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TWO PRE-CONDITIONING ALGORITHMS FOR SOLVING SPLIT EQUALITY PROBLEM IN HILBERT SPACES

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Abstract. In this research, we propose two preconditioning algorithms that incorporate line search and self adaptive techniques to address the split equality problem in real Hilbert spaces. Among these two algorithms, one is designed to obtain the optimal step size, with the aim of accelerating the convergence rate of the algorithm. This method does not necessitate prior knowledge of the operator norms. Moreover, we establish a weak convergence theorem under specific norm conditions and provide a detailed relevant proof. Finally, the results of numerical experiments demonstrate that the proposed algorithms exhibits a faster convergence speed compared to other existing algorithms. This superiority in convergence performance validates the effectiveness and practicality of our proposed approach in solving the split equality problem.

1. Introduction

In this paper, our aim is to explore the split equality problem (SEP) by applying pre-conditioning methods. Suppose that H, H_1, H_2 and H_3 are four real Hilbert spaces, each equipped with an inner product denoted as $\langle\cdot,\cdot\rangle$ and the corresponding induced norm represented by $\|\cdot\|$. Additionally, let C and Q be two nonempty closed convex sets located in the Hilbert spaces H_1 and H_2, respectively.

The split equality problem was first introduced by Moudafi [20] and could be described as follows:

$$\text{find } u \in C \text{ and } v \in Q \text{ such that } Au = Bv, \tag{1.1}$$

where $A : H_1 \longrightarrow H_3$ and $B : H_2 \longrightarrow H_3$ are two bounded linear operators.

In the case where $H_2 = H_3$ and $B=I$, the split equality problem (SEP) is simplified to the split feasibility problem (SFP), which was first proposed by Censor and Elfving [9]. Due to its extensive applications in the realm of applied mathematics, such as signal processing, image reconstruction, inverse problems, and intensity-modulated radiation therapy; see [7, 8, 10]. The application of algorithms to solve the SEP as described in (1.1) has drawn significant attention; see [11, 22–25, 29, 30] for more details.

One of these algorithms for solving the SEP is called alternating CQ algorithm (ACQA), which is a weakly convergent algorithm introduced by Moudafi [20]:

$$\begin{cases} u_{n+1} = P_C(u_n - \gamma_n A^*(Au_n - Bv_n)), \\ v_{n+1} = P_Q(v_n + \gamma_n B^*(Au_{n+1} - Bv_n)), \end{cases} \tag{1.2}$$

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where A^* and B^* are two adjoint operators of A and B respectively, and also P_C (P_Q) is the projection operator onto C (Q). $\{\gamma_n\}$ is a positive non-decreasing sequence, which satisfies $\gamma_n \in (\varepsilon, \min(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}) - \varepsilon)$ for a small enough $\varepsilon > 0$.

In general, this algorithm requires us to calculate the norms of the operators, and this is extremely difficult in the practical implementation process. In order to avoid this, Dong et al. [12] proposed a self-adaptive stepsize as follows:

$$\gamma_n = \sigma_n \min\left(\frac{\|Au_n - Bv_n\|^2}{\|A^*(Au_n - Bv_n)\|^2}, \frac{\|Au_n - Bv_n\|^2}{\|B^*(Au_n - Bv_n)\|^2}\right), \sigma_n \in (0,1).$$

In fact, as early as 2005, Qu and Xiu [21] adopted a linea-search technique to avoid this drawback. Moreover, their method has received widespread attention and many researchers have made improvements upon it; see [4, 13, 16, 26]. On the other hand, Wang et al. [28] applied a preconditioning technique to improve the convergence speed of the algorithm. In fact, the preconditioning method transforms the elliptical contour of a function into a circular one, with the aim of altering the geometric shape of the function to improve the speed of the algorithm. Moreover, many authors have already used this method to solve a wide variety of problems; see [1, 5, 6, 27].

Inspired by the above works, this paper proposes two preconditioning algorithms that combine the line search technique to solve the split equality problem (1.1). Consequently, in each iteration of the algorithms, a variable step size is adopted instead of the traditional constant step size. This method does not require us to calculate the norms of any operators in advance. Meanwhile, we also prove the weak convergence properties of these algorithms. Eventually, through experimental demonstrations, our preconditioning algorithms can improve the convergence speed.

This paper is organized as follows: In Sect. 2, we provide several key definitions and theorems that will be used in proving our conclusions. In Sect. 3, we introduce two pre-conditioning algorithms and analyze the convergence of the proposed algorithms. The applications and benefits of our algorithms will be discussed in Sect. 4. A concise conclusion is at the end of the paper in Sect. 5.

2. Preliminaries

Definition 2.1. Let S be a nonempty closed convex subset of real Hilbert spaces H.

(i) $T : S \to H$ be a mapping. Then

(a) T is non-expansive , if:

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in S.$$

(b) T is firmly non-expansive, if:

$$\langle Tx - Ty, x - y\rangle \leq \|Tx - Ty\|^2, \ \forall x, y \in S.$$

(ii) Let $g : S \to H$ be a mapping. Then

(a) g is said to be co-coercive on S, if

$$\langle g(x) - g(y), x - y\rangle \geq \alpha \|g(x) - g(y)\|^2, \text{where } \forall x, y \in S \text{ and } \alpha > 0.$$

(b) g is called Lipschitz continuous on S with a constant $a > 0$, if

$$\|g(x) - g(y)\| \leq a \|x - y\|, \ \forall x, y \in S.$$

(iii) A differentiable function $f : H \to R$ be a mapping where R denote real numbers space.

(a) f is convex if and only if

$$f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle, \forall x, y \in H. \tag{2.1}$$

(b) f is weakly lower semi-continuou:

$$x_k \rightharpoonup x \Longrightarrow f(x) \leq \lim_{k\to\infty} inf f(x_k)$$

To introduce our preconditioning technique, we first recall the following concepts. Then we give the concepts of self-adjoint operator and positive definite operator.

