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A FORWARD-BACKWARD-FORWARD METHOD FOR VARIATIONAL INCLUSION PROBLEM AND APPLICATIONS

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Abstract. In this work, we propose a forward-backward-forward (FBF) algorithm enhanced with multi-inertial extrapolation and a self-adaptive strategy for solving variational inclusion problems. Under appropriate assumptions, we establish the weak convergence of the generated sequence to a solution of the variational inclusion problem. To demonstrate the effectiveness of the proposed method, we conduct numerical experiments and compare its performance with several existing algorithms reported in the previous works.

1. Introduction

Throughout this paper, let $\mathcal{H}$ be a real Hilbert space, $\langle \cdot, \cdot \rangle$ denotes the inner product and $\|\cdot\|$ denotes the norm that are defined on $\mathcal{H}$. The operator $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ is *$\mathcal{L}$-Lipschitz continuous* if there exists $\mathcal{L} > 0$ such that $\|\mathcal{S}u - \mathcal{S}v\| \leq \mathcal{L}\|u - v\|$ for all $u, v \in \mathcal{H}$ and it is called *monotone* if $\langle \mathcal{S}u - \mathcal{S}v, u - v\rangle \geq 0,\ \forall u, v \in \mathcal{H}$. Moreover, $\mathcal{S}$ is said to be *inverse strongly monotone* if there exists $\alpha > 0$ such that $\langle \mathcal{S}u - \mathcal{S}v, u - v\rangle \geq \alpha\|\mathcal{S}u - \mathcal{S}v\|^2,\ \forall u, v \in \mathcal{H}$.

For a set-valued operator $\mathcal{G} : \mathcal{H} \to 2^{\mathcal{H}}$, the graph of $\mathcal{G}$ defined by [2] $\text{Graph}(\mathcal{G}) = \{(u, v) \in \mathcal{H} \times \mathcal{H} : u \in \text{dom}(\mathcal{G}),\ v \in \mathcal{G}(u)\}$. The operator $\mathcal{G}$ is said to be *monotone* if $\langle u - x, v - y\rangle \geq 0,$ for all $(u, v), (x, y) \in \text{Graph}(\mathcal{G})$. Furthermore, $\mathcal{G}$ is called *maximal monotone* if it is monotone, and its graph is not strictly contained within the graph of any other monotone operator.

For $\lambda > 0$, the resolvent operator associated with a set-valued operator $\mathcal{G} : \mathcal{H} \to 2^{\mathcal{H}}$ is defined as follows (see [16]):

$$\mathcal{J}_\lambda^{\mathcal{G}}(u) = (I + \lambda\mathcal{G})^{-1}(u), \text{ for all } u \in \mathcal{H}.$$

It is well known that $\mathcal{J}_\lambda^{\mathcal{G}}$ is single-valued, firmly nonexpansive and defined everywhere, i.e., $\text{dom}(\mathcal{J}_\lambda^{\mathcal{G}}) = \mathcal{H}$. Given $\lambda > 0$, define the operator $\mathcal{S}_\lambda$ by $\mathcal{S}_\lambda := (I + \lambda\mathcal{G})^{-1}(I - \lambda\mathcal{F})$. According to [2], the set of fixed points of $\mathcal{S}_\lambda$ coincides with the set of zeros of $\mathcal{F} + \mathcal{G}$, that is, $\text{Fix}(\mathcal{S}_\lambda) = (\mathcal{F} + \mathcal{G})^{-1}(0_{\mathcal{H}})$, where $\text{Fix}(\mathcal{S}) = \{u \in \mathcal{H} : u = \mathcal{S}u\}$.

We consider the variational inclusion problem (VIP) of finding a zero of the sum of two operators. The VIP is to find $u \in \mathcal{H}$ such that

$$0_{\mathcal{H}} \in (\mathcal{F} + \mathcal{G})u, \tag{1.1}$$

where $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is *$\mathcal{L}$-Lipschitz continuous monotone* and $\mathcal{G} : \mathcal{H} \to 2^{\mathcal{H}}$ is *maximal monotone*. We denote the solution set of (1.1) by $\Omega := (\mathcal{F} + \mathcal{G})^{-1}(0_{\mathcal{H}})$. One of the most famous algorithms used to solve problem (1.1) is the forward-backward

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algorithm, which was studied by the authors in [13,20]. This algorithm generates a sequence $\{u_k\}$ such that $u_1 \in \mathcal{H}$ and

$$u_{k+1} = \mathcal{J}^{\mathcal{G}}_{\lambda_k}(u_k - \lambda_k \mathcal{F} u_k), \quad k \geq 1, \tag{1.2}$$

where $\lambda_k > 0$, $\mathcal{F}$ is $\frac{1}{\mathcal{L}}$-inverse strongly monotone and $\mathcal{G}$ is maximal monotone. If $\lambda_k \in \left(0, \frac{2}{\mathcal{L}}\right)$, then $\{u_k\}$ in (1.2) converges weakly to a solution of (1.1). Apart from the inverse strongly monotone assumption on $\mathcal{F}$ (which is strict), other assumptions which confirm the convergence of (1.2) are the strong monotonicity of $\mathcal{F} + \mathcal{G}$ [6] or the use of a backtracking technique [3] (which are also strict).

