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STRONG CONVERGENCE OF AN INERTIAL HALPERN-TYPE ALGORITHM FOR MINIMIZATION AND SPLIT NULL POINT PROBLEMS IN HILBERT SPACES

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Abstract. In this paper, we introduce a novel inertial Halpern-type iterative algorithm for solving minimization problems and split null point problems in Hilbert spaces. We establish strong convergence results for the proposed algorithm under suitable conditions. Furthermore, we demonstrate how our main result can be applied to other related problems. Finally, we provide illustrative examples to effectiveness and applicability.

1. Introduction

Throughout this paper, we assume that $\mathcal{H}$, $\mathcal{H}_1$, and $\mathcal{H}_2$ are real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let I denote the identity operator and $g : \mathcal{H}_1 \to (-\infty, \infty]$ be a proper convex function. Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers and real numbers, respectively.

A fundamental problem in optimization is to find $x \in \mathcal{H}_1$ such that

$$g(x) = \min_{y \in \mathcal{H}_1} g(y),$$

and we denote the set of minimizers by $\operatorname{argmin}_{y \in \mathcal{H}_1} g(y)$.

The *proximal point algorithm (PPA)* is a standard tool for convex optimization, introduced by Martinet [27] and studied by Rockafellar [31]. Given a proper, convex, and lower semi-continuous function g, the PPA generates a sequence $\{x_n\}$ via

$$x_{n+1} = \operatorname*{argmin}_{u \in \mathcal{H}_1} \left[g(u) + \frac{1}{2\eta_n} \|u - x_n\|^2 \right], \quad n \geq 1, \tag{1.1}$$

with $\eta_n > 0$. If g has a minimizer and $\sum_{n=1}^{\infty} \eta_n = \infty$, then $\{x_n\}$ converges weakly to a minimizer; see [6]. In general, strong convergence is not guaranteed [4], though variants ensuring strong convergence exist [23, 24].

The PPA has been extended to more complex problems in recent years [29, 37]. Studying minimizers of convex functions in Hilbert spaces is central in nonlinear analysis and differential geometry, and many applications, including machine learning, computer vision, control, and robotics, can be formulated as convex optimization problems [3, 29, 37, 39].

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The *split inverse problem* (SIP) has gained attention for applications in image reconstruction, signal processing, and radiation therapy [1,8–11,13,15,28]. It seeks $x^* \in \mathcal{H}_1$ solving IP_1 such that $y^* := Ax^* \in \mathcal{H}_2$ solves IP_2, where $A\colon \mathcal{H}_1 \to \mathcal{H}_2$ is bounded linear. The first SIP, the split feasibility problem, was introduced by Censor and Elfving [10], followed by variants like the split variational inequality [13], split common null point [9], split common fixed point [12], and split equilibrium problem [25].

The *split null point problem* (SNPP), from Byrne et al. [9], involves multivalued mappings $B_1\colon \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2\colon \mathcal{H}_2 \to 2^{\mathcal{H}_2}$. It seeks $x^* \in \mathcal{H}_1$ such that

$$0 \in B_1(x^*) \quad \text{and} \quad 0 \in B_2(Ax^*). \tag{1.2}$$

Equivalently, $x^* \in B_1^{-1}(0)$ and $Ax^* \in B_2^{-1}(0)$. Byrne et al. [9] proposed a convergent algorithm for maximal monotone B_1, B_2:

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J_\lambda^{B_1}\Big(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n\Big), \tag{1.3}$$

where $J_\lambda^{B_1}, J_\lambda^{B_2}$ are resolvents.

The inertial method, introduced by Polyak [30], accelerates convergence. Alvarez and Attouch [2] applied it to the proximal point algorithm (PPA) for zeros of a maximal monotone operator $\mathcal{B}$:

$$\begin{cases} x_0, x_1 \in \mathcal{H}_1, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + \lambda_n \mathcal{B})^{-1}(y_n), \end{cases} \tag{1.4}$$

and converges weakly provided that $\theta_n \in [0, \theta)$ for some $\theta < 1$, and $\sum_{n=1}^\infty \theta_n \|x_n - x_{n-1}\|^2 < \infty$. Inertial techniques are now widely used in optimization and split feasibility problems for faster convergence and improved stability. [20,22,36].

In 2024, Tiammee *et al.* [36] introduced an iterative method for approximating a solution to the minimization problem, the equilibrium problem, and the common fixed point problem. The iterative scheme is given by

$$\begin{cases} s_n = x_n + \theta_n(x_n - x_{n-1}), \\ w_n = (1-\gamma_n)s_n + \gamma_n T_n s_n, \\ u_n = \operatorname*{arg\,min}_{u \in H} \left[g(u) + \frac{1}{2\lambda_n}\|u - w_n\|^2 \right], \\ y_n = T_{r_n}^f u_n, \\ z_n = (1-\beta_n) y_n + \beta_n T_n y_n, \\ x_{n+1} = (1-\alpha_n) T_n z_n + \alpha_n T_n y_n, \quad \forall\, n \geq 1, \end{cases} \tag{1.5}$$

where $g : H \to (-\infty, \infty]$ denotes a proper, convex, and lower semicontinuous function, and $f : H \times H \to \mathbb{R}$ is a bifunction satisfying conditions $(A1)$–$(A4)$. Furthermore, $T : H \to H$ and $\{T_n : H \to H\}_{n=1}^\infty$ are nonexpansive mappings, while $\{\lambda_n\}$, $\{\theta_n\}$, $\{\gamma_n\}$, $\{\beta_n\}$, and $\{\alpha_n\}$ are sequences in $[0, 1]$. Under the inertial condition $\sum_{n=1}^\infty \theta_n \|x_n - x_{n-1}\| < \infty$, the authors established that the sequence $\{x_n\}$ converges weakly to a solution.