Let G be a bounded linear operator on H. G is said a self-adjoint if $G^* = G$ where G^* is the adjoint operator. Also, G is said to be a positive-definite operator if $\langle x, Gx \rangle > 0$, for all non-zero $x \in H$; see [18] for more details. Then we give the definition of the G-norm $\|\cdot\|_G$, where $\|x\|_G$ satisfies the following condition: $\|x\|_G^2 = \langle x, Gx \rangle = \langle\ x,\ x \rangle_G$ for all $x \in H$; see [3] for more details.

We use this norm to define the G-projection operator on S by P_S^G, i.e.,

$$P_S^G(x) = arg \min_{y \in S} \{\|x - y\|_G\}.$$

It is well known that P_S^G is firmly non-expansive. Moreover, the following inequality relates the G-norm to the standard norm:

$$\lambda_{min}(G)\|x\|^2 \leq \|x\|_G^2 \leq \lambda_{max} \|x\|^2, \tag{2.2}$$

where $\lambda_{min}(G)$ and $\lambda_{max}(G)$ are the minimum and maximum eigenvalues of the operator G, respectively. This holds for all $x \in H$, see [14, 15] for details.

Now some useful lemmas are as follows and we will use to prove our main theorems.

Lemma 2.2 ([17, 19])**.** *Let S be a closed nonempty subset of a real Hilbert space H. For every $x, y \in H$ and $z \in S$, the G-projection operator in S satisfies that the properties are as follows:*

(1) $\langle G(x - P_S^G), z - P_S^G \rangle \leq 0,$
(2) $\|x \pm y\|_G^2 = \|x\|_G^2 \pm 2\langle x, Gy \rangle + \|y\|_G^2.$
(3) $\|P_S^G(x) - z\|_G^2 \leq \|x - z\|_G^2 - \|x - P_S^G(x)\|_G^2.$
(4) $\|P_S^G(x) - P_S^G(y)\|_G^2 \leq \|x - y\|_G^2 - \|(P_S^G(x) - x) - (P_S^G(y) - y)\|_G^2$

The next lemma will be used to prove weak convergence of a sequence.

Lemma 2.3 ([2])**.** *Let S be a nonempty closed and convex subset of real Hilbert space H and $\{x_n\}$ be a sequence in H that satisfies the properties are as follows:*

(1) $\lim_{x\to\infty} \|x_n - x\|$ *exists for all $x \in S$.*
(2) *every weak sequential cluster point of $\{x_n\}$ belongs to S.*

Then the sequence $\{x_n\}$ converges weakly to a point in S.

3. Main results and convergence analysis

In this section, we propose two preconditioning algorithms that incorporate self-adaptive and line-search techniques for solving the split equality problem. Firstly, we begin this paper with the following assumption. Let Ω be the solution set of the SEP (1.1), i.e.

$$\Omega = \{(x, y) \mid (x, y) \in H_1 \times H_2; Ax = By, x \in C, y \in Q\},$$

we assume that the solution set Ω is nonempty, which implies that Ω is closed and convex.

Let $S = C \times Q \subseteq H_1 \times H_2 =: H$. Let $U : H \to H_3$ be an operator defined by $U(x, y) = Ax - By$ for all $(x, y) \in H$. Let U^* denote the adjoint operator of U. Then U and U^*U have the following matrix form

$$U = [A, -B] \text{ and } U^*U = \begin{bmatrix} A^*A & -A^*B \\ -B^*A & B^*B \end{bmatrix}.$$

Then the SEP can be reformulated as finding a point $r = (x, y) \in S$ such that $Ur = 0$. To apply our methods, we rewrite the SEP as the following constrained minimization problem:

$$\min_{r \in S} f(r),$$

where $f(r) = \frac{1}{2} \|Ur\|^2$ and it is clear that $\bigtriangledown f(r) = U^*Ur$.

Next, we combine the objective function with preconditioning methods to define a new function. Let $G : H \to H$ and $D_1 : H_3 \to H_3$ be positive-definite operators such that $U^*D_1 = DU^*$ where $D = G^{-1}$. We can rewrite the function with respect to the norm $\|\cdot\|_{D_1}$, so we define

$$f_{D_1}(r) = \frac{1}{2} \|Ur\|^2_{D_1},$$

and then, the gradient of the operator is obtained by

$$\bigtriangledown f_{D_1}(r) = DU^*Ur.$$

Proposition 3.1. *Assume that the SEP solution set is nonempty, i.e.,* $\Omega \neq \emptyset$. *Then the following are equivalent:*

(1) r^* *is a solution of SEP.*
(2) $r^* \in S$ *and* $f(r^*) = 0$.
(3) $r^* \in S$ *and* $-\nabla f(r^*) \in N_S(r^*)$, *where* $N_S(r^*)$ *is the normal cone to* S *at* r^*.

Equivalently, condition (3) can be expressed as the variational inequality:

$$\langle f(r^*), r - r^* \rangle \geq 0 \quad \text{for all } r \in S.$$

Now we give the following algorithms based on the above works.

Algorithm 1. A Preconditioned CQ Algorithm with Line Search.

Step 0. Let $u_1 \in H$ and $\forall \sigma > 0, \rho \in (0, 1)\, \mu \in \left(0, \frac{\lambda_{min}(G)}{2}\right)$

Step 1. Compute v_n

$$v_n = P^G_S(u_n - \gamma_n DU^*Uu_n),$$

where $\gamma_n = \sigma\rho^{m_n}$ and m_n is the smallest non-negative integer such that

$$\gamma_n \|U^*Uv_n - U^*Uu_n\| \le \mu \|v_n - u_n\| . \tag{3.1}$$

Step 2. Generate u_{n+1} by

$$u_{n+1} = P_S^G(v_n - \tau_n DU^*Uv_n),$$

where

$$\tau_n = \frac{\beta_n f(v_n)}{\|DU^*Uv_n\|_G^2}, 0 < \beta_n < 4.$$

Step 3. The line search in (3.1) terminates after a finite number of steps.

