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EQUALITY-CONSTRAINED OPTIMIZATION WITH SYMMETRY OVER A GROUP OF PERMUTATIONS

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Abstract. We study optimization problems with equality constraints and objective function that are invariant under the action of a group of permutations of the variables. We prove that the set of local optimals that are regular points satisfying necessary and sufficient conditions is invariant under the action of the group. We demonstrate that this framework is useful for certain optimization problems where we can obtain the global optimum as a fixed point of the group action.

1. Introduction

Let G be a subgroup of S_n, the symmetric group of degree n. We consider optimization problems with equality constraints that are invariant under the action of G on the variables. Specifically, for each $\sigma \in G$, the optimization problem remains unchanged under the permutation

$$(x_1, x_2, \ldots, x_n) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).$$

For example, if e is the identity permutation and (ij) transposes i and j, we consider the group $G = \{e, (12), (45), (12)(45)\} \subset S_5$ and the following optimization problem:

$$\begin{aligned} &\text{minimize} && x_1^2 + x_2^2 + 3x_3^3 + x_4^2 + x_5^2 \\ &\text{subject to} && x_1^2 - x_3 = 1, \quad x_2^2 - x_3 = 1, \quad x_4 + x_5 - x_1^3 - x_2^3 = 3. \end{aligned}$$

This problem is invariant under the action of G, as permutations of x_1 with x_2 and x_4 with x_5 do not change the problem. The global minimum occurs at $x_1 = x_2 = 0$, $x_3 = -1$, and $x_4 = x_5 = 1.5$, where the objective value is 1.5. Furthermore, in this case the optimal point is a fixed point of G.

Such optimization problems naturally occur when the objective and constraints form a set of symmetric functions, such as distance functions in clustering problems [5, 9], covariance matrices in portfolio optimization [7, 8], and geometrically symmetric functions in optimal packing problems [1, 2]. In this paper, we demonstrate that for objective and constraint functions satisfying standard conditions, the set of local optimal points for these optimization problems forms an invariant set under the group action.

Optimization problems in which both the objective and constraint functions are symmetric have been widely studied in the literature (For example, see [10]). However, considerably less attention has been devoted to problems that exhibit symmetry only with respect to a proper subgroup of S_n. Such cases arise in practical applications, including circle packing [2]. For instance, the problem of optimally

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packing 15 unit circles within a circular container is not defined by fully symmetric functions, but rather by functions exhibiting subgroup symmetry. The conjectured optimal solution also possesses symmetry over a subgroup [11]. Although these problems have remained open for many years, as illustrated in Section 5, the results presented in this paper offer a foundation that may prove valuable in addressing such challenges.

The outline of the paper is as follows. In Section 2 we provide a precise description of what it means for an equality-constrained optimization problem to be invariant under a subgroup of permutations; we also make precise our main theorem, namely that the local optimals, suitably characterized, form a fixed set under the group action. In Sections 3 and 4, we then prove this main theorem. Finally, in Section 5 we provide examples where the global optimum is a fixed point of the group.

2. Statement of the problem

First recall that for a permutation σ in the symmetric group S_n, the corresponding *permutation matrix* P is defined as $P = [\mathbf{e}_{\sigma(1)} \cdots \mathbf{e}_{\sigma(n)}]^\top$, where $\mathbf{e}_i$ denotes the unit column vector with a 1 in the i-th row. Due to this correspondence, the permutation matrix associated with σ^{-1} is $P^{-1} = P^\top$.

We can now make precise the types of optimization problems we are interested in. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ $(m < n)$ be functions that are at least twice Fr 辿 chet differentiable, and let G be a subgroup of S_n. For $x \in \mathbb{R}^n$, we consider the equality-constrained Optimization Problem:

$$\text{minimize } f(x) \text{ subject to } h(x) = \mathbf{0}. \tag{2.1}$$

Definition 2.1. We say that Optimization Problem 2.1 is **invariant under the action of** G if for each $\sigma \in G$ and corresponding permutation matrix P, there exists $\tau \in S_m$ and corresponding permutation matrix Q, such that $f(Px) = f(x)$ and $h(Px) = Qh(x)$ for all $x \in \mathbb{R}^n$.