In order to weaken the inverse strongly monotone assumption on $\mathcal{F}$, the following forward-backward-forward algorithm (FBFa) was introduced in [26]. This algorithm generates a sequence $\{u_k\}$ such that $u_1 \in \mathcal{H}$ and

$$\begin{cases} y_k = \mathcal{J}^{\mathcal{G}}_{\lambda_k}(u_k - \lambda_k \mathcal{F} u_k), \\ u_{k+1} = y_k - \lambda_k(\mathcal{F} y_k - \mathcal{F} u_k), \end{cases} \tag{1.3}$$

where $\lambda_k > 0$, $\mathcal{F}$ is $\mathcal{L}$-Lipschitz continuous monotone and $\mathcal{G}$ is maximal monotone. If $\lambda_k \in \left(0, \frac{1}{\mathcal{L}}\right)$, then sequence $\{u_k\}$ in (1.3) converges weakly to a solution of (1.1).

In the following, we discuss a technique for accelerating convergence, known as inertial extrapolation. This concept was first introduced and studied by Polyak in [22], where it is referred to as the heavy ball method, as a means to accelerate the solution process for smooth convex minimization problems. Let $\psi : \mathcal{H} \to \mathbb{R}$ be differentiable, this algorithm generates a sequence $\{u_k\}$ such that $u_0, u_1 \in \mathcal{H}$ and

$$u_{k+1} = u_k + \theta_k(u_k - u_{k-1}) - \alpha_k \nabla \psi(u_k), \quad k \geq 1,$$

where θ_k is the extrapolation coefficient and the term $\theta(u_k - u_{k-1})$ constitutes the inertial step. The inertial technique, which is based on a discrete analogue of a second-order dissipative dynamical system [1], is also well known for its effectiveness in enhancing the convergence speed of iterative algorithms. Later, numerous authors have improved, expanded and generalized the inertial extrapolation method since its inception, see [4,11,21,24]. In 2019, the research in [23] demonstrated that the use of a one-step inertial extrapolation may not always yield acceleration benefits when applied within the ADMM framework [5]. Recently, Liang [12] further highlighted that employing multi-step inertial extrapolation, can effectively overcome the limitations of the one-step extrapolation. This has motivated a line of many research (see in [7,8,10,28,29]).

Inspired by the above studies, this paper aims to propose a forward-backward-forward (FBF) algorithm enhanced with multi-inertial extrapolation. By incorporating a self-adaptive strategy with multi-inertial techniques, we introduce a recent method for solving the VIP (1.1) under some suitable conditions. We establish the weak convergence of the proposed method to a solution of the VIP (1.1). Furthermore, we present numerical experiments to demonstrate the efficiency of the method and provide a comparative analysis with existing algorithms in the literature.

Next, we recall the following important lemma that are useful in our convergence analysis.

Lemma 1.1 ([9]). *Let $b \in \mathbb{N}$ and $k \in \{1, 2, \ldots, b\}$. Suppose that the initial values $\Theta_{1-b}, \Theta_{2-b}, \ldots, \Theta_0$ are non-negative real numbers, and let $\{\Theta_k\}_{k=1}^{\infty}$ and $\{\theta_{s,k}\}_{k=1}^{\infty}$ be* *sequences of non-negative real numbers. If*

$$\Theta_{k+1} \leq \Theta_k + \sum_{s=1}^{b} (\Theta_{k-s+1} + \Theta_{k-s})\theta_{s,k}, \quad k \in \mathbb{N},$$

then

$$\Theta_{k+1} \leq \mathcal{M} \cdot \prod_{j=1}^{k} (1 + 2\theta_{1,j} + 2\theta_{2,j} + ... + 2\theta_{b-1,j} + 2\theta_{b,j}), \quad k \in \mathbb{N},$$

where $\mathcal{M} = \max\{\Theta_{1-b}, \Theta_{2-b}, ..., \Theta_0, \Theta_1\}$. *Moreover, if for each* $s \in \{1, \ldots, b\}$ *it holds that* $\sum_{k=1}^{\infty} \theta_{s,k} < +\infty$*, then the sequence* $\{\Theta_k\}$ *is bounded.*

Lemma 1.2 ([19])**.** *Let* $\{b_k\}$*,* $\{\tau_k\}$ *and* $\{a_k\}$ *be sequences of nonnegative real numbers satisfying*

$$b_{k+1} \leq (1 + a_k)b_k + \tau_k, \quad k \geq 1.$$

If $\sum_{k=1}^{\infty} a_k < +\infty$ *and* $\sum_{k=1}^{\infty} \tau_k < +\infty$*, then* $\lim_{k\to\infty} b_k$ *exists.*

Lemma 1.3 ([18])**.** *Let* Ω *be a subset of* $\mathcal{H}$ *and* $\{u_k\} \subseteq \mathcal{H}$ *such that:*

(i) *for every* $z^* \in \Omega$, $\lim_{k\to\infty} \|u_k - z^*\|$ *exists;*
(ii) *each weak cluster point of* $\{u_k\}$ *is in* Ω*. Then* $\{u_k\}$ *converges weakly to a point in* Ω*.*

2. Algorithms and convergence analysis

This section is devoted to the development of an iterative scheme for solving VIP.

