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SOME FIXED POINT THEOREMS BY EMPLOYING GREAT $(\mathfrak{s}, \mathfrak{e})$-SIMULATION FUNCTIONS ON b-METRIC SPACES

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Abstract. This paper explores novel fixed point theorems derived through the application of innovative great (s, e)-simulation functions within the framework of b-metric spaces. By introducing new classes of contractions that account for two distinct arguments, this investigation generalizes existing results and establishes unique fixed point theorems. An illustrative example is provided to demonstrate the practicality of these results.

1. Introduction

One of the fundamental conditions for the existence of fixed points for self-mappings on various spaces having the structure of completeness is the Banach contraction or its more generalized forms. These conditions have been widely studied in different mathematical frameworks. In many cases, such contractive conditions can be extended using control functions, which provide greater flexibility in analyzing the behavior of mappings. One interesting example is the concept of simulation functions, introduced by Khojasteh *et al.* [2], which played a crucial role in defining Z-contraction mappings. Following the introduction of these ideas, many researchers have pursued this line of investigation, resulting in a broad spectrum of generalizations of Banach contractions. For a broader perspective on these developments, the reader may consult the survey provided in [1].

Building on the concept of simulation functions and their extensions in metric spaces, researchers have increasingly explored applications within b-metric spaces. A major development is the inclusion of a multiplicative factor $\mathfrak{s} \geq 1$, which allows the definition of new control functions and generalized contractions. In 2017, Yamaod and Sintunavarat [8] introduced the notion of $\mathfrak{s}$-simulation functions, refining contractivity conditions through sequential properties involving $\mathfrak{s}$. This approach effectively extends fixed point results under broader assumptions. Subsequent works proposed large $\mathfrak{s}$-simulation functions in [3], relaxing restrictive conditions while preserving the essential framework for fixed point theorems. More recently, wide $\mathfrak{s}$-simulation functions and wide $W_{\mathfrak{s}}$-contractions were established in [7], removing the need for non-expansivity assumptions and further expanding the scope of fixed point theory in b-metric spaces.

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In this paper, we extend recent developments by introducing two new classes of functions, termed type-1 and type-2 great (s, e)-simulation functions, and employ them to derive new fixed point results in b-metric spaces. An illustrative example is also presented to demonstrate the effectiveness of the proposed approach.

2. Preliminaries

Throughout the paper, $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real numbers and positive integers, respectively, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Moreover, X denotes a nonempty set and $\mathfrak{s} \geq 1$ a fixed real number. We begin by recalling basic properties of sequences in a set without a metric structure.

Definition 2.1 (cf. [6])**.** A sequence $\{\mathsf{x}_n\}_{n\in\mathbb{N}} \subseteq \mathsf{X}$, where X is a nonempty set, is: *infinite* if $\mathsf{x}_n \neq \mathsf{x}_m$ for each $n, m \in \mathbb{N}$ such that $n \neq m$; *almost periodic* if there are $n_0, p_0 \in \mathbb{N}$ such that $\mathsf{x}_{n_0+r+kp_0} = \mathsf{x}_{n_0+r}$ for each $k \in \mathbb{N}$ and all $r \in \{0, 1, \ldots, p_0 - 1\}$ (this means that $\{\mathsf{x}_n\}_{n\geq n_0}$ is a periodic sequence because the terms $\{\mathsf{x}_{n_0}, \mathsf{x}_{n_0+1}, \mathsf{x}_{n_0+2}, \ldots, \mathsf{x}_{n_0+p_0-1}\}$ are infinitely repeated in the same order); *almost constant* if there is $n_0 \in \mathbb{N}$ such that $\mathsf{x}_n = \mathsf{x}_{n_0}$ for each $n \geq n_0$ (this means that $\{\mathsf{x}_n\}_{n\geq n_0}$ is a constant sequence).

A *Picard sequence* of a mapping $T : \mathsf{X} \to \mathsf{X}$ is a sequence $\{\mathsf{x}_n\}_{n\in\mathbb{N}} \subseteq \mathsf{X}$ such that $\mathsf{x}_{n+1} = T\mathsf{x}_n$ for all $n \in \mathbb{N}$.

A successful generalization of the notion of *metric space* is given by the following class of metric structures.

Definition 2.2. A *b-metric space* is a triple $(\mathsf{X}, \mathsf{d}, \mathfrak{s})$, where $\mathsf{d} : \mathsf{X} \times \mathsf{X} \to [0, \infty)$ satisfies the following properties for each $\mathsf{x}, \mathsf{y}, \mathsf{z} \in \mathsf{X}$:

- $\mathsf{d}(\mathsf{x}, \mathsf{y}) = 0$ if and only if $\mathsf{x} = \mathsf{y}$;
- $\mathsf{d}(\mathsf{y}, \mathsf{x}) = \mathsf{d}(\mathsf{x}, \mathsf{y})$;
- $\mathsf{d}(\mathsf{x}, \mathsf{z}) \leq \mathfrak{s}\, [\, \mathsf{d}(\mathsf{x}, \mathsf{y}) + \mathsf{d}(\mathsf{y}, \mathsf{z})\,]$.

