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CONVERGENCE AND VISUAL ANALYSIS OF THE PICARD-D HYBRID ITERATIVE PROCESS

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Abstract. This paper aims to introduce and study the Picard-D hybrid iterative method (PDHM for short). We also compare the convergence rate between the proposed iteration and some other processes in the literature via a numerical example. Specifically, our main result shows that PDHM converges faster than the Noor and D iterations in the sense of Berinde. Further, we will establish a stable result for our newly developed iterative process. As an application, we apply the proposed method to the visualization of polynomiographs.

1. Introduction and preliminaries

Banach [4] outlined a very basic idea of contraction mapping and proved the well known Banach contraction principle. This result is the basis of fixed point theory, which guarantees not only the fixed point of contraction mapping but also the uniqueness of the fixed point. Browder [7], Gohde [12], and Kirk [18] extended the idea of Banach and introduced new research dimensions in the field of fixed point theory.

Throughout this paper, we will denote C to be a nonempty closed convex subset of a real Banach space X. We will denote the set of fixed points of the operator T: $C \to C$ by $F(T) = \{x \in C : Tx = x\}$. An operator T on C is contraction if, for each $x, y \in C$ and $\theta \in (0,1)$, $\|Tx - Ty\| \leq \theta\|x - y\|$. The fixed point iterative procedure is one of the techniques employed to solve nonlinear equations. Over time, researchers have delved into various iterations to discover fixed points for different equations, aiming to identify the most efficient method to reach these fixed points. In parallel, extensive studies have addressed fixed points of different classes and their generalizations to nonlinear maps [8, 17, 19, 27, 28, 32, 33, 35].

In 2000, Noor [22] introduced a three-step iterative scheme (also known as the Noor Iteration). This scheme extends the results of Banach [4], Mann [21], and Ishikawa [14]. The scheme is defined as follows: $t_1 \in C$,

$$\begin{aligned} v_n &= (1-\gamma_n)t_n + \gamma_n T t_n, \\ u_n &= (1-\beta_n)t_n + \beta_n T v_n, \\ t_{n+1} &= (1-\alpha_n)t_n + \alpha_n T u_n,\ n \geq 1, \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\} \in [0,1)$.

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In 2018, Daengsaen and Khempet [9] introduced the following new three-step iteration process called D-iteration, which generates the sequence $\{u_i\}$ given as: $u_1 \in C$,

$$\begin{aligned} w_n &= (1-\gamma_n)u_n + \gamma_n T u_n, \\ v_n &= (1-\beta_n)Tu_n + \beta_n T w_n, \\ u_{n+1} &= (1-\alpha_n)Tw_n + \alpha_n T v_n, \ n \geq 1. \end{aligned} \tag{1.2}$$

Inspired and motivated by these facts, we introduce a new idea of a hybrid fixed point iterative method, named Picard-D hybrid iterative method (PDHM for short). This new iterative process can be seen as a hybrid of Picard and D iterative processes. The scheme is defined as follow: $x_1 \in C$,

$$\begin{aligned} w_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \\ z_n &= (1-\beta_n)Tx_n + \beta_n T w_n, \\ y_n &= (1-\alpha_n)Tw_n + \alpha_n T z_n, \\ x_{n+1} &= T y_n, \ n \geq 1, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are in (0,1).

Let now state some definitions and lemmas that will be useful in the coming theories.

Lemma 1.1 ([1]). *Let $\{x_n\}$ be a sequence of positive real numbers which satisfies $x_{n+1} \leq (1-\mu_n)x_n$. If $\{\mu_n\} \subset (0,1)$ and $\sum_{n=1}^{\infty} \mu_n = \infty$, then $\lim_{n\to\infty} x_n = 0$.*

Lemma 1.2 ([36]). *Let $\{a_n\}$ and $\{b_n\}$ be non-negative real sequences satisfying the following inequality. $a_{n+1} \leq (1-c_n)a_n + b_n$, where $c_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum_{n=0}^{\infty} c_n = \infty$ and $\frac{b_n}{c_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Let $\{x_n\}$ and $\{y_n\}$ be two fixed point iteration processes that converge to a fixed point p of a given operator T. The sequence $\{x_n\}$ is better than $\{y_n\}$ in the sense of Rhoades [29] if $\|x_n - p\| \leq \|y_n - p\|$ for all $n \in \mathbb{N}$. The definitions presented by Berinde [6] are as follows:

Definition 1.3 ([6]). Let $\{u_n\}$ and $\{v_n\}$ be two sequences of real numbers converging to u and v, respectively. The sequence $\{u_n\}$ is said to converge faster than $\{v_n\}$ if $\lim_{n\to\infty} \left|\frac{u_n - u}{v_n - v}\right| = 0$.

Definition 1.4 ([6]). Let $\{x_n\}$ and $\{y_n\}$ be two fixed point iteration processes that converge to a certain fixed point p of a given operator T. Suppose that the error estimates $\|x_n - p\| \leq u_n$ for all $n \in \mathbb{N}$, $\|y_n - p\| \leq v_n$ for all $n \in \mathbb{N}$, are available, where $\{u_n\}$ and $\{v_n\}$ are two sequences of positive numbers converging to zero. If $\{u_n\}$ converges faster than $\{v_n\}$, then $\{x_n\}$ converges faster than $\{y_n\}$ to p.

