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AN ENHANCED DOUBLE INERTIAL PARALLEL MANN ALGORITHM FOR A FAMILY OF QUASI-NONEXPANSIVE MAPPINGS: APPLICATION TO BLIND IMAGE RESTORATION

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Abstract. This paper presents an enhanced double inertial parallel Mann algorithm designed to solve the fixed-point problem for a family of quasi-nonexpansive mappings. We establish the weak convergence of the proposed method under suitable conditions on the algorithmic parameters. To validate its real-world applicability, the algorithm is applied to blind image restoration, specifically focusing on scenarios where the blurring kernel is unknown. Experimental results confirm that the method effectively recovers degraded images. A notable application of this approach is in improving image clarity for moving vehicles, which plays a vital role in traffic monitoring, accident analysis, and law enforcement. By enhancing visual detail in blurred images, the algorithm supports tasks such as identifying license plates, recognizing occupants, and improving overall situational awareness.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Suppose that C is a nonempty, closed, and convex subset of $\mathcal{H}$. A mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is called

(i) nonexpansive if

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|$$

for all $x, y \in \mathcal{C}$;

(ii) quasi-nonexpansive if

$$\|\mathcal{T}x - y\| \leq \|x - y\|$$

for all $x \in \mathcal{C}$ and $y \in F(\mathcal{T})$ where

$$F(\mathcal{T}) := \{x \in \mathcal{C} : \mathcal{T}x = x\}$$

which is the fixed points set of the mapping $\mathcal{T}$.

Fixed point theory is a fundamental concept in optimization and nonlinear analysis, providing a mathematical framework for ensuring the existence and uniqueness of solutions in iterative methods. It has broad applications across various disciplines, including machine learning, engineering, medicine, economics, and image processing. In artificial intelligence, it aids in training deep neural networks and

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reinforcement learning, while in engineering, it enhances control systems and stability analysis. In medical imaging and computational biology, it supports high-resolution image reconstruction and diagnostic accuracy, whereas in economics, it plays a crucial role in equilibrium analysis and market stability. Additionally, in data science and signal processing, fixed point methods improve noise reduction and image restoration. Its flexibility and robustness make it a powerful tool for addressing complex real-world challenges.

A classical one is Mann iteration [15], which is defined by $x_1 \in \mathcal{C}$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\mathcal{T} x_n, \quad \text{for } n \geq 1. \tag{1.1}$$

It is among the most widely recognized algorithms for this purpose. Under the condition $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, the algorithm has been proven to converge weakly to a fixed point of T [2,20]. However, its convergence is generally slow. To enhance the efficiency of the Mann algorithm, one approach is to integrate inertial extrapolation with the Mann algorithm.

The inertial-type strategy, originally introduced by Polyak [18], has received considerable attention due to its impact on accelerating algorithmic convergence. This technique, commonly referred to as the heavy-ball method, is inspired by second-order dynamical systems and incorporates momentum through the use of two prior iterates.

In recent developments, researchers have extensively explored inertial extrapolation frameworks to enhance the performance of iterative methods. These enhancements have led to notable improvements in both convergence speed and stability, as supported by theoretical analyses and numerical studies. Inertial techniques have been effectively applied in fields such as signal processing and image restoration. Notable examples include inertial forward-backward splitting methods [12, 24], inertial projection schemes [10, 23], inertial extragradient approaches [6, 11, 13], and the widely used Fast Iterative Shrinkage-Thresholding Algorithm (FISTA) [3].

In 2008, Mainge [14] unified the inertial extrapolation with the Mann algorithm (1.1) and introduced the following inertial Mann algorithm: by setting $x_1,\ x_0 \in \mathcal{H}$, and

$$\begin{aligned} y_n &= x_n + \delta_n(x_n - x_{n-1}), \\ x_{n+1} &= \psi_n y_n + (1 - \psi_n)\mathcal{T}_n y_n, \quad n \geq 1, \end{aligned}$$

where $\mathcal{T}_n : \mathcal{H} \to \mathcal{H}$ denotes a countable family of nonexpansive mappings on $\mathcal{H}$; $\delta_n \in [0, 1]$ serves as a damping-type parameter; and $\psi_n \in (0, 2)$ is a relaxation factor, with the iterative sequence $\{x_n\}$ defined above converging weakly to a fixed point under mild assumptions.