The number $\mathfrak{s}$ is called the *coefficient* of the b-metric space.

Definition 2.3. Let $(\mathsf{X}, \mathsf{d}, \mathfrak{s})$ be a b-metric space. A sequence $\{\mathsf{x}_n\} \subseteq \mathsf{X}$ is *convergent to* $\mathsf{z} \in \mathsf{X}$ if $\lim_{n\to\infty} \mathsf{d}(\mathsf{x}_n, \mathsf{z}) = 0$, and it is a *Cauchy sequence* if $\lim_{n,m\to\infty} \mathsf{d}(\mathsf{x}_n, \mathsf{x}_m) = 0$. The space $(\mathsf{X}, \mathsf{d}, \mathfrak{s})$ is *complete* if every Cauchy sequence in X is convergent to a point of X. Moreover, a sequence $\{\mathsf{x}_n\} \subseteq X$ is *asymptotically regular* if $\lim_{n\to\infty} \mathsf{d}(\mathsf{x}_n, \mathsf{x}_{n+1}) = 0$.

In 2015, the following family of functions was introduced to generalize a great class of contractivity condition.

Definition 2.4 ([2])**.** A function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called a *simulation function* if it satisfies the following conditions:

(ζ_1): $\zeta(0, 0) = 0$;
(ζ_2): $\zeta(t, s) < s - t$ for all $t, s > 0$;
(ζ_3): if $\{t_n\}, \{s_n\} \subseteq (0, \infty)$ are sequences with $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

In order to avoid the symmetry between the sequences $\{t_n\}$ and $\{s_n\}$ in the last condition, the following modified version was also introduced:

Definition 2.5 ([5]). A *simulation function* is a function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the conditions (ζ_1), (ζ_2) and

(ζ_3'): if if $\{t_n\}, \{s_n\} \subseteq (0,\infty)$ are sequences with $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Inspired by simulation functions in the previous sense, in [8], the notion of $\mathfrak{s}$-*simulation function* was introduced. Such functions $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ must satisfy (ζ_2) and the following sequential-type condition:

(ζ_4): if $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,\infty)$ such that

$$0 < \liminf_{n\to\infty} \alpha_n \le \mathfrak{s} \limsup_{n\to\infty} \beta_n \le \mathfrak{s}^2 \liminf_{n\to\infty} \alpha_n,$$

$$0 < \liminf_{n\to\infty} \beta_n \le \mathfrak{s} \limsup_{n\to\infty} \alpha_n \le \mathfrak{s}^2 \liminf_{n\to\infty} \beta_n,$$

then $\limsup_{n\to\infty} \zeta(\alpha_n, \beta_n) < 0$.

After that, some researchers realized that the condition (ζ_2) was not necessary to prove some related fixed point theorems, so they defined in [3] the notion of *large* $\mathfrak{s}$-*simulation function* as any function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ only satisfying the condition (ζ_4).

3. Motivation for a new class of contractions

We begin this section with a result that follows directly from applying the triangle inequality twice, whose proof is omitted for brevity.

Proposition 3.1. *If* $(\mathsf{X}, \mathsf{d}, \mathfrak{s})$ *is a b-metric space and* $\mathsf{x}, \mathsf{y}, \mathsf{z}, \mathsf{w} \in \mathsf{X}$*, then*

$$\max\{\, \mathsf{d}(\mathsf{x},\mathsf{z}) - \mathfrak{s}^2 \mathsf{d}(\mathsf{y},\mathsf{w}),\ \mathsf{d}(\mathsf{y},\mathsf{w}) - \mathfrak{s}^2 \mathsf{d}(\mathsf{x},\mathsf{z}) \,\} \le \mathfrak{s}\, \mathsf{d}(\mathsf{x},\mathsf{y}) + \mathfrak{s}^2 \mathsf{d}(\mathsf{z},\mathsf{w}).$$

In particular, if $T : \mathsf{X} \to \mathsf{X}$ *is a mapping and* $x, y \in X$*, then*

$$\max\{\, \mathsf{d}(\mathsf{x},T\mathsf{x}) - \mathfrak{s}^2 \mathsf{d}(\mathsf{y},T\mathsf{y}),\ \mathsf{d}(\mathsf{y},T\mathsf{y}) - \mathfrak{s}^2 \mathsf{d}(\mathsf{x},T\mathsf{x}) \,\} \le \mathfrak{s}\, \mathsf{d}(\mathsf{x},\mathsf{y}) + \mathfrak{s}^2 \mathsf{d}(T\mathsf{x},T\mathsf{y}). \tag{3.1}$$

To describe the behavior of certain sequences that naturally arise in the proofs of the main theorems, we introduce the following result.