Lemma 3.2. *For all $\triangledown f$ is L-Lipschitz continuous on S and co-coercive on S with modulus $1/L$, where L is the largest spectra of the operator U^*U. Therefore, the line search rule (3.1) is well defined. Furthermore, we have the following:*

$$\frac{\mu\rho}{L} < \gamma_n \le \sigma, \text{ for all } n \ge 0. \tag{3.2}$$

Proof. We first show that linea search is well-defined. By the definition of $\triangledown f(r)$, we have for any $u, v \in S$:

$$\begin{aligned} \|\triangledown f(v) - \triangledown f(u)\|^2 &= \|U^*Uv - U^*Uu\|^2 \\ &\le L \|Uv - Uu\|^2 \\ &\le L^2 \|v - u\|^2 , \end{aligned}$$

that is

$$\|\triangledown f(v) - \triangledown f(u)\| \le L \|v - u\| .$$

On the one hand, we have

$$\begin{aligned} \langle \triangledown f(v) - \triangledown f(u), v - u\rangle &= \langle U^*Uv - U^*Uu, v - u\rangle \\ &= \langle Uv - Uu, Uv - Uu\rangle \\ &= \|U(v - u)\|^2 . \end{aligned}$$

Combine the above inequalities, we have

$$\langle \triangledown f(v) - \triangledown f(u), v - u\rangle \ge \left(\frac{1}{L}\right) \|\triangledown f(v) - \triangledown f(u)\|^2 .$$

So that line search is well defined. Below we demonstrate the second half of the lemma.

Obviously, according to the definition of γ_n, it is obvious that we have $\gamma_n \le \sigma$ for all $n = 1, 2, \ldots$. From line search (3.1) we know that $\frac{\gamma_n}{\rho}$ must satisfy that

$$\left\|\triangledown f(u_n) - \triangledown f(P_S^G(u_n - \frac{\gamma_n}{\rho} \triangledown f_{D_1}(u_n)))\right\| > \mu \frac{\left\|u_n - P_S^G(u_n - \frac{\gamma_n}{\rho} \triangledown f_{D_1}(u_n))\right\|}{\frac{\gamma_n}{\rho}}.$$

Combined with the above proofs, we have

$$\frac{\mu\rho}{L} < \gamma_n.$$

This means that we have completed the proof. ☐

Next, we will prove that a sequence generated by Algorithm 1 converges weakly to the solution of the SEP.

Theorem 3.3. *Let $\{u_n\}$ be a sequence generated by Algorithm 1 and suppose that* $\lim_{n\to\infty} inf \beta_n (4-\beta_n) > 0$. *Then $\{u_n\}$ weakly converges to the solution of the SEP.*

Proof. Let $r^* \in \Omega$, then we have $r^* = P_S^G r^*$ and $\bigtriangledown f(r^*) = 0$. From Lemma 2.2 we can have that

$$\begin{aligned} \|u_{n+1} - r^*\|_G^2 &= \left\|P_S^G (v_n - \tau_n DU^*Uv_n) - r^*\right\|_G^2 \\ &\leq \|v_n - \tau_n DU^*Uv_n - r^*\|_G^2 \\ &= \|v_n - r^*\|_G^2 + \tau_n^2 \|DU^*Uv_n\|_G^2 - 2\tau_n \langle v_n - r^*, DU^*Uv_n\rangle_G \\ &= \|v_n - r^*\|_G^2 + \tau_n^2 \|DU^*Uv_n\|_G^2 - 2\tau_n \langle v_n - r^*, U^*Uv_n\rangle . \end{aligned} \tag{3.3}$$

Since the definition of $\bigtriangledown f(r^*)$ and $\bigtriangledown f(r^*) = 0$. We can have that

$$\begin{aligned} \langle v_n - r^*, U^*Uv_n\rangle &= \langle v_n - r^*, U^*Uv_n - \bigtriangledown f(r_*)\rangle \\ &= \langle U(v_n - r^*), U(v_n - r^*)\rangle \\ &= 2f(v_n). \end{aligned} \tag{3.4}$$

Similarly ,we can also have that

$$\langle u_n - r^*, U^*Uu_n\rangle = 2f(u_n). \tag{3.5}$$

On the one hand,by using the inequality (2.1),we have

$$\begin{aligned} 2\gamma_n \langle v_n - u_n, U^*Uu_n\rangle &= 2\gamma_n \langle v_n - u_n, U^*Uu_n - U^*Uv_n\rangle \\ &\quad + 2\gamma_n \langle v_n - u_n, U^*Uv_n\rangle \\ &\geq -2\gamma_n \|v_n - u_n\| \|U^*Uv_n - U^*Uu_n\| \\ &\quad + 2\gamma_n(f(v_n) - f(u_n)) \\ &\geq -2\gamma_n \|v_n - u_n\| \|U^*Uv_n - U^*Uu_n\| - 2\gamma_n f(u_n). \end{aligned} \tag{3.6}$$

On the other hand, it follows that from (2.2), (3.5), (3.6) and Lemma 2.2, we can obtain the inequality as follows

$$\begin{aligned} \|v_n - r^*\|_G^2 &= \left\|P_S^G(u_n - \gamma_n DU^*Uu_n) - r^*\right\|_G^2 \\ &\leq \|u_n - \gamma_n DU^*Uu_n - r^*\|_G^2 - \|v_n - u_n + \gamma_n DU^*Uu_n\|_G^2 \\ &= \|u_n - r^*\|_G^2 - 2\gamma_n \langle u_n - r^*, U^*Uu_n\rangle - \|v_n - u_n\|_G^2 \\ &\quad - 2\gamma_n \langle v_n - u_n, U^*Uu_n\rangle \\ &\leq \|u_n - r^*\|_G^2 - \|v_n - u_n\|_G^2 - 4\gamma_n f(u_n) + 2\gamma_n f(u_n) \\ &\quad + 2\gamma \|v_n - u_n\| \|U^*Uv_n - U^*Uu_n\| \\ &\leq \|u_n - r^*\|_G^2 - \|v_n - u_n\|_G^2 - 2\gamma_n f(u_n) + 2\mu \|v_n - u_n\|^2 \\ &\leq \|u_n - r^*\|_G^2 - 2\gamma_n f(u_n) - (\lambda_{min}(G) - 2\mu) \|v_n - u_n\|^2 . \end{aligned} \tag{3.7}$$