Recent studies have indicated that a one-step inertial term may not always enhance the acceleration of Alternating Direction Method of Multipliers, as discussed by Poon and Liang [19]. Consider two subspaces $\mathcal{H}_1, \mathcal{H}_2 \subset \mathbb{R}^2$ such that their intersection is nonempty:

$$\mathcal{H}_1 \cap \mathcal{H}_2 \neq \emptyset.$$

A fundamental feasibility problem is to determine a point $\hat{x} \in \mathbb{R}^2$ satisfying:

$$\hat{x} \in \mathcal{C}_1 \cap \mathcal{C}_2. \tag{1.2}$$

It has been demonstrated in [19], Section 4, that for this feasibility problem, the two-step inertial fixed-point iteration

$$x_{n+1} = \mathcal{T}(x_n + \theta(x_n - x_{n-1}) + \delta(x_{n-1} - x_{n-2})),$$

where $\mathcal{T} := \frac{1}{2}\Big(I + (2P_{\mathcal{C}_1} - I)(2P_{\mathcal{C}_2} - I)\Big)$ and $P_{\mathcal{C}_1}$, $P_{\mathcal{C}_2}$ are the orthogonal projections onto $\mathcal{C}_1$ and $\mathcal{C}_1$, converges more efficiently in terms of iteration count and computational cost than the one-step inertial fixed-point iteration:

$$x_{n+1} = \mathcal{T}(x_n + \theta(x_n - x_{n-1})). \tag{1.3}$$

Furthermore, empirical results show that sequences generated by the one-step inertial fixed-point iteration (1.3) tend to converge more slowly than their non-inertial counterparts. This suggests that the one-step inertial approach may fail to provide acceleration. However, as detailed by Liang [9], the implementation of multi-step inertial techniques, like the two-step inertial term $x_n + \theta(x_n - x_{n-1}) + \delta(x_{n-1} - x_{n-2})$ where $\theta > 0, \delta < 0$, could mitigate this limitation and contribute to improved acceleration.

Now, Let $\{\mathcal{T}_n\}$ be a family of nonexpansive mappings on $\mathcal{C}$. We define

$$\Omega = \{\mathcal{T}_n : n \in \mathbb{N}\}$$

as the family of such mappings. Then, $F(\Omega)$ represents the set of all common fixed points of Ω, that is,

$$F(\Omega) = \bigcap_{n=1}^{\infty} F(\mathcal{T}_n).$$

For a bounded sequence $\{x_n\} \subset \mathcal{C}$, we define $\omega_w(x_n)$ as the set of all weak-cluster points of $\{x_n\}$. A sequence $\{\mathcal{T}_n\}$ is said to satisfy the NST-condition (I) (proposed by Nakajo, Shimoji, and Takahashi [16]) with respect to Ω if, for every bounded sequence $\{x_n\} \subset \mathcal{C}$, the following implication holds:

$$\lim_{n\to+\infty} \|x_n - \mathcal{T}_n x_n\| = 0 \quad \Rightarrow \quad \lim_{n\to+\infty} \|x_n - \mathcal{T} x_n\| = 0, \quad \forall \mathcal{T} \in \Omega.$$

If Ω is a singleton, i.e., $\Omega = \{\mathcal{T}\}$, then the sequence $\{\mathcal{T}_n\}$ satisfies the NST-condition (I) with respect to $\mathcal{T}$. This condition plays a crucial role in convergence analysis, particularly in ensuring the stability and robustness of iterative algorithms for solving fixed-point problems and convex optimization tasks.

