : \Bigg(\left\langle \xi_1, \bar{x} \ln\left(\frac{y}{\bar{x}}\right) \right\rangle_{\bar{x}} + \beta \bar{x} \left| \ln\left(\frac{y}{\bar{x}}\right) \right|_{\bar{x}}, \\ \left\langle \xi_2, \bar{x} \ln\left(\frac{y}{\bar{x}}\right) \right\rangle_{\bar{x}} + \beta \bar{x} \left| \ln\left(\frac{y}{\bar{x}}\right) \right|_{\bar{x}} \Bigg) \notin \operatorname{int} C',$$

where $C' = -\mathbb{R}^2_+ \backslash \{0\}$.

Now, consider the function $\Theta : K \to \mathbb{R}$, which is defined as follows:

$$\begin{aligned} \Theta(y) &= \inf_{\xi \in \partial_c f(y)} \sup_{x \in K} \min_{1 \leq i \leq 2} \left\{ \left\langle P^1_{0,\Omega} \xi_i, \exp_x^{-1}(y) \right\rangle_x - \beta \left\| \exp_x^{-1}(y) \right\|_x \right\} \\ &\qquad\qquad \text{(on applying parallel translation from x to y)} \\ &= \inf_{\xi \in \partial_c f(y)} \sup_{x \in [1,e]} \min \Big\{ - \left\langle \xi_1, \exp_y^{-1}(x) \right\rangle_y - \beta \left\| \exp_x^{-1}(y) \right\|_x, \\ &\qquad\qquad - \left\langle \xi_2, \exp_y^{-1}(x) \right\rangle_y - \beta \left\| \exp_x^{-1}(y) \right\|_x \Big\} \\ &= \begin{cases} (2y^2 - 1) \ln y, & y > 1, \\ 0, & y = 1, \end{cases} \\ &\geq 0, \forall y \in K. \end{aligned}$$

Notably, $\Theta(\bar{x}) = 0$ as $\bar{x} = 1 \in K$ is a solution of (WAVVIP1) and vice versa. Therefore, Θ is a gap function for (WAVVIP1).

Let $\gamma = 2$ and $\Theta^{\gamma} : K \to \mathbb{R}$ be a real-valued function defined as follows :

$$\begin{aligned}
\Theta^{\gamma}(y) &= \inf_{\xi \in \partial_c f(y)} \sup_{x \in K} \min_{1 \leq i \leq 2} \Big\{ \left\langle P^1_{0,\Omega}\xi_i, \exp_x^{-1}(y) \right\rangle_x - \beta \left\| \exp_x^{-1}(y) \right\|_x \\
&\qquad - \frac{1}{2\gamma} \left\| \exp_x^{-1}(y) \right\|_x^2 \Big\} \\
&\qquad \text{(on applying parallel translation from } x \text{ to } y) \\
&= \inf_{\xi \in \partial_c f(y)} \sup_{x \in [1,e]} \min \Big\{ -\left\langle \xi_1, \exp_y^{-1}(x) \right\rangle_y - \beta \left\| \exp_x^{-1}(y) \right\|_x \\
&\qquad - \frac{1}{2\gamma} \left\| \exp_x^{-1}(y) \right\|_x^2, \\
&\qquad -\left\langle \xi_2, \exp_y^{-1}(x) \right\rangle_y - \beta \left\| \exp_x^{-1}(y) \right\|_x - \frac{1}{2\gamma} \left\| \exp_x^{-1}(y) \right\|_x^2 \Big\} \\
&= \begin{cases} \left(2y^2 - \frac{5}{4}\right) \ln y, & y > 1, \\ 0, & y = 1, \end{cases} \\
&\geq 0, \forall y \in K.
\end{aligned}$$

Notably, $\Theta^{\gamma}(\bar{x}) = 0$ as $\bar{x} = 1 \in K$ is a solution of (WAVVIP1) and vice versa. Therefore, Θ^{γ} is a gap function for (WAVVIP1).

4. Existence of solution of (WAVVIP)

In this section, by employing the KKM-Fan lemma, we derive certain conditions under which the solutions of weak approximate vector variational inequality problem (WAVVIP) exist.