Definition 1.5 ([6]). Let $T, \widetilde{T} : C \to C$ be two operators. We say that $\widetilde{T}$ is an approximate operator for T if, for a fixed $\epsilon > 0$ we have $\left\|Tx - \widetilde{T}\widetilde{x}\right\| \leq \epsilon$.

After the advent of computational mathematics, the iterative aspects of fixed point theory gained unprecedented attention. Following the discussion above, mathematicians recognized the importance of assessing the stability of methods used to approximate fixed points of operators before applying them. In 1967, Ostrowski [25] introduced the pioneering result on T-stability. Following Ostrowski's pioneering result, subsequent researchers made significant contributions to the study of stability. Notably, Harder and Hicks [13] in 1988 and Rhoades [30, 31]. In addition to Ostrowski's work, other notable contributions on stability came from Osilike [23] in 1995, Osilike and Udemene [24] in 1999, and Berinde [5] in 2002. These researchers provided clear explanations of stability concepts and introduced simpler approaches compared to Harder and Hicks [13]. The following definition is credited to Harder and Hicks [13].

Definition 1.6 ([5])**.** Let X be a Banach space and, $T : X \to X$ a self map, $x_0 \in X$ and the iteration procedure defined by

$$x_{n+1} = f(T, x_n),\ n = 0, 1, 2, \dots \tag{1.4}$$

such that the generated sequence $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}$ be an arbitrary sequence in X and the set $\epsilon_n = \|y_{n+1} - f(T, y_n)\|$ for $n = 0, 1, 2, \dots$, then the iteration process (1.4) is said to be T-stable or stable with respect to T if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$.

2. Convergence analysis

In this section, we are now ready to prove the theorem of strong convergence of a Picard-D hybrid iterative method (PDHM) to a fixed point for a contraction mapping in a Banach space. We will also show that PDHM is stable. And finally, we shall prove that PDHM gives the faster rate of convergence than the earlier existing schemes. In addition, we also present an example using MATLAB programing. We have also given a graphical representation for this.

Theorem 2.1. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping. Let $\{x_n\}$ be the sequence generated by (1.3) with real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ satisfying $\sum_{n=1}^{\infty} \alpha_n\beta_n\gamma_n = \infty$. Then $\{x_n\}$ converges strongly to a unique fixed point of T.*

Proof. We know that a unique $p \in F(T)$ exists (by Banach contraction theorem). We will prove that $x_n \to p$ as $n \to \infty$. Using (1.3) we have

$$\begin{aligned}
\|w_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n T x_n - p\| \\
&\le (1-\gamma_n)\|x_n - p\| + \gamma_n\|Tx_n - Tp\| \\
&\le (1-\gamma_n)\|x_n - p\| + \gamma_n\theta\|x_n - p\| \\
&= (1-\gamma_n(1-\theta))\|x_n - p\|.
\end{aligned} \tag{2.1}$$

Using (2.1), we have

$$\begin{aligned}
\|z_n - p\| &= \|(1-\beta_n)Tx_n + \beta_n T w_n - p\| \\
&\le (1-\beta_n)\|Tx_n - p\| + \beta_n\|Tw_n - Tp\| \\
&\le (1-\beta_n)\theta\|x_n - p\| + \beta_n\theta\|w_n - p\| \\
&\le (1-\beta_n)\|x_n - p\| + \beta_n\theta(1-\gamma_n(1-\theta))\|x_n - p\| \\
&\le (1-\beta_n)\|x_n - p\| + \beta_n(1-\gamma_n(1-\theta))\|x_n - p\| \qquad (2.2)
\end{aligned}$$

$$= (1 - \beta_n + \beta_n(1 - \gamma_n(1 - \theta))\|x_n - p\|$$
$$= (1 - \beta_n\gamma_n(1 - \theta))\|x_n - p\|.$$

In addition, using (2.1) and $\gamma_n(1-\theta) > 0$, we have $\|w_n - p\| \leq \|x_n - p\|$. From (2.1), (2.2) and $\|w_n - p\| \leq \|x_n - p\|$, we obtain

$$\begin{aligned}
\|y_n - p\| &= \|(1 - \alpha_n)Tw_n + \alpha_n Tz_n - p\| \\
&\leq (1 - \alpha_n)\|Tw_n - Tp\| + \alpha_n\|Tz_n - Tp\| \\
&\leq (1 - \alpha_n)\theta\|w_n - p\| + \alpha_n\theta\|z_n - p\| \\
&\leq (1 - \alpha_n)\|w_n - p\| + \alpha_n\theta\|z_n - p\| \qquad (2.3) \\
&\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\theta(1 - \beta_n\gamma_n(1 - \theta))\|x_n - p\| \\
&\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n(1 - \beta_n\gamma_n(1 - \theta))\|x_n - p\| \\
&\leq (1 - \alpha_n\beta_n\gamma_n(1 - \theta))\|x_n - p\|.
\end{aligned}$$

By (2.3), which implies that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|Ty_n - p\| \\
&\leq \theta\|y_n - p\| \qquad (2.4) \\
&\leq (1 - \alpha_n\beta_n\gamma_n(1 - \theta))\|x_n - p\|.
\end{aligned}$$

Let $\mu_n = \alpha_n\beta_n\gamma_n(1-\theta)$. We observed that $\mu_n < 1$. Since $\sum_{n=1}^{\infty}\alpha_n\beta_n\gamma_n = \infty$, using Lemma 1.1, we obtain $\lim_{n\to\infty}\|x_n - p\| = 0$. So, $x_n \to p$ as $n \to \infty$. This completes the proof. □

Now, we prove the stability of our iteration process (1.3).