Algorithm 2.1. *Let* $\mu \in (0, 1)$*,* $\lambda_1 > 0$ *and* $y_{1-b}, y_{2-b}, ..., y_0, u_1 \in \mathcal{H}$*. Fix* $b \in \mathbb{N}$*, for each* $s \in \{1, 2, ..., b\}$*, let* $\{\theta_{s,k}\}_{k=1}^{\infty}$ *be a sequence of non-negative real numbers. Choose a nonnegative real sequence* $\{d_k\}$ *and* $\{\beta_k\} \subset (0, 1)$*. Let* $\{u_k\}$ *be defined by*

Step 1*: Calculate*

$$w_k = \mathcal{J}_{\lambda_k}^{\mathcal{G}}(u_k - \lambda_k \mathcal{F} u_k).$$

If $w_k = u_k$ *stop. Else go to* ***Step 2****.*

Step 2*: Calculate*

$$y_k = (1 - \beta_k)u_k + \beta_k(w_k + \lambda_k(\mathcal{F}u_k - \mathcal{F}w_k)),$$

and the self-adaptive stepsizes λ_k *is given by:*

$$\lambda_{k+1} = \begin{cases} \min\left\{\frac{\mu\|u_k - w_k\|}{\|\mathcal{F}u_k - \mathcal{F}w_k\|}, \lambda_k + d_k\right\}, & \text{if } \|\mathcal{F}u_k - \mathcal{F}w_k\| \neq 0; \\ \lambda_k + d_k, & \text{otherwise.} \end{cases}$$

Step 3*: Compute*

$$u_{k+1} = y_k + \sum_{s=1}^{b} \theta_{s,k}(y_{k-s+1} - y_{k-s}).$$

Set k := k + 1 and go back to ***Step 1****.*

Lemma 2.1 ([14]). *Let* $\{\lambda_k\}$ *be generated by Algorithm 2.1. If* $\sum_{k=1}^{\infty} d_k < +\infty$, *then*

$$\lim_{k\to\infty} \lambda_k = \lambda \in \left[\min\left\{\lambda_1, \frac{\mu}{L}\right\}, \lambda_1 + d\right]$$

where $d = \sum_{k=1}^{\infty} d_k$ *and*

$$\|\mathcal{F}u_k - \mathcal{F}w_k\| \leq \frac{\mu}{\lambda_{k+1}}\|u_k - w_k\|. \tag{2.1}$$

Theorem 2.2. *Let* $\{u_k\}$ *be the sequence generated by Algorithm 2.1. Suppose that for each* $s \in \{1, 2, ..., b\}$, *the following conditions hold:* $\sum_{k=1}^{\infty} \theta_{s,k} < +\infty$, $\sum_{k=1}^{\infty} d_k < +\infty$ *and* $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$. *Then the following statements hold:*

(i) *for each* $z^* \in \Omega$, *there is* $k_0 \in \mathbb{N}$ *such that* $\|u_{k+1} - z^*\| \leq \mathcal{M} \cdot \prod_{j=1}^{k}(1 + 2\theta_{1,j} + 2\theta_{2,j} + \ldots + 2\theta_{b-1,j} + 2\theta_{b,j})$, *where* $\mathcal{M} = \max\{\|u_{k_0-b} - z^*\|, \|u_{k_0-b+1} - z^*\|, \ldots, \|u_{k_0-1} - z^*\|, \|u_{k_0} - z^*\|\}$, $\forall k \geq k_0$.

(ii) *The sequence* $\{u_k\}$ *converges weakly to a point in* Ω.

Proof. (i) Let $z^* \in \Omega$. Since $w_k = \mathcal{J}_{\lambda_k}^{\mathcal{G}}(u_k - \lambda_k \mathcal{F}u_k)$, we obtain $(u_k - \lambda_k \mathcal{F}u_k) \in (w_k + \lambda_k \mathcal{G}w_k)$. Thus there exists $v_k \in \mathcal{G}w_k$ such that

$$(I - \lambda_k \mathcal{F})u_k = w_k + \lambda_k v_k.$$

This implies that

$$v_k = \frac{1}{\lambda_k}(u_k - w_k - \lambda_k \mathcal{F}u_k). \tag{2.2}$$

On the other hand, we have $0 \in (\mathcal{F} + \mathcal{G})z^*$ and $\mathcal{F}w_k + v_k \in (\mathcal{F} + \mathcal{G})w_k$. Since $\mathcal{F} + \mathcal{G}$ is a maximal monotone operator, we obtain

$$\langle \mathcal{F}w_k + v_k, w_k - z^* \rangle \geq 0. \tag{2.3}$$

Replacing (2.2) into (2.3), we get

$$\frac{1}{\lambda_k}\langle u_k - w_k - \lambda_k \mathcal{F}u_k + \lambda_k \mathcal{F}w_k, w_k - z^* \rangle \geq 0.$$