Definition 4.1 ([17])**.** The set-valued map $G : K \to 2^K$ is said to be a KKM mapping, if for any finite number of elements $x_1, x_2, \ldots, x_m \in K$ one has

$$\operatorname{conv}\left(\{x_1, \ldots, x_m\}\right) \subseteq \bigcup_{i=1}^{m} G\left(x_i\right).$$

Lemma 4.2 ([17,63])**.** *Let M be a Hadamard manifold and K be a nonempty subset of M. Let $G : K \to 2^K$ be a KKM mapping, such that for each $x \in K, G(x)$ is closed. Moreover, suppose that there exists $x_0 \in K$, such that $G\left(x_0\right)$ is compact. Then,*

$$\bigcap_{x \in K} G(x) \neq \emptyset.$$

In the following theorem, we establish the existence of a solution for (WAVVIP) under relaxed compactness assumption and without any monotonicity on $\partial_c f_i, i \in I$.

Theorem 4.3. *Let M be of constant sectional curvature and K be a nonempty, closed and geodesic convex set. For each $i \in I$, let $f_i : K \to \mathbb{R}$ be locally Lipschitz function on K and let $u \in K$ be any given point. Moreover, suppose that there exists* *a nonempty bounded subset D of K such that for each $x \in K \setminus D$, there exist $\beta > 0$ and $y \in D$, such that for any $\xi_i \in \partial_c f_i(x), i \in I$, one has*

$$\left(\left\langle \xi_1, \exp_x^{-1}(y)\right\rangle_x + \beta \left\|\exp_x^{-1}(y)\right\|_x, \ldots, \left\langle \xi_p, \exp_x^{-1}(y)\right\rangle_x + \beta \left\|\exp_x^{-1}(y)\right\|_x\right) \in \operatorname{int} C.$$

Then (WAVVIP) has a solution.

Proof. Let us choose $K_\delta = K \cap \overline{B(u,\delta)}$, sufficiently large such that $D \subset \operatorname{int}_K K_\delta$, where $\operatorname{int}_K K_\delta$ is the relative interior of K_δ to K and

$$\overline{B(u,\delta)} = \{x \in K : d(u,x) \le \delta\}.$$

Then K_δ is nonempty, compact and convex set. Define the set-valued mapping $\Gamma : K_\delta \to 2^{K_\delta}$ by

$$\Gamma(y) := \left\{x \in K_\delta : \exists \xi_i \in \partial_c f_i(x), i \in I, \left(\left\langle \xi_1, \exp_x^{-1}(y)\right\rangle_x + \beta \left\|\exp_x^{-1}(y)\right\|_x, \ldots, \left\langle \xi_p, \exp_x^{-1}(y)\right\rangle_x + \beta \left\|\exp_x^{-1}(y)\right\|_x\right) \notin \operatorname{int} C\right\}.$$

The set $\Gamma(y)$ is nonempty because it contains y. The proof is divided into the following parts:

(i) The set Γ is a KKM map on K_δ. Suppose to the contrary that there exists $\{x_1, \ldots, x_m\} \subseteq K_\delta$ and $x \in \operatorname{conv}(\{x_1, \ldots, x_m\})$, but $x \notin \bigcup_{j=1}^m \Gamma(x_j)$, thus, for each $j = 1, 2, \ldots, m$

$$\left(\left\langle \xi_1, \exp_x^{-1}(x_j)\right\rangle_x + \beta \left\|\exp_x^{-1}(x_j)\right\|_x, \ldots, \left\langle \xi_p, \exp_x^{-1}(x_j)\right\rangle_x + \beta \left\|\exp_x^{-1}(x_j)\right\|_x\right) \in \operatorname{int} C.$$

That is,

$$\left\langle \xi_i, \exp_x^{-1}(x_j)\right\rangle_x + \beta \left\|\exp_x^{-1}(x_j)\right\|_x < 0, \forall i \in I. \tag{4.1}$$

Let

$$U := \{y \in K_\delta : \forall \xi_i \in \partial_c f_i(x), i \in I, \left\langle \xi_i, \exp_x^{-1}(y)\right\rangle_x + \beta \left\|\exp_x^{-1}(y)\right\|_x < 0\}.$$

For any $i \in \{1, 2, \ldots, m\}$, let

$$W_{x,\xi_i} := \{y \in K_\delta : \forall \xi_i \in \partial_c f_i(x), \langle \xi_i, \exp_x^{-1}(y)\rangle_x + \beta \| \exp_x^{-1}(y)\|_x < 0\}.$$

One can easily see that

$$U := \bigcap_{i=1}^m W_{x,\xi_i}.$$

Therefore, in order to show that U is a geodesic convex set, we will show that W_{x,ξ_i} is a geodesic convex set for every $i = 1, 2, \ldots, m$. Let $p, q \in W_{x,\xi_i}$ such that $p \neq q$. We will show that the geodesic segment $\Omega : [0,1] \to M$ joining $\Omega(0) = p$ and $\Omega(1) = q$ is contained in W_{x,ξ_i}, that is $\Omega(t) \in W_{x,\xi_i}, \forall t \in [0,1]$.