Lemma 3.2. *Let* $\{\mathsf{x}_n\}$ *be an asymptotically regular and infinite sequence in a b-metric space* $(\mathsf{X}, \mathsf{d}, \mathfrak{s})$*. If* $\{\mathsf{x}_n\}$ *is a not Cauchy sequence, then there are* $\varepsilon_0 > 0$ *and two partial subsequences* $\{\mathsf{x}_{n(k)}\}_{k\in\mathbb{N}}$ *and* $\{\mathsf{x}_{m(k)}\}_{k\in\mathbb{N}}$ *of* $\{\mathsf{x}_n\}$ *such that* $k < n(k) < m(k)$ *and* $\mathsf{d}(\mathsf{x}_{n(k)}, \mathsf{x}_{m(k)-1}) < \varepsilon_0 \le \mathsf{d}(\mathsf{x}_{n(k)}, \mathsf{x}_{m(k)})$ *for each* $k \in \mathbb{N}$*. Furthermore, associated to the sequences* $\{\mathsf{x}_{n(k)}\}$ *and* $\{\mathsf{x}_{m(k)}\}$ *of* X*, let consider the sequences of real numbers* $\{t_k\}, \{s_k\}, \{r_k\}, \{u_k\} \subseteq (0,\infty)$ *defined, for each* $k \in \mathbb{N}$*, as*

$$\begin{cases} t_k = \mathsf{d}(\mathsf{x}_{n(k)}, \mathsf{x}_{m(k)}), & s_k = \mathsf{d}(\mathsf{x}_{n(k)-1}, \mathsf{x}_{m(k)-1}), \\ r_k = \mathsf{d}(\mathsf{x}_{n(k)-1}, \mathsf{x}_{n(k)}), & u_k = \mathsf{d}(\mathsf{x}_{m(k)-1}, \mathsf{x}_{m(k)}). \end{cases} \tag{3.2}$$

Then the following properties hold.

1. *All the sequences* $\{t_k\}, \{s_k\}, \{r_k\}, \{u_k\}$ *are bounded.*
2. $r_k \to 0$ *and* $u_k \to 0$.
3. $0 < \limsup_{k\to\infty} s_k \le \mathfrak{s}\,\varepsilon_0 \le \mathfrak{s} \liminf_{k\to\infty} t_k$.
4. $0 < \limsup_{k\to\infty} t_k \le \mathfrak{s}\,\varepsilon_0 \le \mathfrak{s}^3 \liminf_{k\to\infty} s_k$.
5. $\max\{\, r_k - \mathfrak{s}^2 u_k,\ u_k - \mathfrak{s}^2 r_k \,\} \le \mathfrak{s} s_k + \mathfrak{s}^2 t_k$ *for all* $k \in \mathbb{N}$.

Finally, if $t_k \leq s_k$ *for all* $k \in \mathbb{N}$*, then* $0 < \limsup_{k\to\infty} t_k \leq \mathfrak{s} \liminf_{k\to\infty} t_k$.

Proof. As the sequence $\{\mathsf{x}_n\}$ is infinite, then $t_k, s_k, r_k, u_k \in (0,\infty)$ for all $k \in \mathbb{N}$. Since $\mathsf{d}(\mathsf{x}_n, \mathsf{x}_{n+1}) \to 0$, we obtain $\{r_k\}$ and $\{u_k\}$ are bounded sequences and they also converge to 0. Since $\{\mathsf{d}(\mathsf{x}_n, \mathsf{x}_{n+1})\}$ converges, it is bounded; hence, there exists $M > 0$ such that $\mathsf{d}(\mathsf{x}_n, \mathsf{x}_{n+1}) \leq M$ for all $n \in \mathbb{N}_0$. Notice that the sequence $\{\mathsf{t}_k\}_{k\in\mathbb{N}}$ verifies that

$$\begin{aligned} 0 < \varepsilon_0 \leq t_k &= \mathsf{d}(\mathsf{x}_{n(k)}, \mathsf{x}_{m(k)}) \\ &\leq \mathfrak{s}\left[\mathsf{d}(\mathsf{x}_{n(k)}, \mathsf{x}_{m(k)-1}) + \mathsf{d}(\mathsf{x}_{m(k)-1}, \mathsf{x}_{m(k)})\right] \\ &< \mathfrak{s}\left[\varepsilon_0 + \mathsf{d}(\mathsf{x}_{m(k)-1}, \mathsf{x}_{m(k)})\right] \\ &\leq \mathfrak{s}\left[\varepsilon_0 + M\right]. \end{aligned} \tag{3.3}$$