Subsequently, Jun-On and Cholamjiak [22] proposed the *Picard–Mann projection forward–backward splitting algorithm with inertia (OIPMPFBS-I)*, which incorporates an inertial technique. The algorithm is defined as

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = (1-\alpha_n)x_n + \alpha_n J^G_{\lambda_n}(I - \lambda_n \mathrm{F})y_n, \\ x_{n+1} = P_C\left(J^G_{\lambda_n}(I - \lambda_n \mathrm{F})z_n\right), \end{cases} \tag{1.6}$$

where C is a nonempty closed convex subset of a real Hilbert space H, $\mathrm{F} : H \to H$ is an α-inverse strongly monotone operator, and $\mathrm{G} : H \to 2^H$ is a maximal monotone operator such that $(\mathrm{F}+\mathrm{G})^{-1} \cap C \neq \emptyset$. Under the inertial condition $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$, they proved that the sequence $\{x_n\}$ converges weakly to a solution.

In this article, motivated by these works, we study the split null point and minimization problems in Hilbert spaces. In Section 3, we propose an inertial Halpern-type algorithm for finding a common solution for maximal monotone operators and convex lower semi-continuous functions, and prove its strong convergence under suitable conditions. We also show reductions to other split problems. In Section 4, a numerical example illustrates the convergence of the algorithm.

2. Preliminaries

Let C be a nonempty closed convex subset of $\mathcal{H}$. The metric projection operator $P_C \colon H \to C$ is defined such that for each $x \in \mathcal{H}$, $P_C x$ represents the unique element in C satisfying:

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

This projection $P_C x$ can be equivalently characterized by the variational inequality:

$$\langle x - P_C x, y - P_C x\rangle \leq 0 \quad \text{for all } y \in C.$$

We recall that a mapping $T : \mathcal{H} \to \mathcal{H}$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in \mathcal{H}$, and *quasi-nonexpansive* if $\mathcal{F}(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in \mathcal{H}$ and $y \in \mathcal{F}(T)$, where $\mathcal{F}(T) := \{x \in \mathcal{H} : Tx = x\}$. Note that every nonexpansive mapping is quasi-nonexpansive, but the converse is not true in general.

Definition 2.1. Let $T : C \to C$ be a mapping. The mapping $T - I$ is said to be *demiclosed at zero* if for any sequence $\{x_k\}$ in C which $x_k \rightharpoonup x$ and $Tx_k - x_k \to 0$, then $x \in \mathcal{F}(T)$.

Lemma 2.2 ([5])**.** *Let $\mathcal{H}$ be a real Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence in $\mathcal{H}$ such that $x_n \rightharpoonup x$ with $x_n - Tx_n \to 0$, then $x \in \mathcal{F}(T)$.*

Let us recall the definition of a maximal monotone operator. A multivalued mapping $B \colon \mathcal{H} \to 2^{\mathcal{H}}$ is called *maximal monotone* if it satisfies the following conditions:

1. *Monotonicity*:

$$\langle x - y, z - w\rangle \geq 0 \quad \text{for all } x, y \in \mathrm{dom}(B),\ z \in Bx,\ w \in By,$$

where $\mathrm{dom}(B) := \{x \in \mathcal{H} : Bx \neq \emptyset\}$.

2. *Maximality*: The graph

$$G(B) := \{(x, z) \in \mathcal{H} \times \mathcal{H} : z \in Bx\}$$

is not properly contained in the graph of any other monotone operator, which is equivalent to

$$(x, z) \in G(B) \iff \langle x - y, z - w\rangle \geq 0 \quad \text{for all } (y, w) \in G(B).$$

For a maximal monotone operator $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ and $\lambda > 0$, the *resolvent* of B with parameter λ is defined by

$$J_\lambda^B := (I + \lambda B)^{-1}.$$

The resolvent operator has the following important properties [14]:

- $J_\lambda^B\colon \mathcal{H} \to \mathrm{dom}(B)$ is single-valued and *firmly nonexpansive*:

  $$\|J_\lambda^B x - J_\lambda^B y\|^2 \leq \langle J_\lambda^B x - J_\lambda^B y, x - y\rangle \quad \text{for all } x, y \in \mathcal{H}.$$

  This is equivalent to:

  $$\langle J_\lambda^B x - J_\lambda^B y, (J_\lambda^B x - x) - (J_\lambda^B y - y)\rangle \leq 0.$$

- The fixed point set satisfies $F(J_\lambda^B) = B^{-1}(0) = \{x \in \mathcal{H} : 0 \in Bx\}$.
- The operator $I - J_\lambda^B$ is *demiclosed* at zero.

Lemma 2.3 ([33])**.** *The following holds for all $u, w \in \mathcal{H}$ and any $\lambda \in [0, 1]$:*

(1) $\|\lambda u + (1 - \lambda)w\|^2 = \lambda\|u\|^2 + (1 - \lambda)\|w\|^2 - \lambda(1 - \lambda)\|u - w\|^2$*;*
(2) $\|u \pm w\|^2 = \|u\|^2 \pm 2\langle u, w\rangle + \|w\|^2$*;*
(3) $\|u + w\|^2 \leq \|u\|^2 + 2\langle w, u + w\rangle$.