Theorem 2.2. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping. Let $\{x_n\}$ be the sequence generated by (1.3) with real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ satisfying $\sum_{n=1}^{\infty}\gamma_n = \infty$. Then the iterative process (1.3) is T-stable.*

Proof. Suppose (1.3) generate the sequence $x_{n+1} = f(T, x_n)$ which converge to a unique $x^* \in F(T)$ (by Theorem 2.1). Let $\{t_n\}$ be any sequence in C and $\epsilon_n = \|t_{n+1} - f(T, t_n)\|$. We will show that $\lim_{n\to\infty}\epsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} t_n = x^*$.
Let $\lim_{n\to\infty}\epsilon_n = 0$. By using (2.4) we get

$$\begin{aligned}
\|t_{n+1} - x^*\| &\leq \|t_{n+1} - f(T, t_n)\| + \|f(T, t_n) - x^*\| \\
&\leq \epsilon_n + (1 - \alpha_n\beta_n\gamma_n(1 - \theta))\|t_n - x^*\|.
\end{aligned}$$

Define $a_n = \|t_n - x^*\|$, $c_n = \alpha_n\beta_n\gamma_n(1 - \theta) \in (0, 1)$ and $b_n = \epsilon_n$, $\forall n \in \mathbb{N}$ which implies that $\frac{b_n}{c_n} \to 0$ as $n \to \infty$. Thus all the conditions of Lemma 1.2 we get $\lim_{n\to\infty} t_n = x^*$. Conversely, let $\lim_{n\to\infty} t_n = x^*$, we have

$$\begin{aligned}
\epsilon_n &= \|t_{n+1} - f(T, t_n)\| \\
&\leq \|t_{n+1} - x^*\| + \|f(T, t_n) - x^*\| \\
&\leq \|t_{n+1} - x^*\| + (1 - \alpha_n\beta_n\gamma_n(1 - \theta))\|t_n - x^*\|.
\end{aligned}$$

This implies that $\lim_{n\to\infty}\epsilon_n = 0$. Hence, (1.3) is T-stable. The proof is completed. □

In the remainder of this section, we prove that (1.3) converges faster than (1.1) and (1.2) in Berinde's sense.

Theorem 2.3. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping. Suppose that each of the iterative processes (1.1), (1.2) and (1.3) converge to the same fixed point p of T, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha \le \alpha_n < 1$, $\beta \le \beta_n < 1$ and $\gamma \le \gamma_n < 1$ for some $\alpha, \beta, \gamma > 0$ and for all $n \in \mathbb{N}$. Then the Picard-D hybrid iterative process (1.3) converges faster than all the other two iterative processes.*

Proof. Suppose that p is the fixed point of T. By using (1.3), we have

$$\|x_{n+1} - p\| = \|Ty_n - p\| \le \theta\|y_n - p\| \tag{2.5}$$

and

$$\begin{aligned} \|w_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n Tx_n - p\| \\ &\le (1-\gamma_n)\|x_n - p\| + \gamma_n\|Tx_n - Tp\| \\ &\le (1-\gamma_n)\|x_n - p\| + \gamma_n\theta\|x_n - p\| \\ &= (1-\gamma_n(1-\theta))\|x_n - p\|. \end{aligned} \tag{2.6}$$

In addition, using (2.6) and $\gamma_n(1-\theta) > 0$, we have $\|w_n - p\| \le \|x_n - p\|$. Moreover, from (2.6), we have

$$\begin{aligned} \|z_n - p\| &= \|(1-\beta_n)Tx_n + \beta_n Tw_n - p\| \\ &\le (1-\beta_n)\|Tx_n - Tp\| + \beta_n\|Tw_n - Tp\| \\ &\le (1-\beta_n)\theta\|x_n - p\| + \beta_n\theta\|w_n - p\| \\ &\le (1-\beta_n)\theta\|x_n - p\| + \beta_n\theta\left[(1-\gamma_n(1-\theta))\|x_n - p\|\right] \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\theta\left[(1-\gamma_n(1-\theta))\|x_n - p\|\right] \\ &= \big[1-\beta_n+\beta_n\theta(1-\gamma_n(1-\theta))\big]\|x_n - p\| \\ &= \big[1-\beta_n(1-\theta)(1+\gamma_n\theta)\big]\|x_n - p\|. \end{aligned} \tag{2.7}$$

Moreover, from (2.7) and $\|w_n - p\| \le \|x_n - p\|$, we have

$$\begin{aligned} \|y_n - p\| &= \|(1-\alpha_n)Tw_n + \alpha_n Tz_n - p\| \\ &\le (1-\alpha_n)\|Tw_n - Tp\| + \alpha_n\|Tz_n - Tp\| \\ &\le (1-\alpha_n)\theta\|w_n - p\| + \alpha_n\theta\|z_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\theta\left[1-\beta_n(1-\theta)(1+\gamma_n\theta)\right]\|x_n - p\| \\ &= \big[1-\alpha_n+\alpha_n\theta(1-\beta_n(1-\theta)(1+\gamma_n\theta))\big]\|x_n - p\| \\ &= \big[1-\alpha_n(1-\theta)(1+\beta_n\theta(1+\gamma_n\theta)\big]\|x_n - p\|. \end{aligned} \tag{2.8}$$