Combining equations (3.3), (3.4), (3.7) with Lemma 3.2, we can obtain

$$\begin{aligned}\|u_{n+1}-r^*\|_G^2 &\le \|u_n-r^*\|_G^2 - 2\gamma_n f(u_n) - (\lambda_{min}(G)-2\mu)\|v_n-u_n\|^2 \\ &\quad + \tau_n^2\|DU^*Uv_n\|_G^2 - 2\tau_n\langle v_n - x^*, U^*Uv_n\rangle \\ &= \|u_n-r^*\|_G^2 - 2\gamma_n f(u_n) - (\lambda_{min}(G)-2\mu)\|v_n-u_n\|^2 \\ &\quad + \tau_n^2\|DU^*Uv_n\|_G^2 - 4\tau_n f(v_n) \\ &\le \|u_n-r^*\|_G^2 - (\lambda_{min}(G)-2\mu)\|v_n-u_n\|^2 - 2\frac{\mu\rho}{L}f(u_n) \\ &\quad + \frac{\beta_n^2 f^2(v_n)}{\|DU^*Uv_n\|_G^2} - 4\frac{\beta_n f^2(v_n)}{\|DU^*Uv_n\|_G^2} \\ &= \|u_n-r^*\|_G^2 - (\lambda_{min}(G)-2\mu)\|v_n-u_n\|^2 - 2\frac{\mu\rho}{L}f(u_n) \\ &\quad - \beta_n(4-\beta_n)\frac{f^2(v_n)}{\|DU^*Uv_n\|_G^2}.\end{aligned} \tag{3.8}$$

Since $\mu \in \left(0, \frac{\lambda_{min}(G)}{2}\right)$ and $\beta_n \in (0,4)$, it is clear that we can get that the sequence $\{\|u_n - r^*\|_G\}$ is monotonically decreasing. So we also have that the sequence $\{\|u_n - r^*\|_G\}$ is convergent and the sequence $\{u_n\}$ is bounded. Therefore, from (3.8) we have

$$\lim_{n\to\infty} f(u_n) = 0, \tag{3.9}$$

and

$$\lim_{n\to\infty} \|v_n - u_n\| = 0. \tag{3.10}$$

On the one hand, by using the assumption $\lim_{n\to\infty} inf \beta_n(4-\beta_n) > 0$, we can get it follows that

$$\lim_{n\to\infty} \frac{f^2(v_n)}{\|DU^*Uv_n\|_G^2} = 0,$$

and so we get

$$\lim_{n\to\infty} f(v_n) = 0.$$

On the other hand, we obtain the equation from Lemma 2.2 as follows

$$\begin{aligned}\|u_{n+1}-v_n\|_G^2 &= \left\|P_S^G(v_n - \tau_n DU^*Uv_n) - v_n\right\|_G^2 \\ &\le \tau_n^2\|DU^*Uv_n\|_G^2 \\ &= \frac{\beta_n^2 f^2(v_n)}{\|DU^*Uv_n\|_G^2}.\end{aligned} \tag{3.11}$$

From (3.11), we obtain

$$\lim_{n\to\infty} \|u_{n+1} - v_n\|_G = 0, \tag{3.12}$$

and then we get

$$\|u_{n+1}-u_n\|_G \le \|u_{n+1}-v_n\|_G + \|v_n-u_n\|_G.$$

Next, it follows from (3.10) and (3.12) that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\|_G = 0.$$

Let $\bar{r} \in S$ and it is a weak cluster point of the bounded sequence $\{u_n\}$. So it exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \rightharpoonup \bar{r}$ as $k \rightharpoonup \infty$. According to the weak lower semi-continuity of f and equation (3.9), we get

$$0 \le f(\bar{r}) \le \lim_{k\to\infty} inf f(u_{n_k}) = \lim_{n\to\infty} f(u_n) = 0,$$

this means $f(\bar{r})$=0. So $\bar{r}$ is a solution of the SEP and from Lemma 2.3 we can obtain the sequence $\{u_n\}$ converges weakly to a point in S and complete this proof. □

Now, to obtain the optimal step size, we will introduce a new preconditioning algorithm using self-adaptive techniques given by Dong et al. [13]. And before proving the convergence analysis of our algorithm, to simplify our proof process, we commence this section by giving some basic remarks and theorem.

Algorithm 2. A new pre-conditioning algorithm with optimal step size.

Step 0. Let $\forall\sigma > 0, \mu$ and $\rho \in (0, 1)$. Compute u_{n+1} by the following way:

Step 1. Compute

$$v_n = P_S^G(u_n - \alpha_n DU^*Uu_n), \tag{3.13}$$

where $\alpha_n = \sigma\rho^{m_n}$ and m_n is the smallest non-negative integer such that

$$\alpha_n \|DU^*Uu_n - DU^*Uv_n\|_G \le \mu \|u_n - v_n\|_G . \tag{3.14}$$

Step 2. Compute

$$u_{n+1} = P_S^G(u_n - \gamma\alpha_n\theta_n DU^*Uv_n), \tag{3.15}$$

where $\gamma \in (0, 2)$,

$$d(u_n, v_n) = u_n - v_n - \alpha_n(DU^*Uu_n - DU^*Uv_n) \tag{3.16}$$

and

$$\theta_n = \frac{\langle u_n - v_n, d(u_n, v_n)\rangle_G + \alpha_n \|Uv_n\|^2}{\|d(u_n, v_n)\|_G^2}. \tag{3.17}$$

Lemma 3.4. *The (3.14) is well defined. And at the same time* $\alpha \le \alpha_n \le \sigma$ *and* $\alpha = min\left\{\sigma, \frac{\mu\rho}{L}\right\}$.