Lemma 2.4 ([38])**.** *Suppose that $\{t_n\}$ is a sequence of nonnegative real numbers such that*

$$t_{n+1} \leq (1 - \alpha_n)t_n + \alpha_n\beta_n + \delta_n, \quad n \in \mathbb{N},$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_n \beta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\alpha_n\beta_n| < \infty$;
(iii) $\delta_n \geq 0$ *for all* $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then, $\lim_{n\to\infty} t_n = 0$.

Lemma 2.5 ([26])**.** *Let $\{\Gamma_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_{i+1}}$ for all $i \in \mathbb{N}$. Also consider the sequence of positive integers $\{\rho(n)\}$ defined by*

$$\rho(n) := \max\{m \leq n : \Gamma_m < \Gamma_{m+1}\}$$

for all $n \geq n_0$ (for some n_0 large enough). Then, $\{\rho(n)\}$ is a nondecreasing sequence such that $\rho(n) \to \infty$ as $n \to \infty$, and it holds that

$$\Gamma_{\rho(n)} \leq \Gamma_{\rho(n)+1}, \quad \Gamma_n \leq \Gamma_{\rho(n)+1}.$$

Let $f : H \to (-\infty, \infty]$ be a proper, convex, and lower semi-continuous function. For $\lambda > 0$, the Moreau–Yosida resolvent is defined by

$$J_\lambda x = \operatorname*{argmin}_{u \in H} \left[f(u) + \frac{1}{2\lambda}\|u - x\|^2 \right], \quad x \in H.$$

It is well-known that $\mathrm{Fix}(J_\lambda) = \operatorname{argmin} f$ and that J_λ is nonexpansive [18, 21].

Lemma 2.6 (Properties of the resolvent [17, 21])**.** *Let $\mathcal{H}$ be a real Hilbert space and $g : \mathcal{H} \to (-\infty, \infty]$ be a proper, convex, and lower semi-continuous function. For each $x, y \in \mathcal{H}$ and $\lambda > \mu > 0$, the resolvent J_λ satisfies:*

(1) *(Resolvent identity)*

$$J_\lambda x = J_\mu\Big(\frac{\lambda - \mu}{\lambda} J_\lambda x + \frac{\mu}{\lambda} x\Big),$$

(2) *(Subdifferential inequality)*

$$\frac{1}{2\lambda}\|J_\lambda x - y\|^2 - \frac{1}{2\lambda}\|x - y\|^2 + \frac{1}{2\lambda}\|x - J_\lambda x\|^2 \le g(y) - g(J_\lambda x).$$

3. Main results

In this section, we propose an accelerated algorithm combining the inertial technique and Halpern approximation to find a common solution of the split null point and convex minimization problems, and establish its strong convergence under suitable assumptions.

- $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator.
- $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is a maximal monotone operator.
- $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ is a maximal monotone operator.
- $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are the resolvents of B_1 and B_2, respectively.
- $g : \mathcal{H}_1 \to (-\infty, \infty]$ is a proper convex and lower semi-continuous.
- $\Theta := \operatorname{argmin}_{v \in \mathcal{H}_1} g(v) \cap \Omega$ where $\Omega := \{x \in B_1^{-1}0 : Ax \in B_2^{-1}0\}$.

Let $\{\alpha_n\} \subset (0, 1)$ and $\{\eta_n\}, \{\lambda_n\}, \{\tau_n\}, \{\mu_n\} \subset (0, \infty)$.

Algorithm 1 Inertial Modified Halpern Approximation Method (IMHAM)

1: **Initialize:** Take $u, x_0, x_1 \in \mathcal{H}_1$.
2: **for** $n \ge 1$ **do**
3: **Compute the inertial step:**

$$\theta_n = \begin{cases} \min\left\{\mu_n, \frac{\tau_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \ne x_{n-1}; \\ \mu_n, & \text{otherwise.} \end{cases}$$

4: **Compute y_n, z_n and u_n:**

$$y_n = x_n + \theta_n(x_n - x_{n-1})$$

$$z_n = \arg\min_{v \in \mathcal{H}_1} \left[g(v) + \frac{1}{2\eta_n}\|v - y_n\|^2 \right]$$

$$u_n = J_{\lambda_n}^{B_1}\left(z_n + \gamma A^*\left(J_{\lambda_n}^{B_2} - I\right) A z_n\right)$$

5: **Compute x_{n+1}:**

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) u_n$$

Lemma 3.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 1 and $\Theta \neq \emptyset$. Suppose that $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$ and $\lim_{n\to\infty} \tau_n = 0$. Then $\{x_n\}$ is bounded. Furthermore, $\{u_n\}, \{y_n\}$ and $\{z_n\}$ are also bounded.*

Proof. Since Θ is closed and convex. Let $x^* = P_\Theta u$. By characterization of the metric projection, we get

$$\langle u - x^*, p - x^*\rangle \leq 0 \quad \text{for all } p \in \Theta. \tag{3.1}$$

Since $x^* \in \Theta$, we obtain $g(x^*) \leq g(v)$ forall $v \in H$, $J_{\lambda_n}^{B_1} x^* = x^*$ and $J_{\lambda_n}^{B_1} Ax^* = Ax^*$. Let $\{x_n\}$ be a sequence in $\mathcal{H}_1$ generated by Algorithm 1. So, we get

$$\|y_n - x^*\| \leq \|x_n - x^*\| + \theta_n \|x_n - x_{n-1}\|. \tag{3.2}$$

Since $\lim_{n\to\infty} \tau_n = 0$. Then there exists a positive constant $M > 0$ such that $\tau_n < M$ for all $n \in \mathbb{N}$. Therefore, by definition of θ_n, we obtain

$$\begin{aligned} \|y_n - x^*\|^2 &\leq \|x_n - x^*\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \|x_n - x^*\| \|x_n - x_{n-1}\| \\ &\leq \|x_n - x^*\|^2 + \tau_n^2 + 2\tau_n \|x_n - x^*\| \end{aligned} \tag{3.3}$$