It follows from (2.5) and (2.8) that

$$\begin{aligned} \|x_{n+1} - p\| &= \|Ty_n - p\| \\ &\le \theta\|y_n - p\| \\ &\le \theta\big[1-\alpha_n(1-\theta)(1+\beta_n\theta(1+\gamma_n\theta)\big]\|x_n - p\| \\ &\le \big[1-\alpha_n(1-\theta)(1+\beta_n\theta(1+\gamma_n\theta)\big]\|x_n - p\| \end{aligned}$$

$$\leq \big[1-\alpha(1-\theta)(1+\beta\theta(1+\gamma\theta)\big]^n\|x_1-p\|.$$

Let $a_n = \big[1-\alpha(1-\theta)(1+\beta\theta(1+\gamma\theta)\big]^n\|x_1-p\|$. Using (1.1), we have

$$\begin{aligned}\|t_{n+1}-p\| &= \|(1-\alpha_n)t_n-\alpha_n Tu_n-p\|\\ &\leq (1-\alpha_n)\|t_n-p\|+\alpha_n\|Tu_n-Tp\|\\ &\leq (1-\alpha_n)\|t_n-p\|+\alpha_n\theta\|u_n-p\|\\ &\leq (1-\alpha_n)\|t_n-p\|+\alpha_n\|u_n-p\|\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\|v_n-p\| &= \|(1-\gamma_n)t_n+\gamma_n Tt_n-p\|\\ &\leq (1-\gamma_n)\|t_n-p\|+\gamma_n\|Tt_n-Tp\|\\ &\leq (1-\gamma_n)\|t_n-p\|+\gamma_n\theta\|t_n-p\|\\ &= (1-\gamma_n(1-\theta))\|t_n-p\|.\end{aligned} \tag{2.10}$$

It follows from (2.10) that

$$\begin{aligned}\|u_n-p\| &= \|(1-\beta_n)t_n+\beta_n Tv_n-p\|\\ &\leq (1-\beta_n)\|t_n-p\|+\beta_n\|Tv_n-Tp\|\\ &\leq (1-\beta_n)\|t_n-p\|+\beta_n\theta\|v_n-p\|\\ &\leq (1-\beta_n)\|t_n-p\|+\beta_n\theta\big[(1-\gamma_n(1-\theta))\|t_n-p\|\big]\\ &\leq \big[1-\beta_n+\beta_n\theta(1-\gamma_n(1-\theta))\big]\|t_n-p\|\\ &= \big[1-\beta_n(1-\theta)(1+\gamma_n\theta)\big]\|t_n-p\|.\end{aligned} \tag{2.11}$$

Using (2.9) and (2.11), we have

$$\begin{aligned}\|t_{n+1}-p\| &\leq (1-\alpha_n)\|t_n-p\|+\alpha_n\big[1-\beta_n(1-\theta)(1+\gamma_n\theta)\big]\|t_n-p\|\\ &= \big[1-\alpha_n\beta_n(1-\theta)(1+\gamma_n\theta)\big]\|t_n-p\|\\ &\leq \big[1-\alpha\beta(1-\theta)(1+\gamma\theta)\big]^n\|t_1-p\|.\end{aligned}$$

Let $b_n = \big[1-\alpha\beta(1-\theta)(1+\gamma\theta)\big]^n\|t_1-p\|$. Hence

$$\frac{a_n}{b_n} = \frac{\big[1-\alpha(1-\theta)(1+\beta\theta(1+\gamma\theta)\big]^n\|x_1-p\|}{\big[1-\alpha\beta(1-\theta)(1+\gamma\theta)\big]^n\|t_1-p\|} \to 0 \text{ as } n\to\infty.$$

Therefore, the Picard-D hybrid iterative process (1.3) converges faster than the Noor iterative process (1.1).

Now, for the sequence $\{u_n\}$ generated by (1.2), we have the following

$$\begin{aligned}\|w_n-p\| &= \|(1-\gamma_n)u_n+\gamma_n Tu_n-p\|\\ &\leq (1-\gamma_n)\|u_n-p\|+\gamma_n\|Tu_n-Tp\|\\ &\leq (1-\gamma_n)\|u_n-p\|+\gamma_n\theta\|u_n-p\|\\ &= (1-\gamma_n(1-\theta))\|u_n-p\|.\end{aligned} \tag{2.12}$$

Using (2.12), we have

$$
\begin{aligned}
\|v_n - p\| &= \|(1-\beta_n)Tu_n + \beta_n Tw_n - p\| \\
&\le (1-\beta_n)\|Tu_n - Tp\| + \beta_n\|Tw_n - Tp\| \\
&\le (1-\beta_n)\theta\|u_n - p\| + \beta_n\theta\|w_n - p\| \\
&\le (1-\beta_n)\|u_n - p\| + \beta_n\theta\left[(1-\gamma_n(1-\theta))\|u_n - p\|\right] \\
&= \left[1-\beta_n+\beta_n\theta(1-\gamma_n(1-\theta))\right]\|u_n - p\| \\
&= \left[1-\beta_n(1-\theta)(1+\gamma_n\theta)\right]\|u_n - p\|.
\end{aligned}
\tag{2.13}
$$