Since the method of proving this theorem is similar to Lemma 3.2, we will not prove it here. In order to simplify the proof of our main theorem, we first give a simple remark of the algorithm.

Remark 3.5. From (3.13) and (3.16), we can rewrite that

$$v_n = P_S^G(v_n - \alpha_n DU^*Uv_n - d(u_n, v_n)), \tag{3.18}$$

and from Lemma 2.2, it follows that

$$\langle r - v_n, \alpha_n DU^*Uv_n - d(u_n, v_n)\rangle_G \ge 0, \ \forall r \in S. \tag{3.19}$$

Lemma 3.6. *Let $\{u_n\}$ and $\{v_n\}$ be the sequences generated by the Algorithm 2 and according to the definition of $d(u_n, v_n)$ and θ_n, we can have*

$$\langle u_n - v_n, d(u_n, v_n)\rangle_G \geq (1-\mu) \|u_n - v_n\|_G^2 ,$$

and

$$\theta_n \geq \frac{1-\mu}{1+\mu^2}.$$

Proof. On the one hand, from (3.14) we can obtain

$$\begin{aligned}\langle u_n - v_n, d(u_n, v_n)\rangle_G &= \langle u_n - v_n, u_n - v_n - \alpha_n(DU^*Uu_n - DU^*Uv_n)\rangle_G \\ &= \|u_n - v_n\|_G^2 - \alpha_n \langle u_n - v_n, DU^*Uu_n - DU^*Uv_n\rangle_G \\ &\geq \|u_n - v_n\|_G^2 - \alpha_n \|u_n - v_n\|_G \|DU^*Uu_n - DU^*Uv_n\|_G \\ &\geq (1-\mu) \|u_n - v_n\|_G^2 .\end{aligned}$$

On the other hand, by (3.14) we also obtain

$$\begin{aligned}\|d(u_n, v_n)\|_G^2 &= \|u_n - v_n - \alpha_n(DU^*Uu_n - DU^*Uv_n)\|_G^2 \\ &= \|u_n - v_n\|_G^2 + \alpha_n^2 \|DU^*Uu_n - DU^*Uv_n\|_G^2 \\ &\quad - 2\alpha_n \langle u_n - v_n, DU^*Uu_n - DU^*Uv_n\rangle_G \\ &\leq \|u_n - v_n\|_G^2 + \alpha_n^2 \|DU^*Uu_n - DU^*Uv_n\|_G^2 \\ &\leq (1+\mu^2) \|u_n - v_n\|_G^2 .\end{aligned}$$

Combining the above inequalities with (3.17), we can have

$$\theta_n \geq \frac{\langle u_n - v_n, d(u_n, v_n)\rangle_G}{\|d(u_n, v_n)\|_G^2} \geq \frac{1-\mu}{1+\mu^2}.$$

Consequently, we complete our proof. □

Then, we will present our main theorem and provide its proof.

Theorem 3.7. *Assume that the sequences $\{u_n\}$, $\{v_n\}$ are generated by the Algorithm 2. Then the sequence $\{u_n\}$ converges weakly to a solution of the SEP.*

Proof. Let $r^* \in \Omega$, from (3.15) and Lemma 2.2, we have

$$\begin{aligned}\|u_{n+1} - r^*\|_G^2 &\leq \|u_n - \gamma\alpha_n\theta_n DU^*Uv_n - r^*\|_G^2 \\ &\quad - \|u_n - \gamma\alpha_n\theta_n DU^*Uv_n - u_{n+1}\|_G^2 \\ &= \|u_n - r^*\|_G^2 - \|u_n - u_{n+1}\|_G^2 \\ &\quad - 2\gamma\alpha_n\theta_n \langle u_{n+1} - r^*, DU^*Uv_n\rangle_G \\ &\leq \|u_n - r^*\|_G^2 - \|u_n - u_{n+1}\|_G^2 \\ &\quad - 2\gamma\alpha_n\theta_n \langle u_{n+1} - v_n, DU^*Uv_n\rangle_G \\ &\quad - 2\gamma\alpha_n\theta_n \langle v_n - r^*, DU^*Uv_n\rangle_G .\end{aligned} \tag{3.20}$$

Combining the define of $\bigtriangledown f_{D_1}(r)$ with $r^* \in \Omega$, we can obtain

$$\begin{aligned}\langle v_n - r^*, DU^*Uv_n\rangle_G &= \langle v_n - r^*, GDU^*Uv_n\rangle \\ &= \langle Uv_n - Ur^*, Uv_n\rangle \\ &= \langle Uv_n - Ur^*, Uv_n - Ur^*\rangle \\ &= \|Uv_n\|^2 .\end{aligned} \tag{3.21}$$

Since $u_{n+1} \in S$ and let $r = u_{n+1}$ in (3.19), we have

$$\alpha_n \langle u_{n+1} - v_n, DU^*Uv_n\rangle_G \geq \langle u_{n+1} - v_n, d(u_n, v_n)\rangle_G , \tag{3.22}$$

and the substituting (3.21), (3.22) in the equation (3.20), we have

$$\begin{aligned}\|u_{n+1} - r^*\|_G^2 \leq{}& \|u_n - r^*\|_G^2 - \|u_n - u_{n+1}\|_G^2 \\ &- 2\gamma\theta_n \langle u_{n+1} - v_n, d(u_n, v_n)\rangle_G - 2\gamma\alpha_n\theta_n \|Uv_n\|^2 \\ ={}& \|u_n - r^*\|_G^2 - \|u_n - u_{n+1}\|_G^2 \\ &+ 2\gamma\theta_n \langle u_n - u_{n+1}, d(u_n, v_n)\rangle_G \\ &- 2\gamma\theta_n(\langle u_n - v_n, d(u_n, v_n)\rangle_G + \alpha_n \|Uv_n\|^2).\end{aligned} \tag{3.23}$$

By using the equation (3.17) we have

$$\theta_n^2 \|d(u_n, v_n)\|_G^2 = \theta_n(\langle u_n - v_n, d(u_n, v_n)\rangle_G + \alpha_n \|Uv_n\|^2).$$