Then the sequence $\{t_k\} \subseteq (0,\infty)$ is bounded and so it has the limit inferior and the limit superior, and letting $k \to \infty$ in (3.3), we deduce that

$$0 < \varepsilon_0 \leq \liminf_{k\to\infty} t_k \leq \limsup_{k\to\infty} t_k \leq \mathfrak{s}\,\varepsilon_0. \tag{3.4}$$

By a similar technique to the previous step, we obtain that the sequence $\{s_k\} \subseteq (0,\infty)$ is also bounded and

$$0 \leq \liminf_{k\to\infty} s_k \leq \limsup_{k\to\infty} s_k \leq \mathfrak{s}\,\varepsilon_0. \tag{3.5}$$

Furthermore, for each $k \in \mathbb{N}$, we have

$$\begin{aligned} \varepsilon_0 \leq t_k &= \mathsf{d}(\mathsf{x}_{n(k)}, \mathsf{x}_{m(k)}) \\ &\leq \mathfrak{s}\,\mathsf{d}(\mathsf{x}_{n(k)}, \mathsf{x}_{n(k)-1}) + \mathfrak{s}^2\,\mathsf{d}(\mathsf{x}_{n(k)-1}, \mathsf{x}_{m(k)-1}) + \mathfrak{s}^2\,\mathsf{d}(\mathsf{x}_{m(k)-1}, \mathsf{x}_{m(k)}) \\ &= \mathfrak{s}\,\mathsf{d}(\mathsf{x}_{n(k)}, \mathsf{x}_{n(k)-1}) + \mathfrak{s}^2\, s_k + \mathfrak{s}^2\,\mathsf{d}(\mathsf{x}_{m(k)-1}, \mathsf{x}_{m(k)}), \end{aligned}$$

and letting $k \to \infty$ in the previous inequality, we deduce that $\varepsilon_0 \leq \mathfrak{s}^2 \liminf_{k\to\infty} s_k$. In particular, $\liminf_{k\to\infty} s_k \geq \dfrac{\varepsilon_0}{\mathfrak{s}^2} > 0$, which together with (3.5) implies that

$$0 < \frac{\varepsilon_0}{\mathfrak{s}^2} \leq \liminf_{k\to\infty} s_k \leq \limsup_{k\to\infty} s_k \leq \mathfrak{s}\,\varepsilon_0. \tag{3.6}$$

Joining (3.4) and (3.6), we deduce that

$$0 < \limsup_{k\to\infty} s_k \leq \mathfrak{s}\,\varepsilon_0 \leq \mathfrak{s} \liminf_{k\to\infty} t_k. \tag{3.7}$$

In addition to this, also by (3.4) and (3.6), we get

$$0 < \limsup_{k\to\infty} t_k \leq \mathfrak{s}\,\varepsilon_0 = \mathfrak{s}^3\,\frac{\varepsilon_0}{\mathfrak{s}^2} \leq \mathfrak{s}^3 \liminf_{k\to\infty} s_k. \tag{3.8}$$

This proves all the items of this result. Finally, if we assume that $t_k \leq s_k$ for all $k \in \mathbb{N}$, then, by (3.4), we get $0 < \varepsilon_0 \leq \limsup_{k\to\infty} t_k \leq \limsup_{k\to\infty} s_k$, which implies, by (3.7), that $0 < \limsup_{k\to\infty} t_k \leq \limsup_{k\to\infty} s_k \leq \mathfrak{s} \liminf_{k\to\infty} t_k$. □

4. Type-1 great $(\mathfrak{s},\mathfrak{e})$-simulation functions and fixed point theorems for type-1 great $(\mathfrak{s},\mathfrak{e})$-contractions

This section presents a generalization of existing simulation functions, providing a unified framework for new contraction types and fixed point theorems in b-metric spaces. Motivated by large simulation functions and Lemma 3.2, we first introduce the auxiliary functions employed herein.