Since $g(x^*) \leq g(u)$ for all $u \in H$. It follows that

$$g(x^*) + \frac{1}{2\lambda_n}\|x^* - x^*\|^2 \leq g(u) + \frac{1}{2\lambda_n}\|u - x^*\|^2, \quad \text{for all } u \in H,$$

and hence $x^* = J_{\eta_n} x^*$ for all $n \in \mathbb{N}$. Since $z_n = J_{\eta_n} y_n$, it implies by nonexpansiveness of J_{η_n} that

$$\|z_n - x^*\| = \|J_{\eta_n} y_n - J_{\eta_n} x^*\| \leq \|y_n - x^*\|. \tag{3.4}$$

Since $J_{\lambda_n}^{B_1}$ is nonexpansive and A is a bounded linear operator, we have

$$\begin{aligned} \|u_n - x^*\|^2 &= \left\| J_{\lambda_n}^{B_1}(z_n + \gamma A^*(J_{\lambda_n}^{B_2} - I)Az_n) - J_{\lambda_n}^{B_1} x^* \right\|^2 \\ &\leq \|z_n - x^*\|^2 + \gamma^2 \left\| A^*(J_{\lambda_n}^{B_2} - I)Az_n \right\|^2 \\ &\quad + 2\gamma \left\langle z_n - x^*, A^*(J_{\lambda_n}^{B_2} - I)Az_n \right\rangle \\ &\quad + 2\gamma \left\langle Az_n - Ax^*, J_{\lambda_n}^{B_2}(Az_n) - Az_n \right\rangle. \end{aligned} \tag{3.5}$$

Since $J_{\lambda_n}^{B_2}$ is firmly nonexpansive, we have

$$\begin{aligned} \mathcal{E}_n &= 2\gamma \Big\langle Az_n - Ax^* + (J_{\lambda_n}^{B_2}(Az_n) - Az_n) \\ &\quad -(J_{\lambda_n}^{B_2}(Az_n) - Az_n), J_{\lambda_n}^{B_2}(Az_n) - Az_n \Big\rangle \\ &= 2\gamma \Big(\Big\langle J_{\lambda_n}^{B_2}(Az_n) - Ax^*, J_{\lambda_n}^{B_2}(Az_n) - Az_n \Big\rangle \\ &\quad - \left\| J_{\lambda_n}^{B_2}(Az_n) - Az_n \right\|^2 \Big) \\ &\leq -2\gamma \left\| J_{\lambda_n}^{B_2}(Az_n) - Az_n \right\|^2. \end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we obtain that

$$\|u_n - x^*\|^2 \leq \|z_n - x^*\|^2 - \gamma(2-\gamma\|A\|^2)\left\|J_{\lambda_n}^{B_2}(Az_n) - Az_n\right\|^2. \tag{3.7}$$

This implies that $\|u_n - x^*\| \leq \|z_n - x^*\|$. Thus, we have

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \alpha_n\|u - x^*\| + (1-\alpha_n)\|u_n - x^*\| \\ &\leq \alpha_n\|u - x^*\| + (1-\alpha_n)\|z_n - x^*\| \\ &= (1-\alpha_n)\|x_n - x^*\| + \alpha_n\left[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|u - x^*\|\right]\end{aligned} \tag{3.8}$$

According to the definition of θ_n, we have $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \to 0$ as $n \to \infty$. Then there exists a positive constant $M_1 > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq M_1 \quad \forall n \geq 1$. From (3.8), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\left[M_1 + \|u - x^*\|\right] \\ &\leq \max\{\|x_n - x^*\|, M_1 + \|u - x^*\|\}\end{aligned}$$

By induction, we obtain

$$\|x_n - x^*\| \leq \max\{\|x_1 - x^*\|, M_1 + \|u - x^*\|\} \quad \text{for all } n \in \mathbb{N}.$$

Therefore, the sequence $\{x_n\}$ is bounded, which implies that $\{u_n\}, \{z_n\}, \{y_n\}$ are also bounded. □

Theorem 3.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 1 and $\Theta \neq \emptyset$. Suppose that the parameter γ and the sequence $\{\alpha_n\}, \{\lambda_n\}, \{\eta_n\}$ satisfy the following conditions:*

(i) $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$;
(ii) $\lim_{n\to\infty} \tau_n = 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iv) $\liminf_{n\to\infty} \lambda_n > 0$;
(v) $\eta_n \geq \eta > 0$ *for some* η.

Then, the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Theta$, where $x^* = P_\Theta u$.

Proof. It follows from (3.3), (3.4) and (3.7) that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|u - x^*\|^2 + \|z_n - x^*\|^2 - \gamma(2-\gamma\|A\|^2)\left\|J_{\lambda_n}^{B_2}(Az_n) - Az_n\right\|^2 \\ &\leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 + \tau_n^2 + 2\tau_n\|x_n - x^*\| \\ &\quad - \gamma(2-\gamma\|A\|^2)\left\|J_{\lambda_n}^{B_2}(Az_n) - Az_n\right\|^2.\end{aligned} \tag{3.9}$$

Thus, by (3.3), we get the following inequalities

$$\begin{aligned}\gamma(2-\gamma\|A\|^2)\|J_{\lambda_n}^{B_2}(Az_n) - Az_n\|^2 &\leq \alpha_n\|u - x^*\|^2 + \tau_n^2 + 2\tau_n\|x_n - x^*\| \\ &\quad + \left(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\right).\end{aligned} \tag{3.10}$$

Now, we divide the rest of the proof into two cases.