Motivated by recent progress in fixed-point theory and optimization, this study aims to develop an improved iterative algorithm that enhances convergence and reduces computational cost. The objectives are twofold:

(1) to refine existing methods by integrating inertial techniques and parallel processing under weaker assumptions than the NST-condition; and
(2) to demonstrate the algorithm's practical utility through applications in blind image processing.

2. Preliminaries

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{C}$ be a subset of $\mathcal{H}$. Let $\{\mathcal{T}_n\}$ and τ be two families of mappings of C into itself such that $F(\tau)$, $\bigcap_{n=1}^{\infty} F(\mathcal{T}_n)$ are nonempty sets, where $F(\mathcal{T}_n)$ is the set of all fixed points of $\mathcal{T}_n$, $F(\tau)$ is the set of all common fixed points of τ. $\{\mathcal{T}_n\}$ is said to satisfy the $NS\mathcal{T}_w$-condition with respect to τ if for each bounded sequence $\{x_n\}$ in $\mathcal{C}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty} \|x_n - \mathcal{T}_n x_n\| = 0 \quad \Rightarrow \quad \lim_{k\to\infty} \|x_{n_j} - \mathcal{T} x_{n_j}\| = 0, \quad \forall \mathcal{T} \in \tau.$$

It is obvious that if $\{\mathcal{T}_n\}$ satisfies the NST-condition with respect to τ, then $\{\mathcal{T}_n\}$ satisfies the $NS\mathcal{T}_w$-condition with respect to τ. The state of $NS\mathcal{T}_w$-condition has been proposed in [4]

Lemma 2.1 (Opial [17]). *Let $\mathcal{H}$ be a Hilbert space and $\{x_n\}$ be a sequence in $\mathcal{H}$. Suppose that there exists a nonempty subset $\Gamma \subset \mathcal{H}$ satisfying the following conditions:*

(i) *For every $p \in \Gamma$, the limit $\lim\limits_{n\to+\infty} \|x_n - p\|$ exists;*

(ii) *Every weak-cluster point of the sequence $\{x_n\}$ belongs to Γ.*

Then, there exists $x^* \in \Gamma$ such that the sequence $\{x_n\}$ weakly converges to x^*.

Lemma 2.2 ([7]). *Let $T : \mathcal{C} \to \mathcal{C}$ be a nonexpansive mapping on a closed and convex subset $\mathcal{C}$ of a Hilbert space $\mathcal{H}$. If T has a fixed point, then $I - \mathcal{T}$ is demiclosed that is, whenever a sequence $\{x_n\}$ in $\mathcal{C}$ weakly converges to $x \in \mathcal{C}$ and the sequence $\{x_n - \mathcal{T}x_n\}$ strongly converges to y, it follows that $x - \mathcal{T}x = y$.*

Lemma 2.3 ([1]). *Let $\{\gamma_n\}$, $\{\delta_n\}$, and $\{\alpha_n\}$ be the sequences in $[0, +\infty)$ such that*

$$\gamma_{n+1} \le \gamma_n + \alpha_n(\gamma_n - \gamma_{n-1}) + \delta_n \quad \text{for all } n \ge 1,$$

$\sum_{n=1}^{\infty} \delta_n < +\infty$ and there exists a real number α with $0 \le \alpha_n \le \alpha < 1$ for all $n \ge 1$. Then the followings hold:

(i) *$\sum_{n\ge1} [\gamma_n - \gamma_{n-1}]_+ < +\infty$, where $[t]_+ = \max\{t, 0\}$;*

(ii) *There exists $\gamma^* \in [0, +\infty)$ such that $\lim_{n\to+\infty} \gamma_n = \gamma^*$.*

3. Main theorem

In this section, we present a double inertial parallel Mann algorithm and establish its weak convergence for approximating a common fixed point of a countable family of quasi-nonexpansive mappings. The iterative scheme is constructed as follows:

Algorithm 3.1.