Before stating our main theorem, we recall some notation and background. We will denote by $S = \{x \in \mathbb{R}^n : h(x) = \mathbf{0}\}$ the hypersurface along which we are seeking to minimize f. If $x^* \in S$, recall that the *tangent space* of x^* on S is defined as

$$T(x^*) = \{v \in \mathbb{R}^n \mid Dh(x^*)v = \mathbf{0}\},$$

where $Dh(x^*)$ has full rank. We will also denote the *Jacobian matrix* of h as $Dh = [\nabla h_1 \ \ldots \ \nabla h_m]^\top$, where h_i denotes the i-th component of h. We may alternatively write this as

$$Dh = \left[\frac{\partial h}{\partial x_1} \ \cdots \ \frac{\partial h}{\partial x_n}\right].$$

The *Hessian matrix* of f will be denoted as $\nabla^2 f$ where the entry in the i-th row and j-th column is $\frac{\partial^2 f}{\partial x_i \partial x_j}$. By direct computation, $\nabla^2 f = [D(\nabla f)]^\top$.

With this notation, we now recall the standard characterization for solutions of Optimization Problem 2.1; see for example Sections 11.2 and 11.4 in [6]. Specifically, we consider all x^* in S that satisfy the following conditions for being a local minimizer:

R. x^* is a *regular point*, meaning $\nabla h_1(x^*), \nabla h_2(x^*), \ldots, \nabla h_m(x^*)$ are linearly independent.

N1. x^* satisfies the *first-order necessary condition*, namely there exists $\lambda^* \in \mathbb{R}^m$ such that $\nabla f(x^*) + Dh(x^*)^\top \lambda^* = \mathbf{0}$.

N2. x^* satisfies the *second-order necessary condition*, namely for all $y \in T(x^*)$, we have $y^\top \nabla^2 \left[f(x^*) + (\lambda^*)^\top h(x^*)\right] y \geq 0$.

S. x^* satisfies the *sufficient condition*, namely that the inequality in N2 is strict.

As suggested in the statements of these conditions, it is indeed the case that if x^* satisfies Condition R, then Conditions N1 and N2 are necessary conditions for x^* to be a local minimizer, and Condition S is a sufficient condition to ensure local minimality provided the previous conditions are satisfied.

We can now state our main result.

Theorem 2.2. *Let G be a subgroup of S_n such that Optimization Problem 2.1 is invariant under the action of G. Let P be the permutation matrix for $\sigma \in G$, with matrix Q for $\tau \in S_m$ as in Definition 2.1. Then x^* with λ^* satisfies each of the Conditions R, N1, N2, and S, respectively, if and only if Px^* with $Q\lambda^*$ satisfies each of the Conditions R, N1, N2, and S, respectively.*

In the following sections we prove this theorem through a sequence of lemmas and propositions.

3. Regularity and first-order condition

We first recall the chain rule. Let $\alpha : \mathbb{R}^m \to \mathbb{R}$ and $\beta : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable functions. The chain rule states that for $\gamma(x) = \alpha(\beta(x))$,

$$\frac{\partial \gamma}{\partial x_i} = \nabla \alpha(\beta(x))^\top \frac{\partial \beta}{\partial x_i} = \left(\frac{\partial \beta}{\partial x_i}\right)^\top \nabla \alpha(\beta(x)).$$

We now begin by examining ∇f needed for Condition N1.

Lemma 3.1. *For all $x \in \mathbb{R}^n$, $\nabla f(Px) = P\nabla f(x)$.*

Proof. We first observe that

$$\frac{\partial (Px)_i}{\partial x_j} = P_{ij}. \tag{3.1}$$

The reason for this is that since P is the permutation matrix for σ, we know that $(Px)_i = x_{\sigma(i)}$. Differentiating, $\frac{\partial (Px)_i}{\partial x_j} = 1$ if $j = \sigma(i)$ and 0 otherwise.