Case 1. Assume that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|\}_{n \geq n_0}$ is either nonincreasing or nondecreasing. Since $\{\|x_n - x^*\|\}$ is bounded, then it converges i.e. $\|x_n - x^*\| \to d$ as $n \to \infty$ and

$$\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \to 0 \quad \text{as } n \to \infty.$$

By (3.8) we can conclude that $\|y_n - x^*\| \to d$, $\|z_n - x^*\| \to d$ and $\|u_n - x^*\| \to d$ as $n \to \infty$. By Lemma 2.6, we have

$$\frac{1}{2\lambda_n}\|z_n - x^*\|^2 - \frac{1}{2\lambda_n}\|y_n - x^*\|^2 + \frac{1}{2\lambda_n}\|y_n - z_n\|^2 \leq g(x^*) - g(z_n).$$

Since $g(x^*) \leq g(z_n)$ for all $n \geq 1$, we obtain $\|y_n - z_n\|^2 \leq \|y_n - x^*\|^2 - \|z_n - x^*\|^2$. This implies that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.11}$$

By Lemma 2.6 , nonexpansiveness of J_{η_n}, and $\eta_n \geq \eta > 0$ that

$$\begin{aligned}\|y_n - J_\eta y_n\| &= \|y_n - z_n\| + \left\| J_\eta \left(\frac{\eta_n - \eta}{\eta_n} J_{\eta_n} y_n + \frac{\eta}{\eta_n} \right) - J_\eta y_n \right\| \\ &= \left(2 - \frac{\eta}{\eta_n} \right) \|y_n - z_n\| .\end{aligned}$$

This together with (3.11) shows that

$$\lim_{n\to\infty} \|y_n - J_\eta y_n\| = 0. \tag{3.12}$$

Since $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$, $\tau_n \to 0$ and $\alpha_n \to 0$ as $n \to \infty$, then by (3.10) we deduce that

$$\lim_{n\to\infty} \left\| J_{\lambda_n}^{B_2}(Az_n) - Az_n \right\| = 0. \tag{3.13}$$

By (3.5) and (3.6), we also have

$$\left\| z_n + \gamma A^* \left(J_{\lambda_n}^{B_2} - I \right) Az_n - x^* \right\|^2 \leq \|z_n - x^*\|^2. \tag{3.14}$$

$$\|u_n - x^*\|^2 \leq \|z_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\gamma\|u_n - z_n\| \, \|A^*(J_{\lambda_n}^{B_2} - I)Az_n\|,$$

which gives

$$\|u_n - x^*\|^2 \leq \|z_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\gamma\|u_n - z_n\| \, \|A^*(J_{\lambda_n}^{B_2} - I)Az_n\|. \tag{3.15}$$

It follows from (3.4), (3.9), and (3.15) that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|u - x^*\|^2 + \|z_n - x^*\|^2 - \|u_n - z_n\|^2 \\ &\quad + 2\gamma\|u_n - z_n\| \, \|A^*(J_{\lambda_n}^{B_2} - I)Az_n\| \\ &\leq \alpha_n\|u - x^*\|^2 + \|y_n - x^*\|^2 - \|u_n - z_n\|^2 \\ &\quad + 2\gamma\|u_n - z_n\| \, \|A^*(J_{\lambda_n}^{B_2} - I)Az_n\|.\end{aligned}$$

Then, by (3.3), we obtain

$$\begin{aligned}\|u_n - z_n\|^2 &\leq \alpha_n\|u - x^*\|^2 + (\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2) + \tau_n^2 + 2\tau_n\|x_n - x^*\| \\ &\quad + 2\gamma\|u_n - z_n\| \, \|A^*(J_{\lambda_n}^{B_2} - I)Az_n\| \to 0\end{aligned} \tag{3.16}$$

as $n \to \infty$, which implies $\|u_n - z_n\| \to 0$.

We next show that

$$\limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle \le 0.$$

To show this, let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{j\to\infty} \langle u - x^*, x_{n_j} - x^* \rangle = \limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle.$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ and $p \in \mathcal{H}_1$ such that $x_{n_{j_k}} \rightharpoonup p$. Without loss of generality, we can assume that $x_{n_j} \rightharpoonup p$. Since A is a bounded linear operator, we have $\langle z, Ax_{n_j} - Ap \rangle = \langle A^* z, x_{n_j} - p \rangle \to 0$ as $j \to \infty$, for all $z \in \mathcal{H}_2$, this implies that $Ax_{n_j} \rightharpoonup Ap$. From (3.13) and by the demiclosedness of $I - J_{\lambda_n}^{B_2}$ at zero, we get $Ap \in F(J_{\lambda_n}^{B_2}) = B_2^{-1}0$. Since $x_{n_j} \rightharpoonup p$ and $\|y_n - x_n\| \to 0$ as $n \to \infty$, we have $y_{n_j} \rightharpoonup p$. It follows from J_η is a nonexpansive mapping, by (3.12) and Lemma 2.2, we get $p \in F(J_\eta) = \operatorname{argmin}_{u\in H} g(u)$. Now let us show that $p \in B_1^{-1}0$. From $u_n = J_{\lambda_n}^{B_1}(z_n + \gamma A^*(J_{\lambda_n}^{B_2} - I)Az_n)$, then we can easily prove that

$$\frac{1}{\lambda_n}\left(z_n - u_n + \gamma A^* \left(J_{\lambda_n}^{B_2} - I\right) Az_n\right) \in B_1 u_n.$$