Since M is of constant sectional curvature, therefore, there exists a two-dimensional totally geodesic submanifold H of M which contains x, p, and q. It follows that $\exp_x^{-1}(\Omega(t))$ is contained in a two-dimensional pointed convex cone of T_xM, which is spanned by $\exp_x^{-1}(p)$ and $\exp_x^{-1}(q)$ (see, Corollary 3.1 in [20]). Hence, there exist $a, b \ge 0$ for every $t \in [0,1]$ such that

$$\exp_x^{-1}(\Omega(t)) = a \exp_x^{-1}(\Omega(0)) + b \exp_x^{-1}(\Omega(1)).$$

Suppose $a = 0 = b$, which implies that there exists a $\hat{t} \in (0,1)$ such that $\Omega(\hat{t}) = x$ and one can have the following:

$$\hat{t}\exp_x^{-1}(p) + (1-\hat{t})\exp_x^{-1}(q) = 0. \tag{4.2}$$

From (4.2) and in view of the fact that $p, q \in W_{x,\xi_i}$, it follows that

$$0 = \hat{t}\langle \xi_i, \exp_x^{-1}(p)\rangle_x + (1-\hat{t})\langle \xi_i, \exp_x^{-1}(q)\rangle_x < 0,$$

which is a contradiction. Therefore, $a > 0$ or $b > 0$. It follows that

$$\begin{aligned}\langle \xi_i, \exp_x^{-1}(\Omega(t))\rangle_x + \beta\|\exp_x^{-1}(\Omega(t))\|_x &\le a(\langle \xi_i, \exp_x^{-1}(p)\rangle_x + \beta\|\exp_x^{-1}(p)\|) \\ &\quad + b(\langle \xi_i, \exp_x^{-1}(q)\rangle_x + \beta\|\exp_x^{-1}(q)\|_x) \\ &< 0.\end{aligned}$$

Therefore, $\Omega(t) \in W_{x,\xi_i}, \forall t \in [0,1]$. Hence, W_{x,ξ_i} is a geodesic convex set for every $i = 1, 2, \ldots, m$, which implies that U is a geodesic convex set. In view of the geodesic convexity of U and

$$x \in \operatorname{conv}\left(\{x_1, \ldots, x_m\}\right),$$

it follows that $0 < 0$, which is a contradiction. Hence, $\Gamma(y)$ is a KKM map.

(ii) $\Gamma(y)$ is compact for any $y \in K_\delta$. Since, $\Gamma(y) \subseteq K_\delta$ and K_δ is compact. Therefore, we only need to show that Γ is closed valued. Let $\{x_m\}$ be a sequence in $\Gamma(y)$, which converges to $\bar{x} \in K_\delta$, where $x_m = \Omega(t_m) = \exp_{\bar{x}}\left(t_m \exp_{\bar{x}}^{-1}(y)\right) \in K$. Therefore, there exists $\xi_i^m \in \partial_c f_i(x_m)$, for $i \in I$, such that

$$\Big(\left\langle \xi_1^m, \exp_{x_m}^{-1}(y)\right\rangle_{x_m} + \beta\left\|\exp_{x_m}^{-1}(y)\right\|_{x_m}, \ldots, \left\langle \xi_p^m, \exp_{x_m}^{-1}(y)\right\rangle_{x_m} + \beta\left\|\exp_{x_m}^{-1}(y)\right\|_{x_m}\Big) \notin \operatorname{int} C.$$

Hence,

$$\Big(\left\langle P^0_{t_m,\Omega}\xi_1^m, P^0_{t_m,\Omega}\exp_{x_m}^{-1}(y)\right\rangle_{x_m} + \beta\left\|\exp_{x_m}^{-1}(y)\right\|_{x_m}, \ldots, \left\langle P^0_{t_m,\Omega}\xi_p^m, P^0_{t_m,\Omega}\exp_{x_m}^{-1}(y)\right\rangle_{x_m} + \beta\left\|\exp_{x_m}^{-1}(y)\right\|_{x_m}\Big) \notin \operatorname{int} C.$$