This implies that

$$\langle u_k - w_k - \lambda_k(\mathcal{F}u_k - \mathcal{F}w_k), w_k - z^* \rangle \geq 0. \tag{2.4}$$

Consider

$$\begin{aligned}
\|y_k - z^*\|^2 &= \|(1-\beta_k)u_k + \beta_k(w_k + \lambda_k(\mathcal{F}w_k - \mathcal{F}u_k)) - z^*\|^2 \\
&= (1-\beta_k)\|u_k - z^*\|^2 + \beta_k\|w_k + \lambda_k(\mathcal{F}w_k - \mathcal{F}u_k) - z^*\|^2 \\
&\quad -\beta_k(1-\beta_k)\lambda_k^2\|u_k - w_k - \lambda_k(\mathcal{F}w_k - \mathcal{F}u_k)\|^2 \\
&= (1-\beta_k)\|u_k - z^*\|^2 - \beta_k(1-\beta_k)\lambda_k^2\|u_k - w_k - \lambda_k(\mathcal{F}w_k - \mathcal{F}u_k)\|^2 \\
&\quad +\beta_k\left[\|w_k - z^*\|^2 - 2\lambda_k\langle \mathcal{F}w_k - \mathcal{F}u_k, w_k - z^*\rangle + \lambda_k^2\|\mathcal{F}w_k - \mathcal{F}u_k\|^2\right] \\
&= (1-\beta_k)\|u_k - z^*\|^2 - \beta_k(1-\beta_k)\lambda_k^2\|u_k - w_k - \lambda_k(\mathcal{F}w_k - \mathcal{F}u_k)\|^2 \\
&\quad +\beta_k\left[\|u_k - z^*\|^2 + \|u_k - w_k\|^2 + 2\langle w_k - u_k, u_k - z^*\rangle\right. \\
&\quad \left.+\lambda_k^2\|\mathcal{F}w_k - \mathcal{F}u_k\|^2 - 2\lambda_k\langle \mathcal{F}w_k - \mathcal{F}u_k, w_k - z^*\rangle\right] \\
&= (1-\beta_k)\|u_k - z^*\|^2 - \beta_k(1-\beta_k)\lambda_k^2\|u_k - w_k - \lambda_k(\mathcal{F}w_k - \mathcal{F}u_k)\|^2
\end{aligned}$$

$$
\begin{aligned}
& +\beta_k \left[\|u_k - z^*\|^2 + \|u_k - w_k\|^2 - 2\langle w_k - u_k, w_k - u_k\rangle \right. \\
& +2\langle w_k - u_k, w_k - z^*\rangle + \lambda_k^2\|\mathcal{F}w_k - \mathcal{F}u_k\|^2 \\
& \left. -2\lambda_k\langle \mathcal{F}w_k - \mathcal{F}u_k, w_k - z^*\rangle\right] \\
= \; & (1-\beta_k)\|u_k - z^*\|^2 - \beta_k(1-\beta_k)\lambda_k^2\|u_k - w_k - \lambda_k(\mathcal{F}w_k - \mathcal{F}u_k)\|^2 \\
& +\beta_k \left[\|u_k - z^*\|^2 - \|u_k - w_k\|^2 + 2\langle w_k - u_k, w_k - z^*\rangle \right. \\
& \left. +\lambda_k^2\|\mathcal{F}w_k - \mathcal{F}u_k\|^2 - 2\lambda_k\langle \mathcal{F}w_k - \mathcal{F}u_k, w_k - z^*\rangle\right] \\
= \; & (1-\beta_k)\|u_k - z^*\|^2 - \beta_k(1-\beta_k)\lambda_k^2\|u_k - w_k - \lambda_k(\mathcal{F}w_k - \mathcal{F}u_k)\|^2 \\
& +\beta_k \left[\|u_k - z^*\|^2 - \|u_k - w_k\|^2 + \lambda_k^2\|\mathcal{F}w_k - \mathcal{F}u_k\|^2 \right. \\
& \left. -2\lambda_k\langle w_k - u_k - \lambda_k(\mathcal{F}w_k - \mathcal{F}u_k), w_k - z^*\rangle\right]. \qquad (2.5)
\end{aligned}
$$

Since (2.1), (2.4) and (2.5), we get

$$
\begin{aligned}
\|y_k - z^*\|^2 \; = \; & (1-\beta_k)\|u_k - z^*\|^2 - \beta_k(1-\beta_k)\lambda_k^2\|u_k - w_k - \lambda_k(\mathcal{F}w_k - \mathcal{F}u_k)\|^2 \\
& +\beta_k \left[\|u_k - z^*\|^2 - \left(1 - \frac{\mu^2\lambda_k^2}{\lambda_{k+1}^2}\right)\|w_k - u_k\|^2\right] \\
\leq \; & \|u_k - z^*\|^2 - \beta_k(1-\beta_k)\lambda_k^2\|u_k - w_k - \lambda_k(\mathcal{F}w_k - \mathcal{F}u_k)\|^2 \\
& -\beta_k\left(1 - \frac{\mu^2\lambda_k^2}{\lambda_{k+1}^2}\right)\|w_k - u_k\|^2. \qquad (2.6)
\end{aligned}
$$