As a result of Equation 3.1, the j-th column of $D(Px)$ is the j-th column of P, which is precisely the j-th row of $P^\top$. This yields

$$\frac{\partial (Px)}{\partial x_j} = \mathbf{e}_{\sigma^{-1}(j)}. \tag{3.2}$$

We can now prove our lemma. Differentiating $f(Px)$ with respect to x_i, and applying the chain rule along with Equation 3.2, we obtain

$$\frac{\partial f(Px)}{\partial x_i} = \left(\frac{\partial (Px)}{\partial x_i}\right)^\top \nabla f(Px) = \mathbf{e}_{\sigma^{-1}(i)}^\top \nabla f(Px).$$

Since $f(Px) = f(x)$ from Definition 2.1, we also have that

$$\frac{\partial f(Px)}{\partial x_i} = \frac{\partial f(x)}{\partial x_i} = \mathbf{e}_i^\top \nabla f(x).$$

Equating these expressions gives $\mathbf{e}_{\sigma^{-1}(i)}^\top \nabla f(Px) = \mathbf{e}_i^\top \nabla f(x)$, and since this is true for all i, we know that $P^\top \nabla f(Px) = \nabla f(x)$. Left-multiplying by P yields $\nabla f(Px) = P\nabla f(x)$. □

We now turn to considering how derivatives of h interact with P and Q, and begin with an initial lemma.

Lemma 3.2. *For all $x \in \mathbb{R}^n$, $Dh(Px) = QDh(x)P^\top$.*

Proof. We first consider how Q interacts with h. Direct computation shows $\frac{\partial(Q\circ h)}{\partial x_i} = Q\frac{\partial h}{\partial x_i}$ for all i. Collecting columns gives

$$D(Q \circ h) = QDh. \tag{3.3}$$

Now turning to how h interacts with P, by the chain rule,

$$\frac{\partial(h_j \circ P)}{\partial x_i} = \nabla h_j(Px)^\top \frac{\partial(Px)}{\partial x_i}$$

for all j. Thus,

$$\frac{\partial(h \circ P)}{\partial x_i} = Dh(Px)\frac{\partial(Px)}{\partial x_i} = Dh(Px)\mathbf{e}_{\sigma^{-1}(i)}$$

with the last equality by Equation 3.2. But in this last equation, $\frac{\partial(h\circ P)}{\partial x_i}$ is just the i-th column of $D(h \circ P)$, and $\mathbf{e}_{\sigma^{-1}(i)}$ is the i-th column of P, so that collecting columns we obtain

$$D(h \circ P)(x) = Dh(Px)P. \tag{3.4}$$

We can prove the lemma. Since $h \circ P = Q \circ h$ from Definition 2.1, we have $D(h \circ P) = D(Q \circ h)$. By Equations 3.3 and 3.4 we have $Dh(Px)P = QDh(x)$. Right-multiplying by $P^\top$ gives $Dh(Px) = QDh(x)P^\top$. □

We can now establish the first two parts of Theorem 2.2.

Proposition 3.3. *Conditions R and N1 respectively each hold for $x^* \in S$ and $\lambda^* \in \mathbb{R}^m$ if and only if they each hold for $Px^* \in S$ and $Q\lambda^* \in \mathbb{R}^m$.*

Proof. For Condition R, if x^* is a regular point this means that the gradient vectors $\nabla h_1(x^*), \nabla h_2(x^*), \ldots, \nabla h_m(x^*)$ are linearly independent, or equivalently that $Dh(x^*)$ has rank m. By Lemma 3.2 we know that $Dh(Px^*) = QDh(x^*)P^\top$. Since right-multiplication by $P^\top$ just permutes columns, and left-multiplication by Q just permutes rows, it must be that the rank of $Dh(Px^*) = QDh(x^*)P^\top$ equals the rank of $Dh(x^*)$, namely m. Thus Px^* is a regular point as well.

For Condition N1, we assume that $\nabla f(x^*) + Dh(x^*)^\top \lambda^* = \mathbf{0}$ for $\lambda^* \in \mathbb{R}^m$. Left-multiplying by P, we obtain

$$P\nabla f(x^*) + PDh(x^*)^\top \lambda^* = \mathbf{0}. \tag{3.5}$$

Now

$$PDh(x^*)^\top = (P^\top)^\top Dh(x^*)^\top Q^\top Q = (QDh(x^*)P^\top)^\top Q = Dh(Px^*)^T Q$$

where this last equality comes from Lemma 3.2. This, along with the fact that $P\nabla f(x^*) = \nabla f(Px^*)$ from Lemma 3.1, allows us to write Equation 3.5 as $\nabla f(Px^*) + Dh(Px^*)^\top Q\lambda^* = \mathbf{0}$, establishing Condition N1 for Px^* and $Q\lambda^*$. The converses for both Condition R and N1 are then satisfied by allowing $P^\top$ and $Q^\top$ to play the roles of P and Q in the preceding statements and proofs. □