Since each f_i is locally Lipschitz function, therefore, from Lemma 2.22 there exists $L > 0$ such that for sufficiently large m and $i \in I$, $\|\xi_i^m\|_{T_{x_m}M^*} \le L$. It follows that

$$\left\|P^0_{t_m,\Omega}\left(\xi_i^m\right)\right\|_{T_{\bar{x}}M^*} \le L,$$

and so there is a subsequence $P^0_{t_{m_k},\Omega}(\xi_i^{m_k}) \to \xi_i$ in weak* topology. It follows from Lemma 2.22 that $\xi_i \in \partial_c f_i(\bar{x})$ for each $i \in I$. Therefore, there exists $\xi_i \in \partial_c f_i(\bar{x})$ for each $i \in I$, such that

$$\left(\left\langle \xi_1, \exp_{\bar{x}}^{-1}(y)\right\rangle_{\bar{x}} + \beta\left\|\exp_{\bar{x}}^{-1}(y)\right\|_{\bar{x}}, \ldots, \left\langle \xi_p, \exp_{\bar{x}}^{-1}(y)\right\rangle_{\bar{x}} + \beta\left\|\exp_{\bar{x}}^{-1}(y)\right\|_{\bar{x}}\right) \notin \operatorname{int} C.$$

This shows that $\bar{x} \in \Gamma(y)$ and so $\Gamma(y)$ is closed for every $y \in K$.
Therefore, (WAVVIP) has a solution: From Lemma 4.2, we have

$$\bigcap_{y \in K_\delta} \Gamma(y) \neq \emptyset.$$

Therefore, there exists $\bar{x} \in K_\delta$ such that for each $y \in K_\delta$, there exist $\xi_i \in \partial_c f_i(\bar{x})$, $i \in I$ such that

$$\begin{aligned}(4.3)\qquad \big(\left\langle \xi_1, \exp_{\bar{x}}^{-1}(y)\right\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(y)\right\|_{\bar{x}}, \ldots, \left\langle \xi_p, \exp_{\bar{x}}^{-1}(y)\right\rangle_{\bar{x}} \\ +\beta \left\|\exp_{\bar{x}}^{-1}(y)\right\|_{\bar{x}}\big) \notin \operatorname{int} C.\end{aligned}$$

From the assumption, it is obvious that $\bar{x} \in B(u, \delta)$. For any $w \in K \backslash K_\delta$, there exists $\mu \in (0,1)$ such that $\exp_{\bar{x}}(\mu \exp_{\bar{x}}^{-1}(w)) \in K_\delta$. From (4.3), it follows that there exist $\xi_i \in \partial_c f_i(\bar{x}), i \in I$ such that

$$\begin{aligned}&\big(\left\langle \xi_1, \exp_{\bar{x}}^{-1}(\exp_{\bar{x}}(\mu \exp_{\bar{x}}^{-1}(w)))\right\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(\exp_{\bar{x}}(\mu \exp_{\bar{x}}^{-1}(w)))\right\|_{\bar{x}}, \ldots, \\ &\left\langle \xi_p, \exp_{\bar{x}}^{-1}(\exp_{\bar{x}}(\mu \exp_{\bar{x}}^{-1}(w))\right\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(\exp_{\bar{x}}(\mu \exp_{\bar{x}}^{-1}(w)))\right\|_{\bar{x}}\big) \notin \operatorname{int} C.\end{aligned}$$

That is,

$$\begin{aligned}\big(\left\langle \xi_1, \exp_{\bar{x}}^{-1}(w)\right\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(w)\right\|_{\bar{x}}, \ldots, \left\langle \xi_p, \exp_{\bar{x}}^{-1}(w)\right\rangle_{\bar{x}} \\ +\beta \left\|\exp_{\bar{x}}^{-1}(w)\right\|_{\bar{x}}\big) \notin \operatorname{int} C.\end{aligned}$$

For each $y \in K$, there exists $\xi_i \in \partial_c f_i(\bar{x}), i \in I$, such that

$$\begin{aligned}\big(\left\langle \xi_1, \exp_{\bar{x}}^{-1}(y)\right\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(y)\right\|_{\bar{x}}, \ldots, \left\langle \xi_p, \exp_{\bar{x}}^{-1}(y)\right\rangle_{\bar{x}} \\ +\beta \left\|\exp_{\bar{x}}^{-1}(y)\right\|_{\bar{x}}\big) \notin \operatorname{int} C,\end{aligned}$$

which implies that $\bar{x}$ is a solution of (WAVVIP). Hence, the proof is complete. □

Remark 4.4. Theorem 4.3 generalizes Theorem 3.9 of Chen and Huang [15] to a more general problem, namely weak approximate vector variational inequality problems on Hadamard manifolds.