On the one hand, from Lemma 2.2 we have

$$\begin{aligned}2\gamma\theta_n \langle u_n - u_{n+1}, d(u_n, v_n)\rangle_G ={}& - \|(u_n - u_{n+1}) - \gamma\theta_n d(u_n, v_n)\|_G^2 + \|u_n - u_{n+1}\|_G^2 \\ &+ \gamma^2\theta_n^2 \|d(u_n, v_n)\|_G^2 .\end{aligned}$$

On the other hand, substituting the above two equalities in (3.23), we have

$$\begin{aligned}\|u_{n+1} - r^*\|_G^2 \leq{}& \|u_n - r^*\|_G^2 - \|u_n - u_{n+1}\|_G^2 - 2\gamma\theta_n^2 \|d(u_n, v_n)\|_G^2 \\ &- \|(u_n - u_{n+1}) - \gamma\theta_n d(u_n, v_n)\|_G^2 + \|u_n - u_{n+1}\|_G^2 \\ &+ \gamma^2\theta_n^2 \|d(u_n, v_n)\|_G^2 \\ \leq{}& \|u_n - r^*\|_G^2 - \theta_n^2\gamma(2 - \gamma) \|d(u_n, v_n)\|_G^2 .\end{aligned} \tag{3.24}$$

Since $\gamma \in (0, 2)$, it is evident that we can have the sequence $\left\{ \|u_n - r^*\|_G^2 \right\}$ is decreasing and therefore, it must converge. Thus the sequence $\{u_n\}$ is bounded. On the one hand from Lemma 3.6, we have

$$\begin{aligned}\theta_n \|d(u_n, v_n)\|_G^2 &\geq \langle u_n - v_n, d(u_n, v_n)\rangle_G \\ &\geq (1 - \mu) \|u_n - v_n\|_G^2\end{aligned},$$

and

$$\theta_n \geq \frac{1 - \mu}{1 + \mu^2}.$$

On the other hand, we combine the above two inequalities with (3.24), we can obtain

$$\frac{\gamma(2 - \gamma)(1 - \mu)^2}{1 + \mu^2} \|u_n - v_n\|_G^2 \leq \|u_n - r^*\|_G^2 - \|u_{n+1} - r^*\|_G^2 .$$

There will be generate $\sum_{n=1}^{\infty} \|u_n - v_n\|_G^2 < \infty$, so we have

$$\lim_{n\to\infty} \|u_n - v_n\|_G = 0. \tag{3.25}$$

From the inequalities of (3.14) and (3.16), we have

$$\begin{aligned}\|d(u_n, v_n)\|_G &\le \|u_n - v_n\|_G + \alpha_n \|DU^*Uu_n - DU^*Uv_n\|_G \\ &\le (1+\mu)\|u_n - v_n\|_G .\end{aligned}$$

By the above two inequalities, we have

$$\lim_{n\to\infty} \|d(u_n, v_n)\|_G = 0. \tag{3.26}$$

As the sequence $\{u_n\}$ is bounded, it will have a cluster point $\bar{r}$ and the subsequence $\{u_{n_k}\}$ of the sequence, where the subsequence $\{u_{n_k}\}$ convergences weakly to the cluster point $\bar{r}$. The same can be said that the subsequence $\{v_{n_k}\}$ convergences weakly to $\bar{r}$ from (3.25).

Next, we will prove the point $\bar{r}$ is a solution of SEP. Substituting (3.19) via $r = r^*$, and combining with (3.21), we have:

$$\begin{aligned}\langle v_n - r^*, d(u_n, v_n)\rangle_G &\ge \langle v_n - r^*, \alpha_n DU^*Uv_n\rangle_G \\ &= \alpha_n \|Uv_n\|^2 .\end{aligned}$$

By the above inequality and Lemma 3.4, we have

$$\|Uv_{n_k}\|^2 \le \frac{1}{\alpha} \langle v_{n_k} - r^*, d(u_{n_k}, v_{n_k})\rangle_G .$$

From (3.26) and the bounded property of the subsequence $\{v_{n_k}\}$, we have

$$\lim_{k\to\infty} \|Uv_{n_k}\|^2 = 0, \tag{3.27}$$

and from the definition of $\bigtriangledown f_{D_1}(r)$ and (3.27), we have: $\lim_{k\to\infty} \|DU^*Uv_{n_k}\|_G = 0$. By (2.1) and Lemma 2.2, we have

$$\begin{aligned}\left\|v_{n_k} - P_S^G(v_{n_k})\right\|_G &\le \|u_{n_k} - v_{n_k} - \alpha_{n_k} DU^*Uu_{n_k}\|_G \\ &\le \|u_{n_k} - v_{n_k}\|_G + \alpha_{n_k} \|DU^*Uu_{n_k}\|_G \\ &\le \|u_{n_k} - v_{n_k}\|_G + \alpha_{n_k} \|DU^*Uu_{n_k} - DU^*Uv_{n_k})\|_G \\ &\quad + \alpha_{n_k} \|DU^*Uv_{n_k}\|_G \\ &\le (1+\mu)\|u_{n_k} - v_{n_k}\|_G \\ &\quad + \alpha_{n_k} \|DU^*Uv_{n_k}\|_G \to 0 \quad \text{as } k \to \infty.\end{aligned}$$

Which implies $\bar{r} \in S$. So according to the equation (3.27), we have

$$\begin{aligned}0 \le \|U\bar{r}\|^2 = \langle U\bar{r}, U\bar{r}\rangle = \langle \bar{r}, U^*U\bar{r}\rangle &= \lim_{k\to\infty} \langle v_{n_k}, U^*U\bar{r}\rangle = \lim_{k\to\infty} \langle Uv_{n_k}, U\bar{r}\rangle \\ &\le \lim_{k\to\infty} \|Uv_{n_k}\| \, \|U\bar{r}\| = 0,\end{aligned}$$

and it implies that $f(\bar{r})$=0. So from Proposition 3.1, the point $\bar{r}$ is a solution of SEP. So we can substitute $\bar{r} = r^*$ into (3.24), and obtain the sequence $\{\|u_n - \bar{r}\|_G\}$ is convergent. Moreover, the subsequence $\{u_{n_k}\}$ convergences weakly to the cluster point $\bar{r}$. From Lemma 2.3, we complete our proof. □