Definition 4.1. Given $\mathfrak{s}\in[1,\infty)$ and $\mathfrak{e}\in\{1,3\}$, a function $F:(0,\infty)^4\to\mathbb{R}$ is called a *type-1 great* $(\mathfrak{s},\mathfrak{e})$*-simulation function* if the following conditions hold:

$(G^1_{\mathfrak{s},\mathfrak{e}})$**:** If $\{t_k\},\{s_k\},\{r_k\},\{u_k\}\subseteq(0,\infty)$ are bounded from above sequences such that

$$(4.1)\quad \begin{cases} (4.1.a) & 0<\limsup_{k\to\infty} s_k\leq \mathfrak{s}\left(\liminf_{k\to\infty} t_k\right), \\ (4.1.b) & 0<\limsup_{k\to\infty} t_k\leq \mathfrak{s}^{\mathfrak{e}}\left(\liminf_{k\to\infty} s_k\right), \\ (4.1.c) & \{r_k\} \text{ and } \{u_k\} \text{ converge to the same limit } L, \text{ and} \\ & \qquad L\leq\min\{\liminf_{k\to\infty} t_k,\ \liminf_{k\to\infty} s_k\}, \\ (4.1.d) & \max\{r_k-\mathfrak{s}^2u_k,\ u_k-\mathfrak{s}^2r_k\}\leq\mathfrak{s}s_k+\mathfrak{s}^2t_k \quad \text{for all } k\in\mathbb{N}, \\ (4.1.e) & t_k<s_k \quad \text{for all } k\in\mathbb{N}, \end{cases}$$

then

$$(4.2)\qquad \limsup_{k\to\infty} F(t_k,s_k,r_k,u_k)<0.$$

We denote by $\mathcal{G}^1_{\mathfrak{s},\mathfrak{e}}$ the family of all type-1 great $(\mathfrak{s},\mathfrak{e})$-simulation functions.

Some commentaries about the previous definition must be done.

Remark 4.2. (1) A type-1 great $(\mathfrak{s},\mathfrak{e})$-simulation function F is not required to be defined for (t,s,r,u) in the domain of F with any zero argument, since the proposed contractive condition never involves such cases. Consequently, the sequences $\{t_k\}$, $\{s_k\}$, $\{r_k\}$, and $\{u_k\}$ consist solely of positive real numbers.

(2) If $\mathfrak{s}=1$, conditions (4.1.a) and (4.1.b) imply that

$$0<\limsup_{k\to\infty} s_k\leq\liminf_{k\to\infty} t_k\leq\limsup_{k\to\infty} t_k\leq\liminf_{k\to\infty} s_k\leq\limsup_{k\to\infty} s_k<\infty.$$

Hence, $\{t_k\}$ and $\{s_k\}$ converge to the same positive real number.

(3) Since $1\leq\mathfrak{s}\leq\mathfrak{s}^3$, if two sequences $\{t_k\}$ and $\{s_k\}$ verify the condition (4.1.b) for $\mathfrak{e}=1$, then they also satisfy the same condition for $\mathfrak{e}=3$. Therefore, if $\{t_k\}$, $\{s_k\}$, $\{r_k\}$ and $\{u_k\}$ are sequences verifying the hypotheses of condition $(G^1_{\mathfrak{s},1})$, then they also satisfy the hypotheses of condition $(G^1_{\mathfrak{s},3})$. In particular, if $F\in\mathcal{G}^1_{\mathfrak{s},3}$, then necessarily $F\in\mathcal{G}^1_{\mathfrak{s},1}$, that is, $\mathcal{G}^1_{\mathfrak{s},3}\subseteq\mathcal{G}^1_{\mathfrak{s},1}$. However, such families are distinct in nature. If $F\in\mathcal{G}^1_{\mathfrak{s},1}$ and $\{t_k\},\{s_k\},\{r_k\},\{u_k\}\subseteq(0,\infty)$ verify $(G^1_{\mathfrak{s},1})$, then we have

$$\limsup_{k\to\infty} F(t_k,s_k,r_k,u_k)<0 \quad\text{and}\quad \limsup_{k\to\infty} F(s_k,t_k,r_k,u_k)<0,$$

that is, the first two arguments of $\digamma$ play a symmetric role. Nevertheless, if $\digamma \in \mathcal{G}^1_{\mathfrak{s},3}$ and $\{t_k\},\{s_k\},\{r_k\},\{u_k\} \subseteq (0,\infty)$ satify $(G^1_{\mathfrak{s},3})$, one can only conclude that $\limsup_{k\to\infty} \digamma(t_k,s_k,r_k,u_k) < 0$.

(4) In the limit, the condition (4.1.d) is not a restriction. Indeed, for all $k \in \mathbb{N}$,

$$\lim_{h\to\infty} \max\{\, r_h - \mathfrak{s}^2 u_h,\, u_h - \mathfrak{s}^2 r_h \,\} = L - \mathfrak{s}^2 L \leq 0 < \mathfrak{s} s_k + \mathfrak{s}^2 t_k.$$

(5) As a result, the condition (4.1.d) is a restriction to have a control on the sequences $\{r_k\}$ and $\{u_k\}$ in such way that they cannot take excessively large values. Although they are convergent sequences and, consequently, bounded sequences, their first terms could take arbitrary real values, and condition (4.1.d) establishes that the sequences $\{r_k\}$ and $\{u_k\}$ must take values according (in some sense, "respectful") to the sequences $\{t_k\}$ and $\{s_k\}$. Hence, arbitrarily large values must be avoided, and this fact provides us certain control on such sequences.