The following lemma will be useful to prove the existence result for (WAVVIP) with monotonicity.

Lemma 4.5. *For each $i \in I$, let $f_i : K \to \mathbb{R}$ be locally Lipschitz geodesic α_i-convex function on K. Moreover, if we assume that there exists $\bar{x} \in K$, such that*

$$\begin{aligned}&\Big(\left\langle \zeta_1, \exp_y^{-1}(\bar{x})\right\rangle_y - \beta \left\|\exp_y^{-1}(\bar{x})\right\|_y, \ldots, \\ &\qquad\qquad \left\langle \zeta_p, \exp_y^{-1}(\bar{x})\right\rangle_y - \beta \left\|\exp_y^{-1}(\bar{x})\right\|_y\Big) \notin -\operatorname{int} C,\end{aligned}$$

for all $y \in K$ and $\zeta_i \in \partial_c f_i(y), i \in I$, then $\bar{x}$ is a solution of (WAVVIP).

Proof. Assume that there exists $\bar{x} \in K$, such that for all $y \in K$ and $\zeta_i \in \partial_c f_i(y), i \in I$, we have

$$\begin{aligned}(4.4)\qquad &\Big(\left\langle \zeta_1, \exp_y^{-1}(\bar{x})\right\rangle_y - \beta \left\|\exp_y^{-1}(\bar{x})\right\|_y, \ldots, \\ &\qquad\qquad \left\langle \zeta_p, \exp_y^{-1}(\bar{x})\right\rangle_y - \beta \left\|\exp_y^{-1}(\bar{x})\right\|_y\Big) \notin -\operatorname{int} C.\end{aligned}$$

Let $x_t = \Omega(t) := \exp_{\bar{x}}\left(t \exp_{\bar{x}}^{-1}(y)\right), \forall t \in (0,1)$. Since K is a geodesic convex set, therefore $x_t \in K, \forall t \in (0,1)$. Setting $y = x_t$ in (4.4), for all $\zeta_i^t \in \partial_c f_i(x_t), i \in I$ we get

$$\Big(\langle\zeta_1^t, \exp_{x_t}^{-1}(\bar{x})\rangle_{x_t} - \beta\left\|\exp_{x_t}^{-1}(\bar{x})\right\|_{x_t}, \ldots, \langle\zeta_p^t, \exp_{x_t}^{-1}(\bar{x})\rangle_{x_t} - \beta\left\|\exp_{x_t}^{-1}(\bar{x})\right\|_{x_t}\Big) \notin -\operatorname{int} C,$$

and so

$$\Big(\langle P_{t,\Omega}^0\zeta_1^t, P_{t,\Omega}^0\exp_{x_t}^{-1}(\bar{x})\rangle_{x_t} - \beta\left\|\exp_{x_t}^{-1}(\bar{x})\right\|_{x_t}, \ldots, \langle P_{t,\Omega}^0\zeta_p^t, P_{t,\Omega}^0\exp_{x_t}^{-1}(\bar{x})\rangle_{x_t} - \beta\left\|\exp_{x_t}^{-1}(\bar{x})\right\|_{x_t}\Big) \notin -\operatorname{int} C. \tag{4.5}$$

Using the definition of x_t and $\left\|\exp_{\bar{x}}^{-1}(x_t)\right\|_{\bar{x}} = t\left\|\exp_{\bar{x}}^{-1}(y)\right\|_{\bar{x}}$, for $t \in (0,1)$ in (4.5) we have

$$\Big(\langle P_{t,\Omega}^0\zeta_1^t, \exp_{\bar{x}}^{-1}(y)\rangle_{\bar{x}} + \beta\left\|\exp_{\bar{x}}^{-1}(y)\right\|_{\bar{x}}, \ldots, \langle P_{t,\Omega}^0\zeta_p^t, \exp_{\bar{x}}^{-1}(y)\rangle_{\bar{x}} + \beta\left\|\exp_{\bar{x}}^{-1}(y)\right\|_{\bar{x}}\Big) \notin \operatorname{int} C. \tag{4.6}$$