In addition, using (2.12) and $\gamma_n(1-\theta) > 0$, we have $\|w_n - p\| \le \|u_n - p\|$. It follows from (2.13) that

$$
\begin{aligned}
\|u_{n+1} - p\| &= \|(1-\alpha_n)Tw_n - \alpha_n Tv_n - p\| \\
&\le (1-\alpha_n)\|Tw_n - Tp\| + \alpha_n\|Tv_n - Tp\| \\
&\le (1-\alpha_n)\theta\|w_n - p\| + \alpha_n\theta\|v_n - p\| \\
&\le (1-\alpha_n)\theta\|u_n - p\| + \alpha_n\theta\left[1-\beta_n(1-\theta)(1+\gamma_n\theta)\right]\|u_n - p\| \\
&\le (1-\alpha_n)\|u_n - p\| + \alpha_n\left[1-\beta_n(1-\theta)(1+\gamma_n\theta)\right]\|u_n - p\| \\
&= \left[1-\alpha_n\beta_n(1-\theta)(1+\gamma_n\theta)\right]\|u_n - p\| \\
&\le \left[1-\alpha\beta(1-\theta)(1+\gamma\theta)\right]^n\|u_1 - p\|.
\end{aligned}
$$

Let $c_n = \left[1-\alpha\beta(1-\theta)(1+\gamma\theta)\right]^n\|u_1 - p\|$. Thus

$$
\frac{a_n}{c_n} = \frac{\left[1-\alpha(1-\theta)(1+\beta\theta(1+\gamma\theta)\right]^n\|x_1 - p\|}{\left[1-\alpha\beta(1-\theta)(1+\gamma\theta)\right]^n\|u_1 - p\|} \to 0 \text{ as } n \to \infty.
$$

Hence $\{x_n\}$ converges faster than $\{u_n\}$ to p. That is, the Picard-D hybrid iterative process (1.3) converges faster than D iterative process (1.2). □

In order to demonstrate the improved performance of the proposed PDHM (1.3), we consider a numerical example in which we compare our method with the Noor (1.1) and D (1.2) iteration processes.

Example 2.4. Let $C = [1,7] \subseteq X = \mathbb{R}$ and $T : C \to C$ be defined by $Tx = \sqrt[3]{x+6}$ for all $x \in C$. Choose $\alpha_n = \beta_n = \gamma_n = 0.9$ for each $n \in \mathbb{N}$ with initial value $x_1 = 5$. Clearly, T is a contraction mapping and $F(T) = \{2\}$.

Table 1 shows that the PDHM process (1.3) converges faster than all of the D and Noor iterative processes, which found the fixed point in 6 iterations. The second best method is the D-iteration, which needed eight iterations. The worst convergence speed is observed for the Noor iteration, which required more than 8 iterations to find the fixed point.

TABLE 1. The comparison of the convergence rates of the Noor (1.1), D (1.2) and PDHM (1.3) iterative processes.

Step	PDHM	D	Noor
1	5.00000000000000	5.00000000000000	5.00000000000000
2	2.000710476295592	2.008528744565135	2.32491786957709
3	2.000000181624082	2.000026157236961	2.03524174735882
4	2.000000000046431	2.000000080242713	2.00382309861342
5	2.000000000000012	2.000000000246161	2.00041474551741
6	**2.0000000000000**	2.000000000000755	2.00004499339710
7		2.000000000000002	2.00000488108043
8		**2.0000000000000**	2.00000052952096

3. Visual analysis via polynomiography

Visual analysis is a common technique in the contemporary examination of root-finding methods, often utilized to evaluate the stability and convergence of these methods (see [26]). This approach allows for the observation of an area rather than just a single point, providing a broader perspective on the behaviour of the method across that area, which enhances our understanding of it. In the context of polynomials, this type of visual analysis is known as polynomiography, and the individual images produced are termed polynomiographs.

In 2000, the term "polynomiographs" was coined to describe the visuals derived from root-finding methods, with the overarching technique being polynomiography. Kalantari [15] first introduced these concepts (see also [20, 34, 37]). In polynomiography, the core component of the generation algorithm is the method used for finding roots. Numerous root-finding techniques are documented in scholarly literature. Let's review some of these methods for a complex polynomial p. The Newton method [16] $N(z) = z - \frac{p(z)}{p'(z)}$. The Halley method [16] $H(z) = z - \frac{2p'(z)p(z)}{2p'(z)^2 - p''(z)p(z)}$. The B_4 method (the fourth element of the Basic Family introduced by Kalantari [16]) $B_4(z) = z - \frac{6p'(z)^2 p(z) - 3p''(z)p(z)^2}{p'''(z)p(z)^2 + 6p'(z)^3 - 6p''(z)p'(z)p(z))}$. The Ezzati-Saleki method (E_s for short) [10] $E_s(z) = N(z) + p(N(z))\big(\frac{1}{p'(z)} - \frac{4}{p'(z)+p'(N(z))}\big)$.

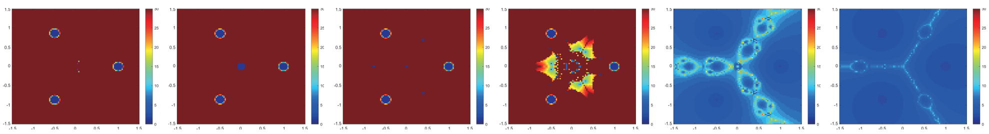
In this section, we visually analyze the PDHM, D, Noor and Picard-Noor methods, comparing their stability and convergence against the Newton, Halley, B_4, and E_s root-finding methods. To assess stability, we employ basins of attraction (see [2]). Within these basins, each root is assigned a unique colour, and we introduce an additional colour (black in our case) to indicate points of divergence. This colouring scheme provides insights into which root each starting point converges towards.