The following result provides extensive examples of type-1 great $(\mathfrak{s},\mathfrak{e})$-simulation functions.

Proposition 4.3. *Given a simulation function* $\zeta : [0,\infty)^2 \to \mathbb{R}$ *(in the sense of Definition 2.4 or 2.5), let define* $\digamma_\zeta : (0,\infty)^4 \to \mathbb{R}$ *by* $\digamma_\zeta(t,s,r,u) = \zeta(t,s)$ *for all* $(t,s,r,u) \in (0,\infty)^4$*. Then* $\digamma_\zeta \in \mathcal{G}^1_{1,\mathfrak{e}}$ *for all* $\mathfrak{e} \in \{1,3\}$.

Proof. Let $\{t_k\},\{s_k\},\{r_k\},\{u_k\} \subseteq (0,\infty)$ be bounded from above sequences verifying the hypotheses (4.1) for $\mathfrak{s} = 1$ and $\mathfrak{e} \in \{1,3\}$. In particular, $t_k < s_k$ for all $k \in \mathbb{N}$. By item 2 of Remark 4.2, as $\mathfrak{s} = 1$, $\{t_k\}$ and $\{s_k\}$ are sequences converging to the same positive real number. Since ζ is a simulation function (in the sense of Definition 2.4 or 2.5), condition (ζ_3) (or (ζ_3')) guarantees that $\limsup_{k\to\infty} \digamma_\zeta(t_k,s_k,r_k,u_k) = \limsup_{k\to\infty} \zeta(t_k,s_k) < 0$. Hence, $\digamma_\zeta \in \mathcal{G}^1_{1,\mathfrak{e}}$. □

Example 4.4. Let $\zeta : [0,\infty)^2 \to \mathbb{R}$ be a simulation function (in the sense of Definition 2.4 or in the sense of Definition 2.5) and let $\digamma : (0,\infty)^4 \to \mathbb{R}$ be a function satisfying $\digamma \leq \digamma_\zeta$, where $\digamma_\zeta$ is defined in the same manner as in Proposition 4.3. Then $\digamma \in \mathcal{G}^1_{1,\mathfrak{e}}$ for all $\mathfrak{e} \in \{1,3\}$.

Proposition 4.5. *If* $\digamma \in \mathcal{G}^1_{\mathfrak{s},\mathfrak{e}}$ *and* $0 < r_0 \leq t_0 < s_0 < \mathfrak{s} t_0$*, then*

$$\digamma(t_0,s_0,r_0,r_0) < 0.$$

Proof. Letting $t_k = t_0 > 0, s_k = s_0 > 0$ and $r_k = u_k = r_0 > 0$ for all $k \in \mathbb{N}$. Notice that the condition (4.1.d) holds because, for all $k \in \mathbb{N}$, $\max\{\, r_k - \mathfrak{s}^2 u_k,\, u_k - \mathfrak{s}^2 r_k \,\} = r_0(1-\mathfrak{s}^2) \leq 0 < \mathfrak{s} s_k + \mathfrak{s}^2 t_k$. □

Definition 4.6. Given $\mathfrak{e} \in \{1,3\}$, a mapping T from a b-metric space $(\mathsf{X},\mathsf{d},\mathfrak{s})$ into itself is a *type-1 great* $\mathcal{G}^1_{\mathfrak{s},\mathfrak{e}}$*-contraction* (or simply a *great* $\mathcal{G}^1_{\mathfrak{s},\mathfrak{e}}$*-contraction*) if it verifies the following two conditions:

(G^1_1)**:** $\mathsf{d}(\mathsf{Tx},\mathsf{Ty}) < \mathsf{d}(\mathsf{x},\mathsf{y})$ for each $\mathsf{x},\mathsf{y} \in \mathsf{X}$ with $\mathsf{x} \neq \mathsf{y}$;

(G^1_2)**:** there is a type-1 great $(\mathfrak{s},\mathfrak{e})$-simulation function $\digamma \in \mathcal{G}^1_{\mathfrak{s},\mathfrak{e}}$ such that, for all $\mathsf{x},\mathsf{y} \in \mathsf{X}$ with $\mathsf{x} \neq T\mathsf{x}$, $\mathsf{y} \neq T\mathsf{y}$ and $T\mathsf{x} \neq T\mathsf{y}$,

$$\digamma\left(\mathsf{d}(T\mathsf{x},T\mathsf{y}),\, \mathsf{d}(\mathsf{x},\mathsf{y}),\, \mathsf{d}(\mathsf{x},T\mathsf{x}),\, \mathsf{d}(\mathsf{y},T\mathsf{y})\right) \geq 0. \tag{4.3}$$