4. Numerical experiment

In this section, we present some numerical experiments to verify the feasibility of the algorithm we proposed to solve the SEP (1.1). Consider the LASSO problem that we all know as follows:

$$\min_{r\in\mathbb{U}^k} \frac{1}{2}\|Tr-b\|_2^2 \text{ subject to } \|r\|_1 \le \alpha, \tag{4.1}$$

where $T\in U^{m\times k}, m<k$, $\alpha>0$ and ℓ_1- norm is defined by $\|r\|_1=\sum_{i=1}^k |r_i|$. Due to the ℓ_1- norm constraint of (4.1), this problem has the possibility of finding a sparse solution of the SPE (1.1).

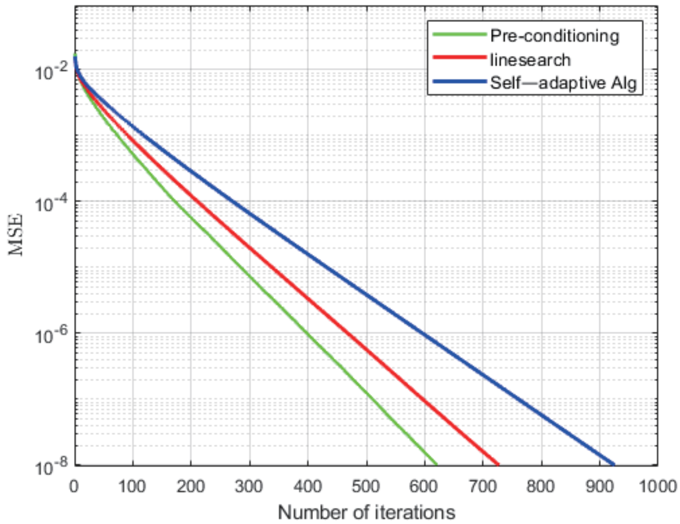

Figure 1. Graph of MSE values and number of iterations for $K=30$, when $MSE<10^{-8}$

We first evaluate the advantages of the proposed preconditioning method compared to other methods in the literature. We generate a matrix T with standard normal distribution. The true sparse signal r^* is constructed by uniform distribution in the interval $[-1,1]$ with K non-zero random elements. The observed signal $b=Tr^*$ (no noise is assumed). Considering the application of SEP in signal recovery experiments, a simple processing of SEP is required. Let $B=I\in U^{m\times m}$, $C=\{r\in\mathbb{U}^k:\|r\|_1\le\alpha\}$, $\alpha=K$, and $Q=\{b\}$. To implement the projection onto the convex sets, we give the definition of the soft threshold function as follows:

$$soft_\alpha(r)=\max(|r|-\alpha,0)\,\mathrm{sgn}(r).$$

In this experiment, the matrix T is the bounded linear operator. U, G and D_1 are randomly generated symmetric positive-definite matrices. The specific values of some of the parameters that we have selected are as follows.

For the constant step size, we choose $\gamma_n=\tau_n=0.7*\left(\frac{2}{\|T^*T\|^2}\right)$, $\sigma=2$, $\rho=0.2$ and $\beta_n=2$. For self-adaptive size, we used $\gamma_n=\tau_n$ and $\sigma=1.5$, $\rho=0.5$. For all numerical experimental sections, we chose m=256, k=512. The mean square error

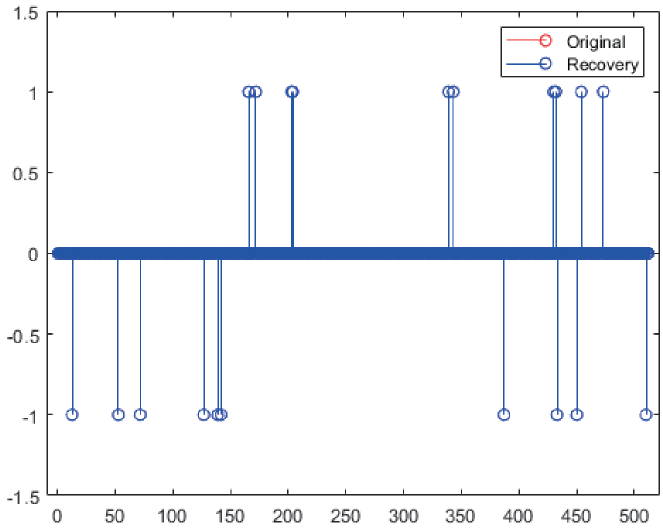

FIGURE 2. Recovered signal by pre-conditioning algorithm

TABLE 1. Numerical results of the algorithms with $m = 256$, $k = 512$

	Methods	$MSE < 10^{-4}$		$MSE < 10^{-6}$		$MSE < 10^{-8}$	
		Iter	CPU time	Iter	CPU time	Iter	CPU time
$K = 10$	pre-conditioning	126	0.0516	240	0.1256	411	0.1024
	line search Alg	129	0.0795	283	0.2207	447	0.3271
	self-adaptive Alg	141	2.3008	332	4.5139	458	8.6321
$K = 20$	pre-conditioning	165	0.0717	315	0.1022	535	0.1782
	line search Alg	203	0.1703	431	0.3352	702	0.4416
	self-adaptive Alg	248	3.0692	531	9.0110	821	13.8518
$K = 30$	pre-conditioning	172	0.0618	338	0.1120	611	0.1813
	line search Alg	232	0.1615	432	0.3682	711	0.5211
	self-adaptive Alg	251	4.2123	530	10.4242	922	14.0760

(MSE) method is used to assess the recovery accuracy: $MSE = \frac{1}{k}\|u_n - r^*\|^2$, where u_n is an estimated signal of r^*.