Initialization : *Let $x_{-1}, x_0, x_1 \in \mathcal{H}$ be arbitrary, θ_n, $\delta_n \in (-\infty, \infty)$, $\alpha_{i,n} \in (0, 1)$ for all $i = 1, 2, ..., N$, with $\sum_{i=0}^{N} \alpha_{i,n} = 1$ for all $n \ge 1$.*

Step 1 *For $k = 1$, compute*

$$y_n = x_n + \theta_n(x_n - x_{n-1}) + \delta_n(x_{n-1} - x_{n-2}).$$

Step 2

$$z_{i,n} = (1 - \alpha_{i,n})y_n + \alpha_{i,n}\mathcal{T}_{i,n}y_n.$$

Step 3

$$x_{n+1} = \arg\max \|z_{i,n} - x_n\|.$$

*Then, update $k = k+1$ in **Step 1.***

Theorem 3.1. *Let $\mathcal{H}$ be a real Hilbert space. Let $\{\mathcal{T}_{i,n}\}_{i=1}^{N}$ and τ be families of quasi-nonexpansive mappings such that $\{\mathcal{T}_{i,n}\}$ satisfies $NS\mathcal{T}_w$-condition with respect to τ for all $i = 1, 2, \ldots, N$. Suppose that $F(\tau)$, $\bigcap_{n=1}^{\infty}\bigcap_{i=1}^{N} F(\mathcal{T}_{i,n})$ are nonempty sets. Let $\{x_n\}$ be defined by Algorithm 1. Assume that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} |\theta_n|\|x_n - x_{n-2}\| < \infty$;
(ii) $\sum_{n=1}^{\infty} |\delta_n|\|x_{n-1} - x_{n-2}\| < \infty$;
(iii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(iv) *$I - \mathcal{T}$ is demiclosed at 0 for all $\mathcal{T} \in \tau$.*

Then, $\{x_n\}$ converges weakly to a point in $F(\tau)$.

Proof. Let $q \in \bigcap_{n=1}^{\infty}\bigcap_{i=1}^{N} F(\mathcal{T}_{i,n})$. Then, this leads to the following bound:

$$\begin{aligned}\|y_n - q\| &= \|x_n + \theta_n(x_{n-1} - x_n) + \delta_n(x_{n-1} - x_{n-2}) - q\| \\ &\leq \|x_n - q\| + \theta_n\|x_{n-1} - x_n\| + \delta_n\|x_{n-1} - x_{n-2}\|.\end{aligned}$$

Additionally, we obtain

$$\begin{aligned}\|z_{i,n} - q\| &= \|(1-\alpha_{i,n})y_n + \alpha_{i,n}\mathcal{T}_{i,n}y_n - q\| \\ &= \|(1-\alpha_{i,n})(y_n - q) + \alpha_{i,n}(\mathcal{T}_{i,n}y_n - q)\| \\ &\leq (1-\alpha_{i,n})\|y_n - q\| + \alpha_{i,n}\|\mathcal{T}_{i,n}y_n - q\| \\ &\leq \|y_n - q\| \\ &\leq \|x_n - q\| + \theta_n\|x_{n-1} - x_n\| + \delta_n\|x_{n-1} - x_{n-2}\|\end{aligned} \tag{3.1}$$

which implies that

$$\|x_{n+1} - q\| \leq \|x_n - q\| + \theta_n\|x_n - x_{n-1}\| + \delta_n\|x_{n-1} - x_{n-2}\|.$$

Applying Lemma 2.3 and assumptions (i) and (ii), we deduce that $\lim_{n\to\infty}\|x_n - q\|$ exists. Consequently, the sequence $\{x_n\}$ is bounded, so are $\{z_{i,n}\}$ and $\{y_n\}$.
Next, consider the following:

$$\begin{aligned}\|z_{i,n} - q\|^2 &= \|(1-\alpha_{i,n})y_n + \alpha_{i,n}\mathcal{T}_{i,n}y_n - q\|^2 \\ &\leq (1-\alpha_{i,n})\|y_n - q\|^2 + \alpha_{i,n}\|\mathcal{T}_{i,n}y_n - q\|^2 \\ &\quad - \alpha_{i,n}(1-\alpha_{i,n})\|\mathcal{T}_{i,n}y_n - y_n\| \\ &\leq \|y_n - q\|^2 - \alpha_{i,n}(1-\alpha_{i,n})\|\mathcal{T}_{i,n}y_n - y_n\| \\ &\leq \|x_n - q\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + \delta_n^2\|x_{n-1} - x_{n-2}\|^2 \\ &\quad + 2\theta_n\|x_n - q\|\|x_n - x_{n-1}\| \\ &\quad + 2\delta_n\|x_n - q\|\|x_{n-1} - x_{n-2}\| \\ &\quad + 2\theta_n\delta_n\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\| \\ &\quad - \alpha_{i,n}(1-\alpha_{i,n})\|\mathcal{T}_{i,n}y_n - y_n\|^2\end{aligned} \tag{3.2}$$

which guarantees the existence of $i_n \in \{1, 2, 3, \ldots, N\}$ such that:

$$\begin{aligned}\alpha_{n,i_n}(1-\alpha_{n,i_n})\|\mathcal{T}_{n,i_n}y_n - y_n\|^2 \le{}& \|x_n - q\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 \\ &+ \delta_n^2\|x_{n-1} - x_{n-2}\|^2 \\ &+ 2\theta_n\|x_n - q\|\|x_n - x_{n-1}\| \\ &+ 2\delta_n\|x_n - q\|\|x_{n-1} - x_{n-2}\| \\ &+ 2\theta_n\delta_n\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\| \\ &- \|x_{n+1} - q\|^2.\end{aligned} \tag{3.3}$$

Applying assumptions (i)-(iii) and equation (3.3), together with the fact that $\lim_{n\to\infty}\|x_n - q\|$ exists, we obtain

$$\lim_{n\to\infty}\|\mathcal{T}_{n,i_n}y_n - y_n\| = 0. \tag{3.4}$$

Using this result, we also derive

$$\|x_{n+1} - y_n\| = \lambda_n\|y_n - \mathcal{T}_{n,i_n}y_n\|. \tag{3.5}$$

By the definition of y_n, we obtain

$$\|x_n - y_n\| \le |\theta_n|\|x_n - x_{n-1}\| + |\delta_n|\|x_{n-1} - x_{n-2}\| \to 0. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain

$$\|x_{n+1} - x_n\| \le \|x_{n+1} - y_n\| + \|x_n - y_n\| \to 0. \tag{3.7}$$

This implies that

$$\|z_{i,n} - x_n\| \le \|x_{n+1} - x_n\| \to 0. \tag{3.8}$$

As $n \to \infty$, for all $i = 1, 2, \ldots, N$, equation (3.6) and (3.8) give

$$\alpha_{i,n}(1-\alpha_{i,n})\|\mathcal{T}_{i,n}y_n - y_n\|^2 \le \|y_n - q\|^2 - \|z_{i,n} - q\|^2 \tag{3.9}$$

which, together with assumptions (i)-(iii) and equations (3.8) and (3.9), lead to

$$\lim_{n\to\infty}\|\mathcal{T}_{i,n}y_n - y_n\| = 0 \tag{3.10}$$

for all $i = 1, 2, \ldots, N$.

Since $\{\mathcal{T}_{i,n}\}$ satisfies the $NS\mathcal{T}_w$-condition with respect to τ, there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\lim_{j\to\infty}\|y_{n_j} - \mathcal{T}y_{n_j}\| = 0 \quad \text{for all } \mathcal{T} \in \tau. \tag{3.11}$$

Since $\{y_{n_j}\}$ is bounded, there exists a subsequence $\{y_{n_{j_k}}\}$ that converges weakly to a point $q^* \in \mathcal{H}$. By assumption (iv), $I - \mathcal{T}$ is demiclosed at zero for all $\mathcal{T} \in \tau$, and using (3.11), it follows that $q^* \in F(\tau)$. Then, by Opial's property in Lemma 2.1 and equation (3.6), we deduce that $\{x_n\}$ converges weakly to an element of $F(\tau)$. □

4. Applications

In this section, we present a recent technique to restore image without blind deconvolution technique. The experiments were conducted by MATLAB R2024A.