Since, each f_i is locally Lipschitz, therefore, from Lemma 2.22 (i), there exists some $L > 0$, such that

$$\left\|\zeta_i^t\right\|_{T_{x_t}M^*} \leq L, \forall \zeta_i^t \in \partial_c f_i\left(x_t\right), i \in I,$$

that is,

$$\left\|P_{t,\Omega}^0\zeta_i^t\right\|_{T_{\bar{x}}M^*} \leq L.$$

Hence, there exists a subsequence $P_{t,\Omega}^0\zeta_i^t$ (without relabeling), which converges to $\xi_i \in \partial_c f_i(\bar{x})$, $i \in I$ in weak* topology. From Lemma 2.22 (ii), it follows that $\xi_i \in \partial_c f_i(\bar{x}), i \in I$. Therefore, for all $y \in K$, there exists $\xi_i \in \partial_c f_i(\bar{x})$, $i \in I$, such that

$$\big(\langle\xi_1, \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} + \beta\left\|\exp_{\bar{x}}^{-1}(x)\right\|_{\bar{x}}, \ldots, \langle\xi_p, \exp_{\bar{x}}^{-1}(x)\rangle_{\bar{x}} + \beta\left\|\exp_{\bar{x}}^{-1}(x)\right\|_{\bar{x}}\big) \notin \operatorname{int} C.$$

Thus, $\bar{x}$ is a solution of (WAVVIP). Hence, the proof is complete. □

In the following theorem, we will prove the existence result for the solution of (WAVVIP) under geodesic α-monotonicity assumption within the framework of $n-$dimensional Hadamard manifold M with constant sectional curvature.

Theorem 4.6. *Let K be a nonempty, compact and geodesic convex set. For each $i \in I$, let $f_i : K \to \mathbb{R}$ be locally Lipschitz geodesic α_i-convex function on K. Then (WAVVIP) has a solution.*

Proof. Define two set-valued mappings $\Gamma, \overline{\Gamma} : K \to 2^K$ by

$$\Gamma(y) := \Big\{x \in K : \forall\zeta_i \in \partial_c f_i(y), \Big(\langle\zeta_1, \exp_y^{-1}(x)\rangle_y - \beta\left\|\exp_y^{-1}(x)\right\|_y, \ldots, \langle\zeta_p, \exp_y^{-1}(x)\rangle_y - \beta\left\|\exp_y^{-1}(x)\right\|_y\Big) \notin -\operatorname{int} C\Big\},$$

$$\bar{\Gamma}(y) := \left\{x \in K : \forall\xi_i \in \partial_c f_i(x), \big(\langle\xi_1, \exp_x^{-1}(y)\rangle_x, \ldots, \langle\xi_p, \exp_x^{-1}(y)\rangle_x\big) \notin \operatorname{int} C\right\},$$

for each $y \in K$. Evidently, $\Gamma(x)$ and $\bar{\Gamma}(x)$ are nonempty, because they contain x. The proof is divided into the following parts:

(i) $\overline{\Gamma}$ is a KKM map on K. Suppose to the contrary that there exists $\{x_1, \ldots, x_m\} \subseteq K$ and $x \in \operatorname{conv}(\{x_1, \ldots, x_m\})$, but $x \notin \bigcup_{j=1}^{m} \overline{\Gamma}(x_j)$, that is, for each $j = 1, 2, \ldots, m$,

$$\left(\left\langle \xi_1, \exp_x^{-1}(x_j)\right\rangle_x, \ldots, \left\langle \xi_p, \exp_x^{-1}(x_j)\right\rangle_x\right) \in \operatorname{int} C.$$

That is,

$$\left\langle \xi_i, \exp_x^{-1}(x_j)\right\rangle_x < 0, \forall i \in I. \tag{4.7}$$

Let

$$U := \left\{y \in K : \forall \xi_i \in \partial_c f_i(x), i \in I, \left\langle \xi_i, \exp_x^{-1}(y)\right\rangle_x < 0\right\}.$$

For any $i \in \{1, 2, \ldots, m\}$, let

$$W_{x,\xi_i} := \{y \in K_\delta : \forall \xi_i \in \partial_c f_i(x), \langle \xi_i, \exp_x^{-1}(y)\rangle_x < 0\}.$$

One can easily see that

$$U := \bigcap_{i=1}^{m} W_{x,\xi_i}.$$

Therefore, in order to show that U is a geodesic convex set, we show W_{x,ξ_i} is a geodesic convex set for every $i = 1, 2, \ldots, m$. Let $p, q \in W_{x,\xi_i}$ such that $p \neq q$. We show that the geodesic $\Omega : [0, 1] \to M$ joining $\Omega(0) = p$ and $\Omega(1) = q$ is contained in W_{x,ξ_i}, that is $\Omega(t) \in W_{x,\xi_i}, \forall t \in [0, 1]$.