Since $\lim_{k\longrightarrow\infty}\left(1 - \frac{\mu^2\lambda_k^2}{\lambda_{k+1}^2}\right) = 1 - \mu^2 > 0$, we can choose $k_0 \in \mathbb{N}$ such that $1 - \frac{\mu^2\lambda_k^2}{\lambda_{k+1}^2} > 0$ for all $k \geq k_0$ and we get

$$
\|y_k - z^*\| \leq \|u_k - z^*\|. \qquad (2.7)
$$

From (2.7), we obtain

$$
\begin{aligned}
\|u_{k+1} - z^*\| \; = \; & \left\| y_k + \sum_{s=1}^{b}\theta_{s,k}(y_{k-s+1} - y_{k-s}) \right\| \\
\leq \; & \|y_k - z^*\| + \sum_{s=1}^{b}\theta_{s,k}\|y_{k-s+1} - y_{k-s}\| \\
\leq \; & \|u_k - z^*\| + \sum_{s=1}^{b}\theta_{s,k}\left[\|u_{k-s+1} - z^*\| + \|u_{k-s} - z^*\|\right]. \qquad (2.8)
\end{aligned}
$$

It follows from Lemma 1.1 and (2.8) that

$$
\|u_{k+1} - z^*\| \leq \mathcal{M} \cdot \prod_{j=1}^{k}(1 + 2\theta_{1,j} + 2\theta_{2,j} + \ldots + 2\theta_{b-1,j} + 2\theta_{b,j}), \quad \forall k \geq k_0
$$

where $\mathcal{M} = \max\{\|u_{k_0-b} - z^*\|, \|u_{k_0-b+1} - z^*\|, \ldots, \|u_{k_0-1} - z^*\|, \|u_{k_0} - z^*\|\}$. Hence $\{u_k\}$ is a bounded sequence. Moreover, $\{y_k\}$ and $\{w_k\}$ are bounded. So, we have $\sum_{s=1}^{b}\theta_{s,k}\|u_{k-s+1} - u_{k-s}\| < \infty$ since $\sum_{s=1}^{b}\theta_{s,k} < \infty$. Applying Lemma 1.2 into

(2.8), we get $\lim_{k\to\infty}\|u_k - z^*\|$ exists. Additionally, from (2.7) we have

$$\begin{aligned}
\|u_{k+1} - z^*\|^2 &= \left\|y_k + \sum_{s=1}^{b}\theta_{s,k}(y_{k-s+1} - y_{k-s}) - z^*\right\|^2 \\
&= \|y_k - z^*\|^2 + \left\|\sum_{s=1}^{b}\theta_{s,k}(y_{k-s+1} - y_{k-s})\right\|^2 \\
&\quad +2\left\langle y_k - z^*, \sum_{s=1}^{b}\theta_{s,k}(y_{k-s+1} - y_{k-s})\right\rangle \\
&\leq \|y_k - z^*\|^2 + \left(\sum_{s=1}^{b}\theta_{s,k}\|y_{k-s+1} - y_{k-s}\|\right)^2 \\
&\quad +2\sum_{s=1}^{b}\theta_{s,k}\|y_k - z^*\|\|y_{k-s+1} - y_{k-s}\|. \qquad (2.9)
\end{aligned}$$

From (2.6) and (2.9), we obtain

$$\begin{aligned}
\|u_{k+1} - z^*\|^2 &\leq \|u_k - z^*\|^2 - \beta_k(1-\beta_k)\lambda_k^2\|u_k - w_k - \lambda_k(\mathcal{F}w_k - \mathcal{F}u_k)\|^2 \\
&\quad - \left(1 - \frac{\mu^2\lambda_k^2}{\lambda_{k+1}^2}\right)\|w_k - u_k\|^2 + \left(\sum_{s=1}^{b}\theta_{s,k}\|y_{k-s+1} - y_{k-s}\|\right)^2 \\
&\quad +2\sum_{s=1}^{b}\theta_{s,k}\|y_k - z^*\|\|y_{k-s+1} - y_{k-s}\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\beta_k(1-\beta_k)\lambda_k^2\|u_k - w_k - \lambda_k(\mathcal{F}w_k - \mathcal{F}u_k)\|^2 + \left(1 - \frac{\mu^2\lambda_k^2}{\lambda_{k+1}^2}\right)\|w_k - u_k\|^2 \\
&\quad\leq [\|u_k - z^*\|^2 - \|u_{k+1} - z^*\|^2] + \left(\sum_{s=1}^{b}\theta_{s,k}\|y_{k-s+1} - y_{k-s}\|\right)^2 \\
&\quad\quad +2\sum_{s=1}^{b}\theta_{s,k}\|y_k - z^*\|\|y_{k-s+1} - y_{k-s}\|.
\end{aligned}$$