Motion blur is a significant challenge in Close Circuit Television (CCTV) surveillance and low-quality imaging, caused by the relative motion between the camera and the subject. This results in streaked images, making it difficult to identify individuals, track objects, or analyze security footage effectively. The issue worsens in low-light conditions, where longer exposure times further degrade image clarity. Unlike other blur types, motion blur creates directional distortions, particularly affecting edges and fine details. As a result, the problem of image restoration has become increasingly important, especially in applications related to security, forensic analysis, and automated surveillance.

The image recovery problem can be thought of as a linear equation given by:

$$b = Dx + v, \tag{4.1}$$

where an original image is denoted as $x \in \mathbb{R}^N$, a degraded image is $b \in \mathbb{R}^M$, a noise term is given by $v \in \mathbb{R}^M$, and $D \in \mathbb{R}^{M\times N}$ is a blurring matrix. It is worth noting that the problem (4.1) can be formulated equivalently as the subsequent convex minimization model:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|b - Dx\|_2^2$$

Recently, several researchers [5, 8, 22, 25] have attempted solutions for image restoration without knowing the specifics of the blurring matrix. This attempt has resulted in the following minimization problem:

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|D_1x - b_1\|_2^2, \min_{x\in\mathbb{R}^n} \frac{1}{2}\|D_2x - b_2\|_2^2, ..., \min_{x\in\mathbb{R}^n} \frac{1}{2}\|D_Nx - b_N\|_2^2 \tag{4.2}$$

where x represents the original image, D_i is the blurring matrix, and b_i is the blurred image resulting from the application of the blurring matrix D_i for all $i = 1, 2, \ldots, N$. However, this technique (4.2) has not yet been applied in practice. Therefore, we define the problem as follows:

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|D_1x - b\|_2^2, \min_{x\in\mathbb{R}^n} \frac{1}{2}\|D_2x - b\|_2^2, ..., \min_{x\in\mathbb{R}^n} \frac{1}{2}\|D_Nx - b\|_2^2. \tag{4.3}$$

D is a randomly selected blurring operator to solve the problem without prior knowledge of the true blurring operator. This approach aims to discover a more effective solution for recovering images in the real world.

To solve the minimization problem of the sum $f + g$, we define the operator $\mathcal{T} := \mathrm{prox}_{\lambda g}(I - \lambda\nabla f)$. Under the assumptions that $f, g : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ are proper, convex, and lower semicontinuous, with ∇f being L-Lipschitz continuous and $\arg\min(f + g) \neq \emptyset$. It follows that $\mathcal{T}$ is quasi-nonexpansive [21] and $I - \mathcal{T}$ is demiclosed at zero [4]. The proposed algorithm can be applied to solve problem (4.3) by setting

$$f_i(x) = \frac{1}{2}\|D_ix - b\|_2^2, \quad g(x) = 0, \quad \text{and} \quad \lambda_i \in \left(0, \frac{2}{\|D_i\|_2^2}\right).$$

To support our main theorem, the parameters θ_k and δ_k are defined as follows:

$$\theta_k = \begin{cases} \frac{1}{k^2\|x_{k+1}-x_k\|}, & \text{if } x_{k+1} \neq x_k,\ k > N \\ \theta & \text{otherwise} \end{cases} \tag{4.4}$$

and

$$\delta_k = \begin{cases} \frac{1}{k^2\|x_{k+1}-x_k\|}, & \text{if } x_{k+1} \neq x_k,\ k > N \\ \delta & \text{otherwise.} \end{cases} \tag{4.5}$$

The parameters of our Algorithm are defined by $\alpha_{1,k} = \alpha_{2,k} = ... = \alpha_{10,k} = 0.9$, $\lambda_i = \dfrac{1.99}{\|D_i\|^2}$ for $i = 1, 2, 3, ..., 10$, and θ_k, δ_k are defined by the same as (4.4) and (4.5) such that $\theta = 0.9$ and $\sigma = 0.1$, when the initial points x_0, x_1, x_2 are blurred image.