Since M is of constant sectional curvature, therefore, there exists a two-dimensional totally geodesic submanifold H which contains x, p, and q. It follows that $\exp_x^{-1}(\Omega(t))$ is contained in a two-dimensional pointed convex cone of T_xM, which is spanned by $\exp_x^{-1}(p)$ and $\exp_x^{-1}(q)$ (see, Corollary 3.1 in [20]). Hence, there exists $a, b \geq 0$ for every $t \in [0, 1]$ such that

$$\exp_x^{-1}(\Omega(t)) = a \exp_x^{-1}(\Omega(0)) + b \exp_x^{-1}(\Omega(1)).$$

Suppose $a = 0 = b$, which implies that there exists a $\hat{t} \in (0, 1)$ such that $\Omega(\hat{t}) = x$ and

$$\hat{t} \exp_x^{-1}(p) + (1 - \hat{t}) \exp_x^{-1}(q) = 0. \tag{4.8}$$

From (4.8) and in view of the fact that $p, q \in W_{x,\xi_i}$, it follows that

$$0 = \hat{t}\langle \xi_i, \exp_x^{-1}(p)\rangle_x + (1 - \hat{t})\langle \xi_i, \exp_x^{-1}(q)\rangle_x < 0,$$

which is a contradiction. Therefore, either $a > 0$ or $b > 0$. Now,

$$\begin{aligned}\langle \xi_i, \exp_x^{-1}(\Omega(t))\rangle_x &= a\langle \xi_i, \exp_x^{-1}(p)\rangle_x + b\langle \xi_i, \exp_x^{-1}(q)\rangle_x \\ &< 0.\end{aligned}$$

Therefore, $\Omega(t) \in W_{x,\xi_i}, \forall t \in [0, 1]$. Hence, W_{x,ξ_i} is a geodesic convex set for every $i = 1, 2, \ldots, m$ which implies that U is geodesic convex. In view of the geodesic convexity of U and

$$x \in \operatorname{conv}(\{x_1, \ldots, x_m\}),$$

it follows that $0 < 0$, which is a contradiction. Hence, $\overline{\Gamma}(y)$ is a KKM map.

(ii) We claim that Γ is a KKM map on K. Since $f_i, i \in I$, is α_i-convex on K, it follows that $\partial_c f_i$ is geodesic α_i-monotone on K. Hence, $\overline{\Gamma}(y) \subseteq \Gamma(y)$, for all $y \in K$, and therefore, $\Gamma(y)$ is a KKM mapping.

(iii) Γ is closed valued. Let $\{x_m\}$ be a sequence in $\Gamma(y)$, which converges to $\bar{x}$, where $x_m = \Omega(t_m) := \exp_{\bar{x}}\left(t_m \exp_{\bar{x}}^{-1}(y)\right) \in K$. Therefore, for $\zeta_i \in \partial_c f_i(y)$, for $i \in I$, one has

$$\begin{aligned}(4.9)\qquad &\Big(\left\langle \zeta_1, \exp_y^{-1}(x_m)\right\rangle_y - \beta \left\|\exp_y^{-1}(x_m)\right\|_y, \dots, \left\langle \zeta_p, \exp_y^{-1}(x_m)\right\rangle_y \\ &\qquad -\beta \left\|\exp_y^{-1}(x_m)\right\|_y\Big) \notin -\operatorname{int} C.\end{aligned}$$

Therefore, as $m \to \infty$, from (4.9) we get

$$\begin{aligned}&\Big(\left\langle \zeta_1, \exp_y^{-1}(\bar{x})\right\rangle_y - \beta \left\|\exp_y^{-1}(\bar{x})\right\|_y, \dots, \left\langle \zeta_p, \exp_y^{-1}(\bar{x})\right\rangle_y \\ &\qquad -\beta \left\|\exp_y^{-1}(\bar{x})\right\|_y\Big) \notin -\operatorname{int} C.\end{aligned}$$

This shows that $\bar{x} \in \Gamma(y)$ and so $\Gamma(y)$ is closed for every $y \in K$.

(iv) (WAVVIP) has a solution: Since K is compact and $\Gamma(y) \subseteq K$, and from (iv) $\Gamma(y)$ is closed, it follows that $\Gamma(y)$ is compact, for every $y \in K$. Therefore, from Lemma 4.2 we have

$$\bigcap_{y \in K} \Gamma(y) \neq \emptyset.$$

Hence, there exists $\bar{x} \in K$, such that for every $y \in K$ and $\zeta_i \in \partial_c f_i(y)$, $i \in I$,

$$\begin{aligned}&\Big(\left\langle \zeta_1, \exp_y^{-1}(\bar{x})\right\rangle_y - \beta \left\|\exp_y^{-1}(\bar{x})\right\|_y, \dots, \left\langle \zeta_p, \exp_y^{-1}(\bar{x})\right\rangle_y \\ &\qquad -\beta \left\|\exp_y^{-1}(\bar{x})\right\|_y\Big) \notin -\operatorname{int} C.\end{aligned}$$

From Lemma 4.5 , it follows that $\bar{x}$ is a solution of (WAVVIP). Hence, the proof is complete. □

To illustrate the significance of Theorem 4.6, we have the following example.