Remark 3.4. Let f be subdifferentiable. Then, Condition N1 can be replaced by the inclusion $0 \in \partial f(x^*) + Dh(x^*)^\top \lambda^*$, where $\partial f(x)$ denotes the subdifferential of f. For an affine transformation, it holds that $\partial f(x) = P^\top \partial f(Px)$ whenever $f(Px) = f(x)$. Accordingly, Proposition 3.3 follows, with the original equality replaced by this inclusion. Since every convex function admits a subgradient, the ability to compute subgradients enables us to (almost) always solve Optimization Problem 2.1 when f is convex.

We now turn to Conditions N2 and S.

4. Second-order conditions

First recall that for any twice differentiable functions $\alpha : \mathbb{R}^m \to \mathbb{R}$ and $\theta : \mathbb{R}^m \to \mathbb{R}$, we have $\nabla^2(\alpha + \theta) = \nabla^2\alpha + \nabla^2\theta$. We therefore approach the inequality in Conditions N2 and S one piece at a time, beginning with how the Hessian interacts with the function f and the permutation matrix P.

Lemma 4.1. *For all $x \in \mathbb{R}^n$, $\nabla^2 f(Px) = P\nabla^2 f(x)P^\top$.*

Proof. By Lemma 3.1, $(\nabla f \circ P)(x) = (P \circ \nabla f)(x)$ for all x. Taking the Jacobian of both sides yields $D(\nabla f \circ P)(x) = D(P \circ \nabla f)(x)$. To the right-side of this equation we can apply Equation 3.3, which holds for P and ∇f as well as for Q and h; and to the left side of the equation we can apply Equation 3.4, which holds for ∇f as well as h, to obtain

$$D(\nabla f(Px))P = PD(\nabla f(x)).$$

Taking transposes of both sides yields

$$P^\top \nabla^2 f(Px) = \nabla^2 f(x)P^\top,$$

so that left-multiplying by P we obtain our desired result of $\nabla^2 f(Px) = P\nabla^2 f(x)P^\top$. □

We now consider how the Hessian interacts with h and P in the inequalities for Conditions N2 and S. Recall that $\tau \in S_m$ is the permutation corresponding to the matrix Q.

Lemma 4.2. *For all $x \in \mathbb{R}^n$ and $k \in \{1, \dots, m\}$, $\nabla^2 h_k(Px) = P\nabla^2 h_{\tau(k)}(x)P^\top$.*

Proof. Since $h(Px) = Qh(x)$ from Definition 2.1, we have that $h_k(Px) = h_{\tau(k)}(x)$ for all k. By differentiating both sides with respect to x_j and applying the chain rule to the left side, we obtain

$$\left(\frac{\partial(Px)}{\partial x_j}\right)^\top \nabla h_k(Px) = \frac{\partial h_{\tau(k)}(x)}{\partial x_j}.$$

By Equation 3.2 this becomes

$$\mathbf{e}_{\sigma^{-1}(j)}^{\top}\nabla h_k(Px) = \frac{\partial h_{\tau(k)}(x)}{\partial x_j}$$

or

$$\frac{\partial h_k(Px)}{\partial x_{\sigma^{-1}(j)}} = \frac{\partial h_{\tau(k)}(x)}{\partial x_j}. \tag{4.1}$$

But now differentiating Equation 4.1 by x_i and applying the chain rule and Equation 3.2 to the left side just as above yields

$$\frac{\partial^2 h_k(Px)}{\partial x_{\sigma^{-1}(i)}\partial x_{\sigma^{-1}(j)}} = \frac{\partial^2 h_{\tau(k)}(x)}{\partial x_i \partial x_j}. \tag{4.2}$$