We set 10 matrix blurs D_i for a group of motion blur as follow:

$$\begin{aligned} D_1 &= (L, \Theta), & D_6 &= (L+1, \Theta-1), \\ D_2 &= (L+1, \Theta), & D_7 &= (L+2, \Theta-1), \\ D_3 &= (L+2, \Theta), & D_8 &= (L, \Theta+1), \\ D_4 &= (L, \Theta), & D_9 &= (L+1, \Theta+1), \\ D_5 &= (L, \Theta-1), & D_{10} &= (L+2, \Theta+1), \end{aligned}$$

Where L is the length and Θ is the angle of motion blur. We set $\Theta = 180°$ because the vehicle is moving horizontally on the road.

We consider 4 cases.

- Case I : set $L = 26$;
- Case II : set $L = 28$;
- Case III : set $L = 30$;
- Case IV : set $L = 32$.

We begin with a degraded image that has been corrupted by motion blur. To recover the original image, we apply the proposed algorithm using a family of motion blur matrices with varying blur lengths L. The restored images after 15,000 iterations for different values of L are presented in Figure 1. Furthermore, to verify the convergence of the algorithm, we present the Cauchy error, measured by the condition $\|x_{n+1} - x_n\| < 10^{-4}$, as shown in Figure 2.

(A) original image (B) L=26, 7000 iterations (C) L=26, 15000 iterations

(D) original image (E) L=28, 7000 iterations (F) L=28, 15000 iterations

(G) original image (H) L=30, 7000 iterations (I) L=30, 15000 iterations

(J) original image (K) L=26, 7000 iterations (L) L=32, 15000 iterations

FIGURE 1. Images restored using different motion blur lengths L and iteration.

From Figure 2, it is observed that the Cauchy error for all values of L converges within the range (0, 1), indicating that the process converges to a certain solution. Furthermore, from Figure 1, it is shown that the images restored with 15,000 iterations at $L = 30$ and $L = 32$ appear significantly clearer compared to the original image. The details are well-preserved, making it possible to distinguish individuals in the images. In contrast, at $L = 26$, the restoration quality is noticeably lower

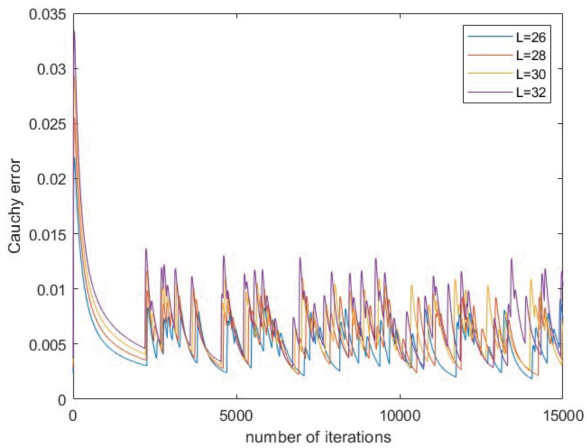

FIGURE 2. Cauchy error

compared to $L = 30$ and $L = 32$. Based on visual assessment, the image at $L = 30$ appears to be the sharpest.

5. Conclusion

In this study, we investigate the fixed-point problem using the inertial parallel modified Mann algorithm. We establish the weak convergence of the algorithm under appropriate parameter conditions. To evaluate its practical performance, we apply the algorithm to image recovery without requiring prior knowledge of the blurring process. Experimental results demonstrate that the proposed algorithm effectively restores images. One key advantage of image recovery, particularly for moving vehicles, is enhancing visual clarity and identifying individuals or objects within the scene. This can be crucial in various applications such as traffic surveillance, accident analysis, and law enforcement. By improving the quality of blurred images, the algorithm aids in recognizing vehicle occupants, license plates, and surrounding details, which can be essential for security and forensic investigations.

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