Example 4.7. Let M be the manifold as considered in Example 3.6. Moreover, let $K := \left\{y : y = e^t, t \in [-4, 0]\right\}$ and $f_1, f_2 : K \subseteq M \to \mathbb{R}$ be defined as:

$$f_1(x) := \begin{cases} -x+1, & x > \frac{1}{2}, \\ x, & x \leq \frac{1}{2}, \end{cases} \text{ and } f_2(x) := \begin{cases} x^2, & x > \frac{1}{2}, \\ -x + \frac{3}{4}, & x \leq \frac{1}{2}, \end{cases}$$

It is clear that the functions f_1 and f_2 are locally Lipschitz functions on K. The subdifferentials of $f_i, i = 1, 2$, are given by

$$\partial_c f_1(x) = \begin{cases} -x^2, & x > \frac{1}{2}, \\ \left[-\frac{1}{4}, \frac{1}{4}\right], & x = \frac{1}{2}, \\ x^2, & x < \frac{1}{2}, \end{cases} \text{ and } \partial_c f_2(x) = \begin{cases} 2x^3, & x > \frac{1}{2}, \\ \left[-\frac{1}{4}, \frac{1}{4}\right], & x = \frac{1}{2}, \\ -x^2, & x < \frac{1}{2}. \end{cases}$$

We can verify that the function f_1 is geodesic α_1-convex on K for $\alpha_1 = \frac{3}{2}$, as for all $x, y \in K$ and any $\xi_1 \in \partial_c f_1(x)$, we have

$$f_1(y) - f_1(x) \geq \left\langle \xi_1, \exp_y^{-1}(x)\right\rangle_y - \alpha_1 \left\|\exp_y^{-1}(x)\right\|_y.$$

Similarly, the function f_2 is geodesic α_2-convex on K for $\alpha_2 = 1$, as for all $x, y \in K$ and any $\xi_2 \in \partial_c f_2(x)$, we have

$$f_2(y) - f_2(x) \geq \left\langle \xi_1, \exp_y^{-1}(x)\right\rangle_y - \alpha_2 \left\|\exp_y^{-1}(x)\right\|_y.$$

Moreover, we can see that $\bar{x} = \frac{1}{2}$ is a solution of the following weak approximate vector variational inequality problem for $\beta = 1$:

$$\begin{aligned}\text{WAVVIP2)}: \ \Big(\big\langle \xi_1, \exp_{\bar{x}}^{-1}(y)\big\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(y)\right\|_{\bar{x}}, \\ \big\langle \xi_2, \exp_{\bar{x}}^{-1}(y)\big\rangle_{\bar{x}} + \beta \left\|\exp_{\bar{x}}^{-1}(y)\right\|_{\bar{x}}\Big) \notin \operatorname{int} C',\end{aligned}$$

where $C' := -\mathbb{R}^2_+ \backslash \{0\}$.

5. Conclusion

In this paper, we have considered a new class of weak approximate vector variational inequality problems (WAVVIP) on Hadamard manifolds. We have defined a gap function and a regularized gap function to solve the considered problem (WAVVIP). Moreover, an analogous to KKM-Fan lemma is employed to derive conditions for the existence of solutions of (WAVVIP) under relaxed compactness and without monotonicity assumptions. Furthermore, we have also established the existence results for the solutions of (WAVVIP) under geodesic α-monotonicity conditions. Nontrivial numerical examples have been given to justify the significance of these results. The results of the paper extend and generalize some earlier results of Charitha et al. [13], Lee [31], Mishra and Upadhyay [38], Upadhyay et al. [59], and Yamashita and Fukushima [62] to the more general space as well as to a more general class of functions. Furthermore, the results established in the paper generalize some existence results derived by Chen and Huang [15] from Stampacchia weak vector variational inequality problem to a more general problem, namely (WAVVIP).

The results presented in this paper open up several avenues for future research. For instance, we intend to extend the results derived in this paper for the existence of a solution to the considered (WAVVIP) within the framework of Hadamard manifold with non-constant sectional curvature.

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