Table 1 shows that the preconditioning algorithm requires fewer iterations and less CPU time than the line search algorithm and the self-adaptive algorithm for different values of K and stopping criteria. It is evident from Fig 1 that the preconditioning algorithm has a faster convergence speed. Fig 2 demonstrates the application of the algorithm to signal recovery, showing a comparison between the original signal and the signal recovered by the preconditioning algorithm when $MSE < 10^{-6}$.

5. CONCLUSION

This article presents the application of line search and preconditioning techniques for addressing the split equality problem. Under specific circumstances, the weak convergence of the algorithm is proven. The preconditioning step offers two significant advantages. First, it improves the convergence speed. Second, it facilitates the use of line-search and self-adaptive techniques, thereby eliminating the need for prior knowledge of the operator norms. Finally, numerical experiments are conducted to verify the effectiveness of the proposed algorithms.

References

[1] H. H. Bauschke and J. M. Borwein, *On projection algorithms for solving convex feasibility problems*, SIAM Rev. **38** (1996), 367–426.

[2] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, CMS Books in Mathematics, 2011.

[3] D. P. Bertsekas and E. Gafni, *Projection methods for variational inequalities with application to the traffic assignment problem*, in: Nondifferential and Variational Techniques in Optimization, D. C. Sorensen, R. J. B. Wets, Mathematical Programming Studies, vol 17. Springer, Berlin, Heidelberg, 1982.

[4] A. Bnouhachem, M. A. Noor, M. Khalfaoui and S. Zhaohan, *On descent-projection method for solving the split feasibility problems*, J Glob Optim. **54** (2012), 627–639.

[5] C. L. Byrne, *Iterative projection onto convex sets using multiple Bregman distances*, Inverse Probl. **15** (1999), 1295–1313.

[6] C. L. Byrne, *Bregman-Legendre multidistance projection algorithms for convex feasibility and optimization*, Studies in Computational Mathematics **8** (2001), 87–99.

[7] C. L. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Problems **20** (2004), 103–120.

[8] Y. Censor, T. Bortfeld, B. Martin and A. Trofimov, *A unified approach for inversion problems in intensity-modulated radiation therapy*, Physics in Medicine and Biology **51** (2006), 2353–2365.

[9] Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer Algor **8** (1994), 221–239.

[10] Y. Censor, T. Elfving, N. Kopf and T. Bortfeld, *The multiple-sets split feasibility problem and its applications for inverse problems*, Inverse Problems **21** (2005), 2071–2084.

[11] S. S. Chang and R. P. Agarwal, *Strong convergence theorems of general split equality problems for quasi-nonexpansive mappings*, J. Inequal. Appl. **2014** (2014): 367.

[12] Q. Dong, S. He and J. Zhao, *Solving the split equality problem without prior knowledge of operator norms*, Optimization **64** (2015), 1887–1906.

[13] Q. L. Dong, Y. C. Tang, Y. J. Cho and Th. M. Rassias, *"Optimal" choice of the step length of the projection and contraction methods for solving the split feasibility problem*, J. Glob. Optim. **71** (2018), 341–360.

[14] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequality and Complementarity Problems*, vol. I., Spinger, New York, 2003.

[15] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequality and Complementarity Problems*, vol. II., Springer, New York, 2003.

[16] A. Gibali, L. Liu and Y. Tang, *Note on the modified relaxation CQ algorithm for the split feasibility problem*, Optimization Letters **12** (2018), 817–830.

[17] B. He, *Inexact implicit methods for monotone general variational inequalities*, Math. Program. **86** (1999), 199–217.

[18] B. V. Limaye, *Functional Analysis*, New Age, New Delhi, 1996.

[19] P. Marcotte and J. H. Wu, *On the convergence of projection methods: Application to the decomposition of affine variational inequalities*, J. Optim. Theory Appl. **85** (1995), 347–362.

[20] A. Moudafi, *Alternating CQ-algorithms for convex feasibility and split fixed-point problems*, J. Nonlilnear and Convex Anal. **15** (2014), 809–818.

[21] B. Qu and N. Xiu, *A note on the CQ algorithm for the split feasibility problem*, Inverse Problems **21** (2005), 1655–1665.

[22] L. Y. Shi, Q. H. Ansari, J. C. Yao and C. F. Wen, *Linear convergence of gradient projection algorithm for split equality problems*, Optimization **67** (2018), 2347–2358.
[23] L. Y. Shi, R. Chen and Y. Wu, *Strong convergence of iterative algorithms for the split equality problem*, J. Inequal. Appl. **2014** (2014): 478.
[24] T. Tian, L. Shi and R. Chen, *Linear convergence of the relaxed gradient projection algorithm for solving the split equality problems in Hilbert spaces*, J. Inequal. Appl. **2019** (2019): 80.
[25] T. M. Tuyen, *Regularization methods for the split equality problems in Hilbert spaces*, Bulletin of the Malaysian Mathematical Sciences Society **46** (2022), 1–20.
[26] P. T. Vuong, J. J. Strodiot and V. H. Nguyen, *A gradient projection method for solving split equality and split feasibility problems in Hilbert spaces*, Optimization, **64** (2014), 2321–2341.
[27] P. Wang and H. Zhou, *A preconditioning method of the CQ algorithm for solving an extended split feasibility problem*, J. Inequal. Appl. **2014** (2014): 163.
[28] P. Wang, H. Zhou and Y. Zhou, *Preconditioning methods for solving a general split feasibility problem*, J. Inequal. Appl. **2014** (2014): 435.
[29] Y. H. Wang, X. T. Li and B. N. Jiang, *Two new inertial relaxed gradient CQ algorithms on the split equality problem*, J. Appl. Anal. Comput. **12** (2022), 436–454.
[30] J. Zhao, Y. Li and X. Wang, *An accelerate algorithm for the split equality common fixed-point problem of directed operators*, Optimization **72** (2022), 2925–2950.

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