In our analysis, we compute specific numerical metrics, among which the convergence area index (CAI) is notably prevalent. The CAI is defined as the proportion of starting points that successfully converged to a root relative to the total number of points within the specified area, as discussed in [3]. The formula for the CAI is as follows: $CAI = \frac{N_c}{N}$, where N_c is the number of points in the polynomiograph that have converged, and N is the overall count of points in the polynomiograph. The CAI values range from 0 (indicating no convergence among the points) to 1 (all points have converged). Moreover, using the polynomiograph, we can calculate an average number of iterations (ANI) (see [11]). We use four root-finding methods in the considered example: Newton, Halley, B_4 and the E_s family. And we generate polynomiographs for a cubic polynomial $p_3(z) = z^3 - 1$, with roots: 1, $-0.5000 \pm 0.8660i$. After each iteration, we proceed with the iteration process till the convergence test is satisfied or the maximum number of iterations is reached. The standard convergence test has the form $|x_{n+1} - x_n| < \varepsilon$, where $\varepsilon > 0$ is the accuracy of the computations.

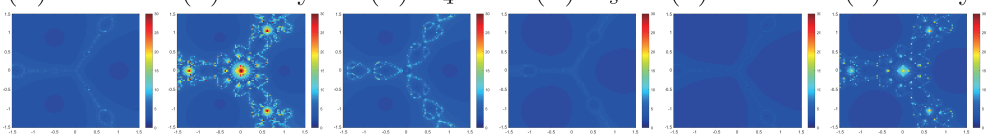
The polynomiographs were generated for three different settings of values of the iterations parameters: (1) $\alpha_n = 0.01$, $\beta_n = 0.01$, $\gamma_n = 0.01$, (2) $\alpha_n = 0.5$, $\beta_n = 0.5$, $\gamma_n = 0.5$, (3) $\alpha_n = 0.95$, $\beta_n = 0.95$, $\gamma_n = 0.95$. All the other parameters needed to generate the polynomiographs were $\varepsilon = 0.001$, resolution of 100×100 pixels and K is maximum number of iterations. (A), (B), (C), (D) come from the use of Noor iteration, (E), (F), (G), (H) come from the use of D-iteration and (I), (J), (K), (L) come from the of PDHM.

TABLE 2. CAI and ANI values calculated from polynomiographs for $p_3(z) = z^3 - 1$ presented in Figures 1, 2, 3, 4, 5, and 6.

Iterations	Root-finding methods	$\alpha_n = \beta_n = \gamma_n = 0.01$		$\alpha_n = \beta_n = \gamma_n = 0.5$		$\alpha_n = \beta_n = \gamma_n = 0.95$	
		CAI	ANI	CAI	ANI	CAI	ANI
Noor	Newton	0.0182	29.5852	1	10.1394	1	4.2372
	Hallay	0.0214	29.5086	1	9.8398	1	3.8840
	B_4	0.0198	29.5554	1	9.8072	1	3.8356
	E_s	0.0946	28.7020	0.9620	11.6820	1	4.6058
D	Newton	1	6.2832	1	3.9444	1	3.3072
	Hallay	1	4.0828	1	3.0734	1	2.4658
	B_4	1	3.3844	1	2.6814	1	2.2100
	E_s	0.9976	5.1670	0.9998	3.9294	1	3.1818
PDHM	Newton	1	**3.9602**	1	**3.1024**	1	**2.7354**
	Hallay	1	**2.7824**	1	**2.2640**	1	**2.1258**
	B_4	1	**2.3636**	1	**2.0604**	1	**2.0264**
	E_s	1	**3.4070**	1	**2.8806**	1	**2.6716**



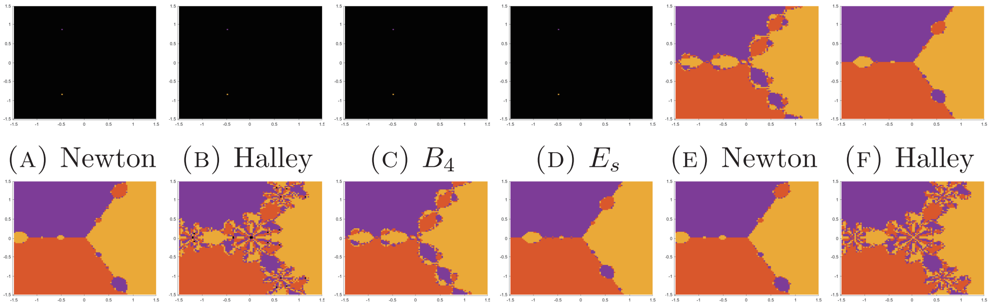
(A) Newton (B) Halley (C) B_4 (D) E_s (E) Newton (F) Halley



(G) B_4 (H) E_s (I) Newton (J) Halley (K) B_4 (L) E_s

FIGURE 1. Polynomiographs for $p_3(z) = z^3 - 1$ generated using various root finding methods with the parameters $\alpha_n = 0.01$, $\beta_n = 0.01$ and $\gamma_n = 0.01$ for $K = 30$.