Since $\lim_{k\to\infty}\|u_k - z^*\|$ exists, $\sum_{k=1}^{\infty}\theta_{s,k} < +\infty$ and $0 < \liminf_{k\to\infty}\beta_k \leq \limsup_{k\to\infty}\beta_k < 1$, we obtain

$$\lim_{k\to\infty}\|u_k - w_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty}\|u_k - w_k - \lambda_k(\mathcal{F}w_k - \mathcal{F}u_k)\| = 0. \qquad (2.10)$$

So, we obtain

$$\lim_{k\to\infty}\lambda_k\|\mathcal{F}w_k - \mathcal{F}u_k\| = 0. \qquad (2.11)$$

From (2.10) and (2.11), we obtain

$$\begin{aligned}
\|y_k - u_k\| &= \|(1-\beta_k)u_k + \beta_k(w_k + \lambda_k(\mathcal{F}u_k - \mathcal{F}w_k)) - u_k\| \\
&\leq \beta_k\|w_k + \lambda_k(\mathcal{F}u_k - \mathcal{F}w_k) - u_k\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_k (\|w_k - u_k\| + \lambda_k \|\mathcal{F}w_k - \mathcal{F}u_k\|) \\
&\to 0. \qquad (2.12)
\end{aligned}
$$

From (2.10) and (2.12), we get

$$
\begin{aligned}
\|y_k - w_k\| &\leq \|y_k - u_k\| + \|u_k - w_k\| \\
&\to 0.
\end{aligned}
$$

Let $(m, n) \in \text{Graph}(\mathcal{F} + \mathcal{G})$, i.e. $n \in (\mathcal{F} + \mathcal{G})m$. For a sequence $\{k_i\}$ of $\{k\}$, we see that $(I - \lambda_{k_i}\mathcal{F})u_{k_i} \in (I + \lambda_{k_i}\mathcal{G})w_{k_i}$, which implies that

$$
\frac{1}{\lambda_{k_i}}(u_{k_i} - w_{k_i} - \lambda_{k_i}\mathcal{F}u_{k_i}) \in \mathcal{G}w_{k_i}.
$$

By the maximal monotone of $\mathcal{G}$, we have

$$
\left\langle m - w_{k_i}, n - \mathcal{F}m - \frac{1}{\lambda_{k_i}}(u_{k_i} - w_{k_i} - \lambda_{k_i}\mathcal{F}u_{k_i}) \right\rangle \geq 0.
$$

Thus

$$
\begin{aligned}
\langle m - w_{k_i}, n \rangle &\geq \left\langle m - w_{k_i}, \mathcal{F}m + \frac{1}{\lambda_{k_i}}(u_{k_i} - w_{k_i} - \lambda_{k_i}\mathcal{F}u_{k_i}) \right\rangle \\
&= \langle m - w_{k_i}, \mathcal{F}m - \mathcal{F}w_{k_i} \rangle + \langle m - w_{k_i}, \mathcal{F}w_{k_i} - \mathcal{F}u_{k_i} \rangle \\
&\quad + \left\langle m - w_{k_i}, \frac{1}{\lambda_{k_i}}(u_{k_i} - w_{k_i}) \right\rangle \\
&\geq \langle m - w_{k_i}, \mathcal{F}w_{k_i} - \mathcal{F}u_{k_i} \rangle + \left\langle m - w_{k_i}, \frac{1}{\lambda_{k_i}}(u_{k_i} - w_{k_i}) \right\rangle. \qquad (2.13)
\end{aligned}
$$

From (2.10) and $\mathcal{F}$ is L-Lipschitz continuous, we obtain $\lim_{i\to\infty} \|\mathcal{F}w_{k_i} - \mathcal{F}u_{k_i}\| = 0$. Since the sequence $\{u_k\}$ is bounded, there exists a subsequence $\{u_{k_j}\}$ and a point $z \in \mathcal{H}$ such that $u_{k_j} \rightharpoonup z$. Furthermore, from (2.10), we also have $w_{k_j} \rightharpoonup z$. Thus, from (2.13) with a sequence $\{k_j\}$, we have

$$
\langle m - z, n \rangle \geq 0.
$$

Then, by the maximal monotonicity of the operator $\mathcal{F} + \mathcal{G}$, it follows that $0_{\mathcal{H}} \in (\mathcal{F} + \mathcal{G})z$ which implies that $z \in (\mathcal{F} + \mathcal{G})^{-1}(0_{\mathcal{H}}) = \Omega$. Finally, by Lemma 1.3, we conclude that the sequence $\{u_k\}$ converges weakly to a point in Ω. This completes the proof. □

3. Image deblurring

In this section, we present the restoration results of the Panoramic Dental X-ray image [15] (Figure 1) obtained using our main method (Algorithm 2.1), and compare them with those achieved by the method of Mzimela et al. ([17], Algorithm 3.3) and the method of Dong et al. ([8], Algorithm (29)). All computations were carried out using MATLAB R2025a on an HP laptop equipped with an Intel(R) Core(TM) i7-1165G7 processor and 16 GB of RAM.

