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A NEW RECURRENCE FORMULA FOR THE GENERALIZED RAMANUJAN τ-FUNCTION AND ITS SPECIAL CASES

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Abstract. In this article, we introduce and investigate the generalized Ramanujan τ-function $\tau_s(n)$, which by the following generating function:

$$\prod_{m=1}^{\infty}(1-z^m)^s=\sum_{n=1}^{\infty}\tau_s(n)\, z^{n-1} \qquad \big(z\in\mathbb{C};\ |z|<1;\ z\neq 0;\ s\in\mathbb{Z}\setminus\{0\}\big).$$

Based on the properties of triangular numbers, we give a new recurrence relation for the function $\tau_s(n)$. By applying this general formula, we derive presumably new recurrence relations for k-colored partitions and the partition function.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of natural numbers, integers, and complex numbers, respectively. Also let

$$\mathbb{N}_0:=\mathbb{N}\cup\{0\} \quad \text{and} \quad \mathbb{Z}_{\neq 0}:=\mathbb{Z}\setminus\{0\}.$$

We denote by $\lfloor x\rfloor$ the floor function of a real number x.

Motivated essentially by several recent developments on the celebrated Ramanujan τ-functions including (for example) [2], [4], [5], [7] and [12] (see also [10]), here we introduce and investigate the generalized Ramanujan τ-function $\tau_s(n)$ as follows:

$$\prod_{m=1}^{\infty}(1-z^m)^s=\sum_{n=1}^{\infty}\tau_s(n)z^{n-1} \qquad \big(z\in\mathbb{C};\ |z|<1;\ z\neq 0;\ s\in\mathbb{Z}_{\neq 0}\big). \tag{1.1}$$

On the set $s\in\mathbb{N}\cup\{-1\}$ in [10, Th. 1] the following recurrence relation for the function $\tau_s(n)$ function was proved in the above-cited paper [10]:

$$\begin{aligned}(n-1)\tau_s(n)=&\sum_{1\leqq m\leqq d_n}(-1)^{m+1}\left(n-1-\frac{(s+1)m(3m+1)}{2}\right)\\ &\cdot\tau_s\left(n-\frac{m(3m+1)}{2}\right)\\ &+\sum_{1\leqq m\leqq b_n}(-1)^{m+1}\left(n-1-\frac{(s+1)m(3m-1)}{2}\right)\\ &\cdot\tau_s\left(n-\frac{m(3m-1)}{2}\right),\end{aligned} \tag{1.2}$$

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where

$$\tau_s(1) = 1 \quad \text{and} \quad d_n = \frac{\sqrt{24n-23}-1}{6} \quad \text{and} \quad b_n = \frac{\sqrt{24n-23}+1}{6}.$$

We did not notice in [10] that the proof of the statement (1.2) also holds true for the set $s \in \mathbb{Z}_{\neq 0}$. Therefore, if we denote by $p_k(n)$ the k-colored partitions: $p_k(n) = \tau_{-k}(n+1)$, we obtain the following presumably new recurrence relation as a corollary.

Corollary 1.1. *For $n, k \in \mathbb{N}$, the following assertion holds true*:

$$\begin{aligned} np_k(n) = & \sum_{m=1}^{\left\lfloor \frac{\sqrt{24n+1}-1}{6} \right\rfloor} (-1)^{m+1} \left(n - \frac{(1-k)m(3m+1)}{2}\right) \\ & \cdot p_k\left(n - \frac{m(3m+1)}{2}\right) \\ & + \sum_{m=1}^{\left\lfloor \frac{\sqrt{24n+1}+1}{6} \right\rfloor} (-1)^{m+1} \left(n - \frac{(1-k)m(3m-1)}{2}\right) \\ & \cdot p_k\left(n - \frac{m(3m-1)}{2}\right). \end{aligned} \tag{1.3}$$

Remark 1.2. As a consequence of the identity (1.2), for the partition function given by (see [3, A000041])

$$p(n) = \tau_{-1}(n+1),$$

we obtain the classical Euler recurrence relation (see, for example, [1, p. 12])

$$p(n) = \sum_{j \in \mathbb{Z}_{\neq 0}} (-1)^{j-1}\, p(n - \delta_j),$$

where

$$\delta_j = \frac{3j^2 + j}{2}$$

are the pentagonal numbers (see, for details, [3, A000326]).

For $s = 24$, that is, for $\tau(n) = \tau_{24}(n)$, we obtain a recurrence relation for Ramanujan's τ-numbers (see [3, A000594]). Our formula corrects a typographical error in Lehmer's formula given in [6, p. 873] and, more recently, in [9, p. 14]. For further details, see [10, p. 3].

2. Main result

In the following theorem, we present our new recurrence relation for the generalized Ramanujan τ-function $\tau_s(n)$.