The right side of Equation 4.2 is the entry in the i-th row and j-th column of $\nabla^2 h_{\tau(k)}(x)$. The left side of Equation 4.2 is in fact the entry in the i-th row and j-th column of $P^{\top}\nabla^2 h_k(Px)P$, since right-multiplying by P permutes column indices by σ^{-1}, and left-multiplying by $P^{\top}$ permutes row indices by σ^{-1}. We thus obtain

$$P^{\top}\nabla^2 h_k(Px)P = \nabla^2 h_{\tau(k)}(x),$$

so that multiplying on the left and right by P and $P^{\top}$, respectively, yields the desired equality $\nabla^2 h_k(Px) = P\nabla^2 h_{\tau(k)}(x)P^{\top}$. □

We can now establish the second two parts of Theorem 2.2; along with Proposition 3.3 this then concludes the proof of Theorem 2.2.

Proposition 4.3. *Conditions N2 and S respectively each hold for $x^* \in S$ and $\lambda^* \in \mathbb{R}^m$ if and only if they each hold for $Px^* \in S$ and $Q\lambda^* \in \mathbb{R}^m$.*

Proof. We first consider Condition N2, and assume that for all $y \in T(x^*)$ we have

$$y^{\top}\nabla^2\left[f(x^*) + (\lambda^*)^{\top}h(x^*)\right]y \geq 0. \tag{4.3}$$

We need to show that for all $y \in T(Px^*)$, we also have

$$y^{\top}\nabla^2\left[f(Px^*) + (Q\lambda^*)^{\top}h(Px^*)\right]y \geq 0. \tag{4.4}$$

Throughout the proof x^* and λ^* are constant.

To this end let $y \in T(Px^*)$, which means that $Dh(Px^*)y = \mathbf{0}$, which by Lemma 3.2 is equivalent to $QDh(x^*)P^{\top}y = \mathbf{0}$. Thus $Dh(x^*)P^{\top}y = \mathbf{0}$ and $P^{\top}y \in T(x^*)$, so that Inequality 4.3 holds for $P^{\top}y$.

We now pay attention to the individual terms in Inequality 4.3. Observe that $(\lambda^*)^{\top}h(x^*) = \sum_{i=1}^{m}\lambda_i^* h_i(x^*) = \sum_{i=1}^{m}\lambda_{\tau(i)}^* h_{\tau(i)}(x^*)$ by rearranging terms. Thus for $P^{\top}y \in T(x^*)$, Inequality 4.3 can be written as

$$y^{\top}P\nabla^2\left[f(x^*) + \sum_{i=1}^{m}\lambda_{\tau(i)}^* h_{\tau(i)}(x^*)\right]P^{\top}y \geq 0$$

or

$$y^{\top}P\nabla^2 f(x^*)P^{\top}y + \sum_{i=1}^{m}\lambda_{\tau(i)}^* y^{\top}P\nabla^2 h_{\tau(i)}(x^*)P^{\top}y \geq 0.$$

Applying Lemmas 4.1 and 4.2 to the terms of the expression then yields

$$y^\top \nabla^2 f(Px^*)y + \sum_{i=1}^{m} \lambda^*_{\tau(i)} y^\top \nabla^2 h_i(Px^*)y \geq 0,$$

or

$$y^\top \nabla^2 \left[f(Px^*) + \sum_{i=1}^{m} \lambda^*_{\tau(i)} h_i(Px^*) \right] y \geq 0,$$

Finally, observing that $(Q\lambda^*)^\top h(Px^*) = \sum_{i=1}^{m} \lambda^*_{\tau(i)} h_i(Px^*)$, we obtain the desired inequality $y^\top \nabla^2 \left[f(Px^*) + (Q\lambda^*)^\top h(Px^*) \right] y \geq 0$.

The converse statement for Condition N1 is then satisfied by allowing $P^\top$ and $Q^\top$ to play the roles of P and Q in the preceding statements and proofs. Finally, Condition S follows from the fact that Inequality 4.3 is strict if and only if Inequality 4.4 is strict. □

5. Examples

We provide some examples demonstrating the utility of studying optimization problems via their invariance under the action of permutation groups. The key observation in these examples is to show that if Optimization Problem 2.1 is invariant under the action of a group G of permutations, then in certain circumstances we can guarantee that a global optimum exists and occurs at a fixed point for the action of G. Then, if we can identify all of these fixed points, and this list is finite, the Optimization Problem 2.1 reduces to a finite search.