Next, we consider the image recovery problem, which can be modelled by the following linear system:

$$
b = \mathcal{K}x + \upsilon, \qquad (3.1)
$$

where $b \in \mathbb{R}^{M\times 1}$ is the observed (degraded) image, $\mathcal{K} \in \mathbb{R}^{M\times M}$ is a blurring matrix, $x \in \mathbb{R}^{M\times 1}$ denotes the original image, and $\upsilon \in \mathbb{R}^{M\times 1}$ is the noise component.

The model in (3.1) can be equivalently reformulated as the following convex optimization problem:

$$\min_{x\in\mathbb{R}^M} \frac{1}{2}\|b - \mathcal{K}x\|_2^2 + \rho\|x\|_1, \tag{3.2}$$

where $\rho > 0$ is a regularization parameter. By defining the operators $\mathcal{F} := \mathcal{K}^T(\mathcal{K}x - b)$ and $\mathcal{G} := \partial(\rho\|x\|_1)$, this problem can be viewed as a variational inclusion of the form (1.1). To evaluate the quality of the restored images, we use two standard metrics: PSNR (Peak Signal-to-Noise Ratio) [25] and SSIM (Structural Similarity Index Measure) [27], defined respectively as

$$\text{PSNR} := 20\log_{10}\left(\frac{255^2}{\|x_r - x\|_2^2}\right),$$

and

$$\text{SSIM} := \frac{(2\theta_x\theta_{x_r} + c_1)(2\sigma_{xx_r} + c_2)}{(\theta_x^2 + \theta_{x_r}^2 + c_1)(\sigma_x^2 + \sigma_{x_r}^2 + c_2)}.$$

where all parameter are defined as in earlier studies [25, 27].

In this example, the starting points y_{-4}, y_{-3}, y_{-2}, y_{-1}, y_0, x_{-3}, x_{-2}, x_{-1}, x_0, and u_1 are initialized as the blurred image. The parameters of Algorithm 2.1 are set as $\mu = 0.9$, $\beta_k = 0.9$, $\lambda_1 = 0.9$, and $t_1 = 1$ such that $t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2}$

$$\theta_{1,k} = \begin{cases} \frac{t_k-1}{t_{k+1}}, & 1 \le k \le 100; \\ \frac{1}{(3k+1)^2}, & k > 100. \end{cases}$$

Moreover, we set $\theta_{2,k} = \frac{1}{(10k+1)^5}$, $\theta_{3,k} = \frac{1}{2k^3+1}$, $\theta_{4,k} = \frac{1}{(4k+1)^5}$, $\theta_{5,k} = \frac{1}{(3k+1)^6}$ and $d_k = \frac{0.01k}{k+1}$. For the method of Mzimela et al., we set $\lambda_1 = 0.9$, $\tau_n = \frac{n-0.1}{n}$, $\epsilon = 10^{-6}$. The parameter θ_n is defined as $\theta_n = \sqrt{\rho_n^2 + \phi_n} - \rho_n$ where $\rho_n = \frac{1}{2(2\tau_{n+1}-1)}\left(\frac{1}{\tau_n} + \frac{1}{\tau_{n+1}} - 1\right)$ and $\phi_n = \frac{\tau_{n+1}}{2\tau_{n+1}-1}\left(\frac{1}{\tau_n} - \epsilon - 1\right)$. For the method of Dong et al., the parameters are chosen as $a_{1,k} = \frac{1}{2k^2+1}$, $a_{2,k} = \frac{1}{(k+1)^3}$, $a_{3,k} = \frac{1}{3k^4+1}$, $b_{1,k} = \frac{1}{3k^2+1}$, $b_{2,k} = \frac{1}{(4k+1)^2}$, $b_{3,k} = \frac{1}{(5k+1)^2}$, $\lambda = \frac{1.9}{\|D\|^2}$ and $\beta = 0.98$.

To generate the degraded (observed) images, we apply different motion blur kernels: **Case I** with a length of 40 and an angle of 90 degrees, and **Case II** with a length of 38 and an angle of 183 degrees. The deblurred images obtained after 1000 iterations for each method are presented in Figures 2 and 3. Furthermore, the PSNR and SSIM performance metrics are reported in Table 1 and illustrated in Figures 4 and 5.

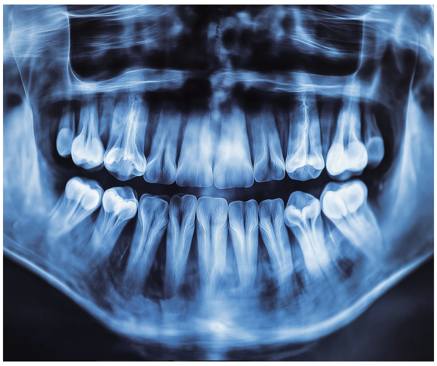


FIGURE 1. Panoramic Dental X-Ray image [15] (size 338×337)

TABLE 1. Numerical comparison of different inertial and other method

Method	Case I		Case II	
	PSNR	SSIM	PSNR	SSIM
Algorithm 2.1				
- 1 inertial	38.5818	0.9679	36.1559	0.9636
- 2 inertial	38.5860	0.9679	36.1611	0.9636
- 3 inertial	38.6048	0.9680	36.1782	0.9637
- 4 inertial	38.6048	0.9680	36.1782	0.9637
- 5 inertial	38.6048	0.9680	36.1782	0.9637
Mzimela et al.	26.8576	0.8799	24.7664	0.8417
Dong et al.	31.0984	0.9211	26.1848	0.8709

Table 1 demonstrates that Algorithm 2.1 yields the best performance in image deblurring, achieving the highest PSNR and SSIM values in Case I and Case II. Notably, the results become stable from 3 inertial steps onward. In contrast, the methods by Mzimela et al. and Dong et al. produce significantly lower performance, especially in the case of severe blurring, highlighting the superiority of Algorithm 2.1 in image restoration.