For low values of the parameters ($\alpha_n = 0.01$, $\beta_n = 0.01$, $\gamma_n = 0.01$), we see on Figure 1 that the Noor iteration has not converged to any of the roots of $p_3(z)$, i.e., we see a uniform red colour, which corresponds to the maximal of iterations. We see a different speed convergence for the other two iterations (D and PDHM). Based on the visual analysis, we can observe that the fastest convergence speed is obtained by the proposed PDHM, followed by the D iteration. The ANI values confirm these observations in Table 2. In Table 2, the lowest ANI value, 2.3636 for the B_4, is obtained by the PDHM, followed by the D (3.3844) iteration. Furthermore, it is noteworthy that the ANI values for PDHM, which are 3.9602, 2.7824, 2.3636, and 3.4070 for the Newton, Halley, B_4, and E_s methods, respectively, also yield better results than the ANI values of the D and Noor iterations. Tables 2 also found that the ANI values for the Noor iteration for the Newton, Halley, B_4, and E_s methods are relatively high compared to the D and PDHM iterations.



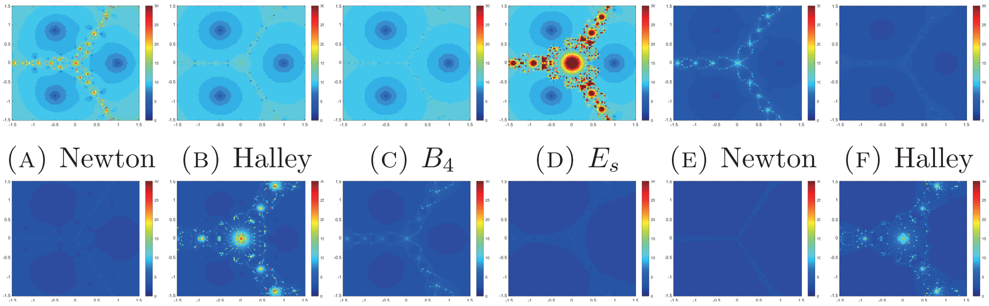
(A) Newton (B) Halley (C) B_4 (D) E_s (E) Newton (F) Halley

(G) B_4 (H) E_s (I) Newton (J) Halley (K) B_4 (L) E_s

FIGURE 2. Basins of attraction for $p_3(z) = z^3 - 1$ generated using various root finding methods with $\alpha_n = 0.01$, $\beta_n = 0.01$ and $\gamma_n = 0.01$ for $K = 30$.

Upon analyzing the polynomiographs presented in Figure 2, we see that the best stability in finding the roots is the PDHM. This phenomenon is especially pronounced in the PDHM mode for the Newton, Halley and B_4 methods, where CAI achieves a perfect score of 1 (see Table 2). While characteristic braids are visible in each case, their shapes vary among the methods. The most intricate braids are observed with the E_s method, resulting in the largest interweaving of basins. Except in Noor iteration, a small percentage of the starting points did not converge to any roots (black colour indicate points of divergence).

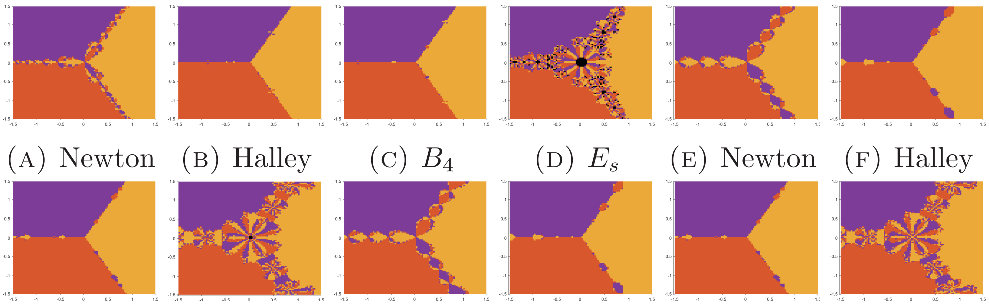
For polynomiographs for the second parameters setting ($\alpha_n = 0.5$, $\beta_n = 0.5$, $\gamma_n = 0.5$) presented in Figure 3, we see that the Noor iteration obtains the slowest speed of convergence. In Figure 3, the polynomiograph contains red colours, indicating a high number of performed iterations. When we look at the polynomiographs presented in Figure 3, we see that the fastest among the analyzed iterations is the PDHM. In the polynomiographs, we can observe darker blue colours than in the case of the other iteration processes, which shows a smaller number of performed iterations. The ANI values in Table 2 confirm this observation because the PDHM obtains the lowest value. The lowest value of ANI equal to 2.0604 for the B_4 is obtained by the PDHM.



(A) Newton (B) Halley (C) B_4 (D) E_s (E) Newton (F) Halley

(G) B_4 (H) E_s (I) Newton (J) Halley (K) B_4 (L) E_s

FIGURE 3. Polynomiographs for $p_3(z) = z^3 - 1$ generated using various root finding methods with the parameters $\alpha_n = 0.5$, $\beta_n = 0.5$ and $\gamma_n = 0.5$ for $K = 30$.