Remark 4.7. The contractivity condition (G_1^1) immediately implies that if a great $\mathcal{G}^1_{\mathfrak{s},\mathfrak{e}}$-contraction has a fixed point, then it is unique. Also notice that a type-1 great $(\mathfrak{s},\mathfrak{e})$-simulation function $\digamma$ needs not to be defined for each (t,s,r,u) in the domain of $\digamma$ with $t=0$ or $s=0$ or $r=0$ or $u=0$ because the contractivity condition (4.3) has not to be satisfied if one of its arguments is zero.

The main result in this section is the following one.

Theorem 4.8. *Each type-1 great $\mathcal{G}^1_{\mathfrak{s},\mathfrak{e}}$-contraction from a complete b-metric $(\mathsf{X},\mathsf{d},\mathfrak{s})$ into itself space has a unique fixed point, whatever $\mathfrak{e}\in\{1,3\}$. In fact, it is a Picard operator (that is, all its Picard sequences converge to its unique fixed point).*

Proof. Let $(\mathsf{X},\mathsf{d},\mathfrak{s})$ be a complete b-metric space and let $T:\mathsf{X}\to\mathsf{X}$ be a mapping satisfying the condition (4.3) associated to a type-1 great $(\mathfrak{s},\mathfrak{e})$-simulation function $\digamma:[0,\infty)^4\to\mathbb{R}$. Given a point $\mathsf{x}_0\in\mathsf{X}$, let $\{\mathsf{x}_n=\mathsf{T}^n\mathsf{x}_0\}_{n\in\mathbb{N}_0}$ be the Picard sequence of T starting from x_0. If there is $n_0\in\mathbb{N}_0$ with $\mathsf{x}_{n_0}=\mathsf{x}_{n_0+1}$, then x_{n_0} is a fixed point of T, and the uniqueness of such fixed point follows from Remark 4.7. On the contrary case, suppose that

$$\mathsf{d}(\mathsf{x}_n,\mathsf{x}_{n+1})>0 \quad \text{for all } n\in\mathbb{N}_0. \tag{4.4}$$

By the hypothesis (G_1^1), the sequence $\{\mathsf{d}(\mathsf{x}_n,\mathsf{x}_{n+1})\}$ verifies

$$0<\mathsf{d}(\mathsf{x}_{n+1},\mathsf{x}_{n+2})=\mathsf{d}(T\mathsf{x}_n,T\mathsf{x}_{n+1})<\mathsf{d}(\mathsf{x}_n,\mathsf{x}_{n+1}) \quad \text{for all } n\in\mathbb{N}_0.$$

As it is strictly decreasing and bounded from below, it is convergent. Let $L:=\lim_{n\to\infty}\mathsf{d}(\mathsf{x}_n,\mathsf{x}_{n+1})\geq 0$. To prove that $L=0$, suppose, by contradiction, that $L>0$. Let consider the sequences $\{t_n\},\{s_n\},\{r_n\},\{u_n\}$ defined by $t_n=u_n=\mathsf{d}(\mathsf{x}_{n+1},\mathsf{x}_{n+2})>0$ and $s_n=r_n=\mathsf{d}(\mathsf{x}_n,\mathsf{x}_{n+1})>0$ for all $n\in\mathbb{N}$. Then $\lim_{n\to\infty}t_n=\lim_{n\to\infty}s_n=\lim_{n\to\infty}r_n=\lim_{n\to\infty}u_n=L>0$. In particular, the first three conditions of (4.1) immediately hold, and the fourth follows from (3.1) applied to $\mathsf{x}=\mathsf{x}_n$ and $\mathsf{y}=\mathsf{x}_{n+1}$, that is,