Example 5.1 (Convex objective functions and global optima)**.** Suppose in Optimization Problem 2.1 our objective function f is convex, meaning

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) \qquad \text{for all } 0 \leq t \leq 1. \tag{5.1}$$

Suppose x^* is a global minimizer, and consider the orbit of x^* under the action of a permutation group G for which Optimization Problem 2.1 is invariant. This orbit will be a set of points $\{P_\sigma x^*\}_{\sigma \in G}$, all of which will be global minimizers. Now consider the point $y^* = \frac{1}{|G|} \sum_{\sigma \in G} P_\sigma x^*$. We know that y^* will be a fixed point of the action, as G will simply permute its summands. But due to convexity of f, all of the convex combinations of the global optimal points, including y^*, must themselves be optimal. Hence y^* is a fixed point of the action of G which is also a global optimal for f.

Example 5.2 (Packing circles in a circle)**.** We present an alternative proof for the optimal packing of five unit circles into the smallest possible circular container; see [3] for the original proof. As shown in Figure 1, such a packing can be represented by an equilateral pentagon whose sides all have length 2, where the sides join centers of adjacent tangential circles.

Referring to Figure 1, let O be an interior point in this pentagon, and we denote by $r_1, \ldots, r_5$ the lengths of each of the circles' centers from O. The angles $\theta_1, \ldots, \theta_5$ measure the corresponding central angles. This setup is in line with Optimization Problem 2.1, where $x = [r_1, \ldots, r_5, \theta_1, \ldots, \theta_5]^\top$ and the objective function is $f(x) = \max_i r_i(x)$, which is convex and continuous. The domain Ω of the objective function

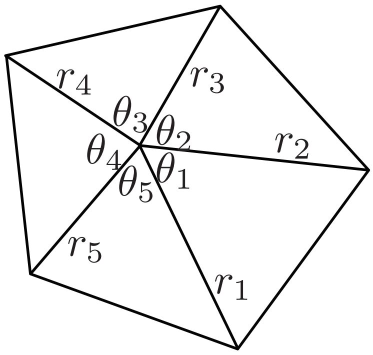

FIGURE 1. Packing five circles in a circular container.

is the hyperrectangle defined by $0 \le r_i \le 4$ and $0 \le \theta_i \le 2\pi$ for $i \in \{1, \ldots, 5\}$, where the upper bound for r_i is achieved if the pentagon degenerates to a quadrilateral. The constraint equations come from the Law of Cosines applied to each triangle, and are

$$\begin{aligned}
r_1^2 + r_2^2 - 2r_1r_2\cos\theta_1 &= 4\\
r_2^2 + r_3^2 - 2r_2r_3\cos\theta_2 &= 4\\
r_3^2 + r_4^2 - 2r_3r_4\cos\theta_3 &= 4\\
r_4^2 + r_5^2 - 2r_4r_5\cos\theta_4 &= 4\\
r_5^2 + r_1^2 - 2r_5r_1\cos\theta_5 &= 4.
\end{aligned}$$

Because the feasible set is non-empty, closed and bounded, we are assured that a global optimum exists. Now we let

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so that the optimization problem is invariant under the action of the group G generated by the permutation matrix

$$P = \left[\begin{array}{c|c} P_1 & \mathbf{0}\\ \hline \mathbf{0} & P_1 \end{array}\right]$$

where $P^5 = I$. Thus the elements of G just cyclically permute indices of the r_i and θ_i synchronously.

Now suppose x^* is a global minimizer. Then $P^k x^*$ is also a global minimizer for $k \in \{1, \ldots, 4\}$ with the result that $y^* = \frac{1}{5}\sum_{k=0}^{4} P^k x^*$ is the unique fixed point of G that is a global minimizer, since we must have $r_1 = \cdots = r_5$ and $\theta_1 = \cdots = \theta_5$ at y^* as G is cyclic of order 5. It follows that $r_i = \csc(\pi/10)$ and the container radius is $1 + \sqrt{2 + \frac{2}{\sqrt{5}}}$.

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