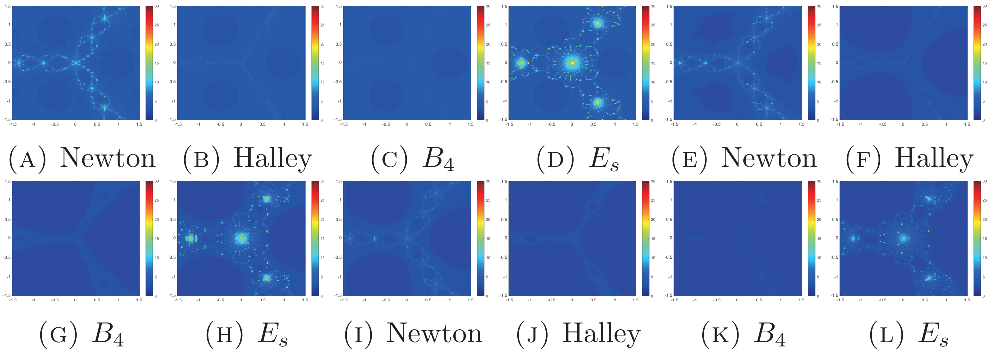
(A) Newton (B) Halley (C) B_4 (D) E_s (E) Newton (F) Halley

(G) B_4 (H) E_s (I) Newton (J) Halley (K) B_4 (L) E_s

FIGURE 4. Basins of attraction for $p_3(z) = z^3 - 1$ generated using various root finding methods with $\alpha_n = 0.5$, $\beta_n = 0.5$ and $\gamma_n = 0.5$ for $K = 30$.

In Figure 4, the interweaving of the basins around the braids is minimal for the Halley and B_4 methods, and the braids appear similar in Noor, D and PDHM iterations. Outside the braided regions, the behaviour of the methods is quite similar. Thus, the Halley and B_4 methods exhibit the most stable behaviour. We can also observe this by looking at the values of CAI in Tables 2. Regarding CAI value in Table 2, the best two methods was PDHM, which obtained convergence of all starting points, i.e., CAI value equal to 1.

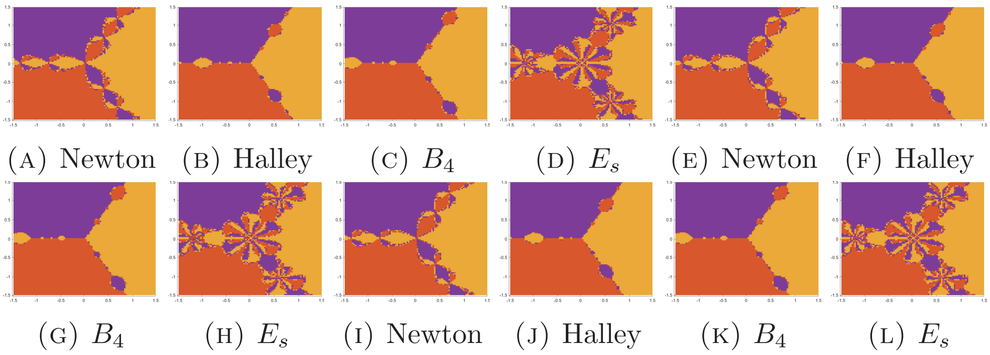
In the last parameter setting, we use high values of the parameters ($\alpha_n = 0.95$, $\beta_n = 0.95$, $\gamma_n = 0.95$). Like for the other two parameter settings, for the polynomiographs, we see that the Noor iteration obtains the slowest speed of convergence. On the other hand, the PDHM again obtains the fastest convergence speed. In the case of each polynomiograph, we can observe darker blue colours than for the two other parameter settings, which shows a smaller number of performed iterations. This shows that for higher values of the parameters, all the iterations need fewer iterations to find the roots. We can also observe this by looking at the values of ANI in Table 2. We see that the PDHM obtains the lowest ANI value for high values of the parameters. The lowest values for the other iterations are also obtained for high values of the parameters. Examining the values in Table 2, we find that CAI indicates the best performance for every method in Noor, D and PDHM iterations. It achieves convergence for all starting points within the area, with a CAI value 1. It is noted that the E_s method produces beautiful images with more artistic value.



(A) Newton (B) Halley (C) B_4 (D) E_s (E) Newton (F) Halley

(G) B_4 (H) E_s (I) Newton (J) Halley (K) B_4 (L) E_s

FIGURE 5. Polynomiographs for $p_3(z) = z^3 - 1$ generated using various root finding methods with the parameters $\alpha_n = 0.95$, $\beta_n = 0.95$ and $\gamma_n = 0.95$ for $K = 30$.



(A) Newton (B) Halley (C) B_4 (D) E_s (E) Newton (F) Halley

(G) B_4 (H) E_s (I) Newton (J) Halley (K) B_4 (L) E_s

FIGURE 6. Basins of attraction for $p_3(z) = z^3 - 1$ generated using various root finding methods with $\alpha_n = 0.95$, $\beta_n = 0.95$ and $\gamma_n = 0.95$ for $K = 30$.

4. CONCLUSIONS

In this study, we conducted a convergence and stability analysis of the newly developed Picard-D hybrid method (PDHM). We demonstrated its effective application for determining fixed points of a contraction operator. Both numerical and theoretical approaches verify its effectiveness. Our findings reveal that PDHM outperforms existing methods, such as the Noor, and D iterations. Additionally, we conducted empirical comparisons of PDHM with these iteration methods in solving root-finding problems using techniques like Newton, Halley, B_4, and E_s methods, employing polynomiography for this purpose. The results consistently indicate that PDHM offers a better convergence speed and stability.

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