Theorem 2.1. *Let $n \in \mathbb{N} \setminus 1$ and $s \in \mathbb{Z}_{\neq 0}$. Then the following recurrence relation holds true:*

$$\tau_s(n) = \frac{1}{n-1} \sum_{m=1}^{\left\lfloor \frac{-1+\sqrt{1+8(n-1)}}{2} \right\rfloor} (-1)^{m+1}(2m+1) \left[n - 1 - \frac{m(m+1)}{2} \left(1 + \frac{s}{3}\right) \right] \cdot \tau_s \left(n - \frac{m(m+1)}{2} \right). \tag{2.1}$$

Proof. We recall that the triangular numbers $\{\omega(n)\}_{\mathbb{N}_0}$ are defined by the following sequence (see [3, A000217]):

$$\omega(m) = \begin{cases} (-1)^k (2k+1) & \left(m = \frac{k(k+1)}{2};\ k \in \mathbb{N}_0 \right) \\ 0 & \text{(otherwise)}. \end{cases} \tag{2.2}$$

By using Jacobi's identity:

$$\prod_{m=1}^{\infty} (1 - z^m)^3 = \sum_{n=0}^{\infty} \omega(n) z^n,$$

we have

$$\frac{d}{dz} \left\{ \ln \left(\sum_{n=1}^{\infty} \tau_s(n) z^{n-1} \right) \right\} = \frac{d}{dz} \left\{ \ln \left(\sum_{n=0}^{\infty} \omega(n) z^n \right)^{s/3} \right\},$$

which yields

$$\frac{\sum\limits_{n=1}^{\infty} (n-1)\tau_s(n) z^{n-2}}{\sum\limits_{n=1}^{\infty} \tau_s(n) z^{n-1}} = \frac{s}{3} \cdot \frac{\sum\limits_{n=1}^{\infty} n\omega(n) z^{n-1}}{\sum\limits_{n=0}^{\infty} \omega(n) z^n}$$

or, equivalently,

$$\left(\sum_{n=1}^{\infty} (n-1)\tau_s(n) z^{n-1} \right) \left(\sum_{n=0}^{\infty} \omega(n) z^n \right) = \frac{s}{3} \left(\sum_{n=1}^{\infty} \tau_s(n) z^{n-1} \right) \left(\sum_{n=1}^{\infty} n\omega(n) z^n \right). \tag{2.3}$$

Now, by using the Cauchy product and comparing the coefficients of z^{n-1}, we obtain

$$\sum_{m=0}^{n-1} (n-1-m)\tau_s(n-m)\omega(m) = \frac{s}{3} \sum_{m=0}^{n-1} m\tau_s(n-m)\omega(m),$$

so that

$$\tau_s(n) = -\frac{1}{n-1} \sum_{m=1}^{n-1} \left[n - 1 - m \left(1 + \frac{s}{3}\right) \right] \omega(m)\tau_s(n-m).$$

The statement of our theorem follows from this last result. ∎

As applications of the result asserted by the theorem, we obtain following presumably new recurrence relations for k-colored partitions and the partition function.

Corollary 2.2. *For $n, k \in \mathbb{N}$, the following recursion formulas hold ture*:

$$np_k(n) = \sum_{m=1}^{\left\lfloor \frac{-1+\sqrt{1+8n}}{2} \right\rfloor} (-1)^{m+1}(2m+1)\left[n - \frac{m(m+1)}{2}\left(1 - \frac{k}{3}\right)\right] \cdot p_k\left(n - \frac{m(m+1)}{2}\right) \tag{2.4}$$

and

$$np(n) = \sum_{m=1}^{\left\lfloor \frac{-1+\sqrt{1+8n}}{2} \right\rfloor} (-1)^{m+1}(2m+1)\left(n - \frac{m(m+1)}{3}\right) \cdot p\left(n - \frac{m(m+1)}{2}\right). \tag{2.5}$$

Remark 2.3. When computing $p(n)$ for large values of n, our formula has approximately $\sqrt{2n}$ terms, while Euler's formula has approximately $\sqrt{8n/3}$. Furthermore, for $s = 24.$ we obtain the classical Ramanujan recurrence relation for Ramanujan's τ-numbers (see [11]).

Example 2.4. For $k = 2$ (2-colored partitions) and $n = 2$, we have

$$\begin{aligned} 2 \cdot p_2(2) &= \sum_{m=1}^{1} (-1)^{m+1}(2m+1)\left[2 - \frac{m(m+1)}{2}\left(1 - \frac{2}{3}\right)\right] \cdot p_2\left(2 - \frac{m(m+1)}{2}\right) \\ &= (-1)^2 \cdot 3 \cdot \left(2 - \frac{1 \cdot 2}{2} \cdot \frac{1}{3}\right) \cdot p_2(1) = 1 \cdot 3 \cdot \left(2 - 1 \cdot \frac{1}{3}\right) \cdot 2 = 10. \end{aligned} \tag{2.6}$$

Therefore, $p_2(2) = 5$, which agrees with the known value given in [3, A000712].

3. Conclusion

After taking the logarithm and differentiating the generating function (1.1), we obtain the following recurrence relation:

$$\tau_s(n) = -\frac{s}{n-1}\sum_{m=1}^{n-1} \sigma(m)\tau_s(n-m). \tag{3.1}$$

A special case happens to be the well-known identity [8, p. 55, Eq. (1)]:

$$p_k(n) = \frac{k}{n}\sum_{m=1}^{n} \sigma(m)p_k(n-m). \tag{3.2}$$

Therefore, by an analogous procedure as in the proof of Theorem 4 in [10], we obtain the generating function for $p_k(n)$ in the following form:

$$\sum_{n=0}^{\infty} p_k(n)q^n = \exp\left(k\sum_{n=1}^{\infty}\frac{\sigma(n)}{n}q^n\right) \qquad (|q|<1), \tag{3.3}$$

where $\sigma(n) = \sum_{d|n} d$ denotes the sum of positive divisors function considered in [3, A000203].

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