By the monotonicity of B_1, we have

$$\left\langle u_n - v, \frac{1}{\lambda_n}\left(z_n - u_n + \gamma A^* \left(J_{\lambda_n}^{B_2} - I\right) Az_n\right) - w \right\rangle \ge 0$$

for all $(v, w) \in G(B_1)$. Thus, we also have

$$\left\langle u_{n_j} - v, \frac{1}{\lambda_{n_j}}\left(z_{n_j} - u_{n_j} + \gamma A^* \left(J_{\lambda_{n_j}}^{B_2} - I\right) Az_{n_j}\right) - w \right\rangle \ge 0 \tag{3.17}$$

for all $(v, w) \in G(B_1)$. Since $u_{n_j} \rightharpoonup p$, $\|z_{n_j} - u_{n_j}\| \to 0$ and $\|(J_{\lambda_{n_j}}^{B_2} - I)Az_{n_j}\| \to 0$ as $j \to \infty$, then by taking the limit as $j \to \infty$ in (3.17) yields

$$\langle p - v, -w \rangle \ge 0$$

for all $(v, w) \in G(B_1)$. By the maximal monotonicity of B_1, we get $0 \in B_1 p$, i.e., $p \in B_1^{-1}0$. Therefore, $p \in \Theta$. Since x^* satisfies the inequality (3.2), we have

$$\limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle = \lim_{j\to\infty} \langle u - x^*, x_{n_j} - x^* \rangle = \langle u - x^*, p - x^* \rangle \le 0.$$

By using Lemma 2.3, we obtain

$$\|x_{n+1} - x^*\|^2 \le (1 - \alpha_n)\|x_n - x^*\|^2 + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle.$$

Therefore, by Lemma 2.4, we can conclude that $x_n \to x^*$ as $n \to \infty$.

Case 2. Suppose that $\{\|x_n - x^*\|\}$ is not a monotone sequence. Hence, there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\|x_{n_i} - x^*\| < \|x_{n_i+1} - x^*\|$ for all $i \in \mathbb{N}$. We now define a sequence $\{\rho(n)\}$ by

$$\rho(n) := \max\{m \le n : \|x_m - x^*\| < \|x_{m+1} - x^*\|\}$$

for all $n \ge n_0$ (for some n_0 large enough). By Lemma 2.5, we obtain that $\{\rho(n)\}$ is a nondecreasing sequence such that $\rho(n) \to \infty$ as $n \to \infty$ and

$$\|x_{\rho(n)} - x^*\|^2 - \|x_{\rho(n)+1} - x^*\|^2 \le 0$$

for all $n \geq n_0$. From (3.10), we get

$$\lim_{n\to\infty} \left\| \left(J^{B_2}_{\lambda_{\rho(n)}} - I \right) A z_{\rho(n)} \right\| = 0. \tag{3.18}$$

From (3.16), we obtain that

$$\lim_{n\to\infty} \|u_{\rho(n)} - z_{\rho(n)}\| = 0. \tag{3.19}$$

By (3.18), (3.19), together with the same argument as in Case 1, we conclude that

$$\limsup_{n\to\infty} \langle u - x^*, x_{\rho(n)} - x^* \rangle \leq 0.$$

In the same way in Case 1, we obtain that

$$\|x_{\rho(n)+1} - x^*\|^2 \leq (1 - \alpha_{\rho(n)})\|x_{\rho(n)} - x^*\|^2 + 2\alpha_{\rho(n)}\langle u - x^*, x_{\rho(n)+1} - x^*\rangle.$$

By applying Lemma 2.4 again, we obtain that $\|x_{\rho(n)} - x^*\| \to 0$ as $n \to \infty$. It follows from Lemma 2.5 that $0 \leq \|x_n - x^*\| \leq \|x_{\rho(n)+1} - x^*\| \to 0$ as $n \to \infty$. Therefore $\{x_n\}$ converges strongly to x^*. This completes the proof.

□

We first present a result on convex minimization, which extends the classical algorithm of Rockafellar [31].

Theorem 3.3. *Let $\mathcal{H}$ be a real Hilbert space and $g : \mathcal{H} \to (-\infty, \infty]$ proper, convex, and lower semi-continuous. Let $u \in \mathcal{H}$ and $\{x_n\}$ be generated by $x_0, x_1 \in \mathcal{H}$ and*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = \arg\min_{v\in\mathcal{H}} \left[g(v) + \frac{1}{2\eta_n}\|v - y_n\|^2 \right], \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n, \end{cases} \tag{3.20}$$

with $\theta_n = \min\{\mu_n, \tau_n/\|x_n - x_{n-1}\|\}$ if $x_n \neq x_{n-1}$, and $\theta_n = \mu_n$ otherwise. Assume $\lim_{n\to\infty} \tau_n = 0$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\eta_n \geq \eta > 0$. Then $\{x_n\}$ converges strongly to a minimizer of g.

Proof. Apply Theorem 3.2 with $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $A = I$, and $B_1 = B_2 = \{0\}$. Then $J^{B_i}_{\lambda} = I$, $\Omega = \mathcal{H}$, and $\Theta = \arg\min g$. The result follows. □

Similarly, by setting $g := I$, we obtain Byrne et al.'s [9] split null point algorithm, now enhanced with an inertial term.