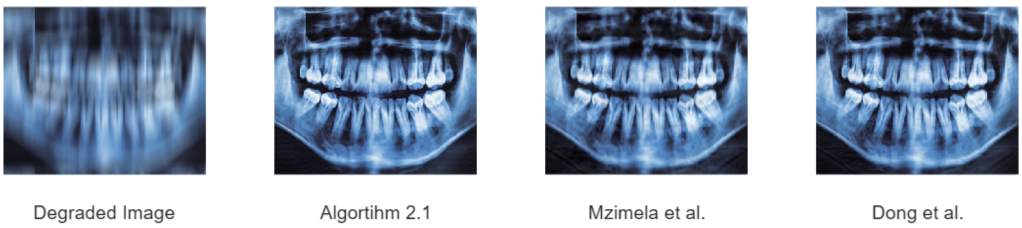
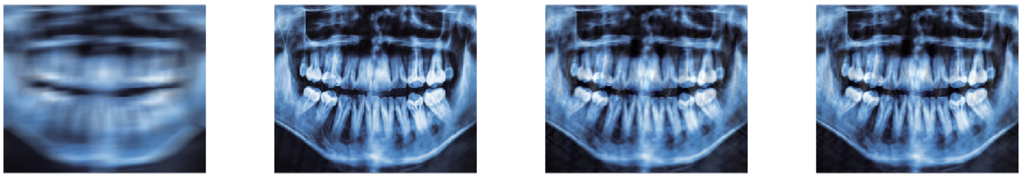
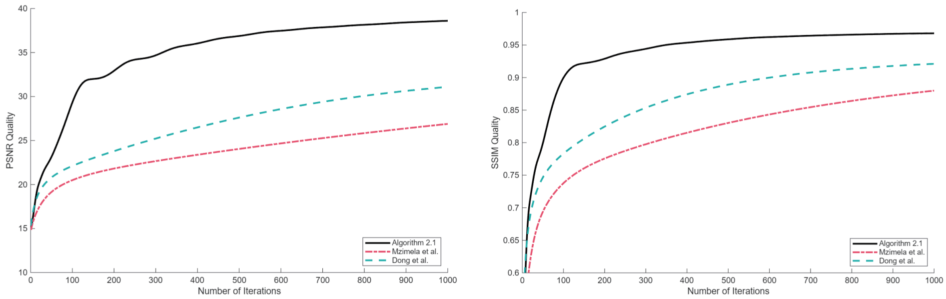


FIGURE 2. The blurred image in **Case I** and deblurred images



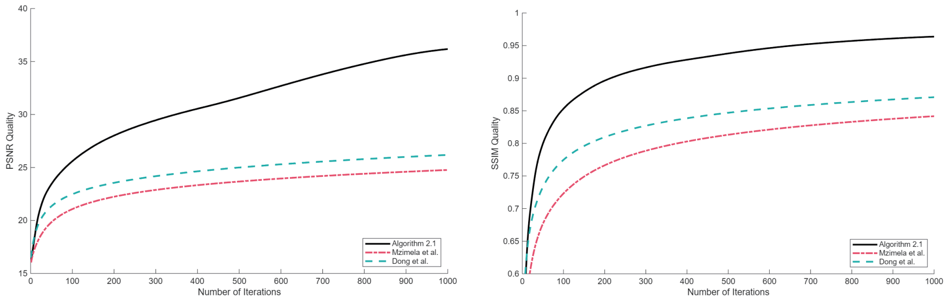
Degraded Image | Algortihm 2.1 | Mzimela et al. | Dong et al.

FIGURE 3. The blurred image in **Case II** and deblurred images



(A) PSNR (B) SSIM

FIGURE 4. PSNR and SSIM of each method by **Case I**



(A) PSNR (B) SSIM

FIGURE 5. PSNR and SSIM of each method by **Case II**

4. CONCLUSION

In this research, we developed a forward-backward-forward algorithm that combines multi-inertial extrapolation with a self-adaptive technique to solve variational inclusion problems. The proposed method is shown to achieve weak convergence under suitable conditions. Numerical experiments on image restoration problems show that using more than one inertial step significantly improves performance compared to a single step. As shown in Table 1, both PSNR and SSIM values increase with the number of inertial steps and stabilize from the third step onward. These results highlight the benefit of using a multi-inertial framework to enhance the efficiency and stability of iterative algorithms in image restoration tasks.

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