Corollary 3.4. *Let $\mathcal{H}_1, \mathcal{H}_2$ be real Hilbert spaces, $A : \mathcal{H}_1 \to \mathcal{H}_2$ bounded linear, and B_1, B_2 maximal monotone with resolvents $J^{B_1}_{\lambda}, J^{B_2}_{\lambda}$ for $\lambda > 0$. Assume $\Omega := \{x \in B_1^{-1}0 : Ax \in B_2^{-1}0\} \neq \emptyset$. Let $u \in \mathcal{H}_1$ and define $\{x_n\}$ by*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) J^{B_1}_{\lambda} \left(y_n + \gamma A^*(J^{B_2}_{\lambda} - I) A y_n \right), \end{cases}$$

where $\theta_n = \min\{\mu_n, \tau_n/\|x_n - x_{n-1}\|\}$ if $x_n \neq x_{n-1}$, and $\theta_n = \mu_n$ otherwise. Suppose $\gamma \in (0, 2/\|A\|^2)$, $\lim_{n\to\infty} \tau_n = 0$, $\lim_{n\to\infty} \alpha_n = 0$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\liminf_{n\to\infty} \lambda_n > 0$. Then $\{x_n\}$ converges strongly to $x^* \in \Omega$.

Finally, for the split feasibility problem (SFP) [10], finding $x^* \in C$ with $Ax^* \in Q$, we obtain:

Theorem 3.5. *Let $C \subset \mathcal{H}_1$, $Q \subset \mathcal{H}_2$ closed convex, A bounded linear, and g proper convex l.s.c. Assume $\Theta := \arg\min g \cap A^{-1}(Q) \neq \emptyset$. Let $\{x_n\}$ be generated by*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = \arg\min[g(v) + \frac{1}{2\eta_n}\|v - y_n\|^2], \\ u_n = P_C(z_n + \gamma A^*(P_Q - I)Az_n), \\ x_{n+1} = \alpha_n u + (1-\alpha_n)u_n, \end{cases}$$

with standard parameters. Then $x_n \to x^* = P_\Theta u$.

Proof. Set $B_1 = \partial i_C$, $B_2 = \partial i_Q$ and apply Theorem 3.2. Then $J_\lambda^{B_1} = P_C$, $J_\lambda^{B_2} = P_Q$, $B_1^{-1}0 = C$, $B_2^{-1}0 = Q$, giving the result. □

4. A numerical example

This section provides a numerical experiment that illustrates the effectiveness of our proposed algorithm.

Example 4.1. Let $\mathcal{H}_1 = \mathbb{R}$ and $\mathcal{H}_2 = \mathbb{R}^3$ with the usual norms. Define a function $g : \mathcal{H}_1 \to (-\infty, \infty]$ by

$$g(x) := \frac{1}{2}\|x\|^2.$$

Using the proximity operator [16], we know that

$$\underset{v\in\mathcal{H}}{\text{argmin}} \left[g(v) + \frac{1}{2\eta_n}\|v - x\|^2\right] = \text{prox}_{\eta_n g} x = \frac{x}{1+\eta_n}.$$

Let $h : [-9, 3] \times [-9, 3] \to \mathbb{R}$ be a bifunction defined by $h(x, y) = y^2 + xy - 2x^2$ and let $f : \mathbb{R}^3 \to (-\infty, \infty]$ be a function defined by $f(z) = \frac{1}{2}\|Pz\|^2$, where

$$P = \begin{bmatrix} -4 & 2 & 7 \\ 1 & -5 & 8 \end{bmatrix}.$$

We define two maximal monotone operators $B_1 : \mathbb{R} \to 2^{\mathbb{R}}$ and $B_2 : \mathbb{R}^3 \to 2^{\mathbb{R}^3}$. In particular, we define $B_1 := A_h$ (see [32]) where $A_h : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is given by

$$A_h(x) = \{y \in \mathbb{R} : h(x, z) \geq \langle z - x, y\rangle, \forall z \in [-9, 3]\},$$

and $B_2 := \partial f$. By [32] and [14], we can write the explicit resolvents of B_1 and B_2 in the following forms:

$$J_1^{B_1} x = \frac{x}{4} \quad \text{and} \quad J_1^{B_2} z = (P^T P + I)^{-1} z$$

for all $x \in \mathbb{R}$ and $z \in \mathbb{R}^3$. Define a bounded linear operator $A : \mathbb{R} \to \mathbb{R}^3$ by $Ax := (5x, -3x, 7x)$. Let $\Theta := \text{argmin}_{v\in\mathcal{H}_1} g(v) \cap \Omega$, where $\Omega = \{x \in B_1^{-1}0 : Ax \in B_2^{-1}0\}$. Take $\alpha_n = \frac{1}{10^5 n}, \lambda_n = 1, \gamma = \frac{1}{\|A\|^2}, u = \frac{1}{2}, \tau_n = \frac{10^{11}}{n^2}, \eta_n = 1$. We first start with the initial points $x_0 = 50$ and $x_1 = 32$, and the stopping criterion for our testing process is set as: $|x_n - x_{n-1}| < 10^{-10}$. The convergence of Algorithm 1 is demonstrated in Table 1, where the iterates converge to the solution $0 \in \Theta$ with a total CPU time of 0.0469 seconds.

n	x_n	$\lvert x_n - x_{n-1}\rvert$
2	0.5836265539	31.4163734460
3	-0.5166478262	1.1002743801
4	-0.0340478946	0.4825999316
5	0.0089340775	0.0429819722
⋮	⋮	⋮
110	4.706E-08	4.36E-10
⋮	⋮	⋮
227	2.269E-08	1.01E-10
228	2.259E-08	1.00E-10

TABLE 1. Numerical experiment of Algorithm 1

Moreover, we compare the algorithm from Theorem 3.3 with the proximal point algorithm (PPA). The results, using the chosen parameters, are illustrated in Figure 1.

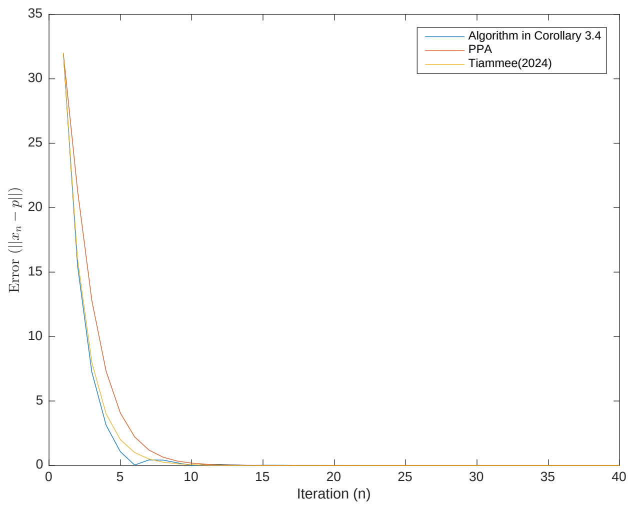

FIGURE 1. Comparison of the PPA, Tiammee 2024 (Algorithm (1.5)) and the algorithm from Corollary 3.4. Parameters: Algorithm in Corollary 3.4: $x_0 = 50, x_1 = 32, u = 0.0001, \alpha_n = 1/10^5 n, \eta_n = \lambda_n = 1, \tau_n = 10^{11}/n, \mu_n = n/(n+1)$; PPA: $x_0 = 32, \lambda_n = (0.001)^n$; Algorithm (1.5): $x_0 = 50, x_1 = 32, \theta_n = n/(n+1)$ with $T_n := I$.

Next, we compare the algorithm presented in Corollary 3.4 with Byrne's algorithm [7]. The results, using the chosen parameters, are shown in Figure 2.

Next, we compare IMHAM with (OIPMPFBS-I) [22] by setting $A := 0, g := 0$ and $B := A_h$. The results, using the chosen parameters, are shown in Figure 3.

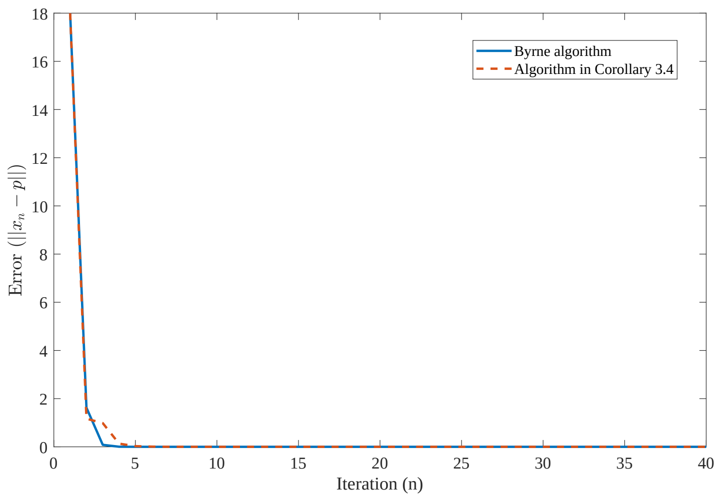

FIGURE 2. Comparison between Algorithm (1.3) and the algorithm in Corollary 3.4. Parameters: Algorithm in Corollary 3.4: $x_0 = 50, x_1 = 32, u = 0.5, \alpha_n = 1/10^5 n, \lambda_n = 1$; Byrne's Algorithm: $x_0 = 32, u = 0.5, \alpha_n = 1/10^5 n, \lambda_n = 1$.

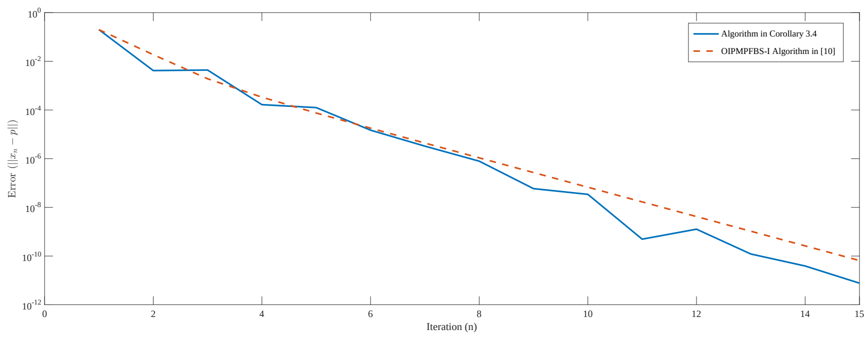

FIGURE 3. Comparison between IMHAM (Algorithm 1) and OIPMPFBS-I (1.6). Parameters: IMHAM: $x_0 = 1, x_1 = 0.5, u = -e10^{-9}, \alpha_n = 1/2^n, \lambda_n = 1, \mu_n = n/(n+1), \tau_n = 10^{11}/n$; OIPMPFBS-I: $x_0 = 1, x_1 = 0.5, \alpha_n = 1/2^n, \lambda_n = 1, \theta_n = 1/2^n$.

5. Conclusion

In this paper, we proposed a novel inertial Halpern-type iterative method to solve convex minimization and split null point problems, and established its strong convergence under suitable conditions. By specifying certain functions, the method can solve these problems separately and can be extended to address split feasibility problems. Numerical examples illustrate the convergence behavior and confirm the practical effectiveness of the algorithm. Comparisons with existing methods show faster convergence in convex minimization problems.

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