$$\begin{aligned}
&\max\{\,r_n-\mathfrak{s}^2u_n,\,u_n-\mathfrak{s}^2r_n\,\}\\
&\quad=\max\{\,\mathsf{d}(\mathsf{x}_n,\mathsf{x}_{n+1})-\mathfrak{s}^2\,\mathsf{d}(\mathsf{x}_{n+1},\mathsf{x}_{n+2}),\,\mathsf{d}(\mathsf{x}_{n+1},\mathsf{x}_{n+2})-\mathfrak{s}^2\,\mathsf{d}(\mathsf{x}_n,\mathsf{x}_{n+1})\,\}\\
&\quad=\max\{\,\mathsf{d}(\mathsf{x}_n,T\mathsf{x}_n)-\mathfrak{s}^2\,\mathsf{d}(\mathsf{x}_{n+1},T\mathsf{x}_{n+1}),\,\mathsf{d}(\mathsf{x}_{n+1},T\mathsf{x}_{n+1})-\mathfrak{s}^2\,\mathsf{d}(\mathsf{x}_n,T\mathsf{x}_n)\,\}\\
&\quad\leq\mathfrak{s}\,\mathsf{d}(\mathsf{x}_n,\mathsf{x}_{n+1})+\mathfrak{s}^2\,\mathsf{d}(T\mathsf{x}_n,T\mathsf{x}_{n+1})\\
&\quad=\mathfrak{s}\,\mathsf{d}(\mathsf{x}_n,\mathsf{x}_{n+1})+\mathfrak{s}^2\,\mathsf{d}(\mathsf{x}_{n+1},\mathsf{x}_{n+2})\\
&\quad=\mathfrak{s}s_n+\mathfrak{s}^2t_n.
\end{aligned}$$

Now, (4.2) means that $\limsup_{n\to\infty}\digamma(t_n,s_n,r_n,u_n)<0$. However, this fact contradicts that, for all $n\in\mathbb{N}$, by (4.3),

$$\begin{aligned}
\digamma(t_n,s_n,r_n,u_n)&=\digamma\left(\mathsf{d}(\mathsf{x}_{n+1},\mathsf{x}_{n+2}),\,\mathsf{d}(\mathsf{x}_n,\mathsf{x}_{n+1}),\,\mathsf{d}(\mathsf{x}_n,\mathsf{x}_{n+1}),\,\mathsf{d}(\mathsf{x}_{n+1},\mathsf{x}_{n+2})\right)\\
&=\digamma\left(\mathsf{d}(T\mathsf{x}_n,T\mathsf{x}_{n+1}),\,\mathsf{d}(\mathsf{x}_n,\mathsf{x}_{n+1}),\,\mathsf{d}(\mathsf{x}_n,T\mathsf{x}_n),\,\mathsf{d}(\mathsf{x}_{n+1},T\mathsf{x}_{n+1})\right)\\
&\geq 0.
\end{aligned}$$

This contradiction proves that $L=0$, that is, $\lim_{n\to\infty}\mathsf{d}(\mathsf{x}_n,\mathsf{x}_{n+1})=0$. In particular, the sequence $\{\mathsf{x}_n\}$ is asymptotically regular, implying that it is either infinite or eventually constant (see [4,6,9] for details). By (4.4), the second case is impossible (it cannot be constant from a term onward). Then it is infinite, that is, $\mathsf{x}_n \neq \mathsf{x}_m$ for each $n, m \in \mathbb{N}$ such that $n \neq m$.

Next, we prove, by contradiction, that $\{\mathsf{x}_n\}$ is a Cauchy sequence in $(\mathsf{X}, \mathsf{d}, \mathfrak{s})$. If this property is false, Lemma 3.2 guarantees that there are $\varepsilon_0 > 0$ and two partial subsequences $\{\mathsf{x}_{n(k)}\}_{k\in\mathbb{N}}$ and $\{\mathsf{x}_{m(k)}\}_{k\in\mathbb{N}}$ of $\{\mathsf{x}_n\}$ such that the sequences $\{t_k\}, \{s_k\}, \{r_k\}, \{u_k\} \subseteq (0, \infty)$ defined, for each $k \in \mathbb{N}$, as (3.2) verify the following assertions:

$$\begin{cases} \bullet\ \{t_k\}, \{s_k\}, \{r_k\}, \{u_k\} \text{ are bounded;} \\ \bullet\ r_k \to 0 \text{ and } u_k \to 0; \\ \bullet\ 0 < \limsup_{k\to\infty} s_k \leq \mathfrak{s}\,\varepsilon_0 \leq \mathfrak{s} \liminf_{k\to\infty} t_k; \\ \bullet\ 0 < \limsup_{k\to\infty} t_k \leq \mathfrak{s}\,\varepsilon_0 \leq \mathfrak{s}^3 \liminf_{k\to\infty} s_k; \\ \bullet\ \max\{\, r_k - \mathfrak{s}^2 u_k,\, u_k - \mathfrak{s}^2 r_k \,\} \leq \mathfrak{s} s_k + \mathfrak{s}^2 t_k \quad \text{for all } k \in \mathbb{N}. \end{cases}$$

In fact, by condition (G_1^1), for all $k \in \mathbb{N}$,

$$t_k = \mathsf{d}(\mathsf{x}_{n(k)}, \mathsf{x}_{m(k)}) = \mathsf{d}(T\mathsf{x}_{n(k)-1}, T\mathsf{x}_{m(k)-1}) < \mathsf{d}(\mathsf{x}_{n(k)-1}, \mathsf{x}_{m(k)-1}) = s_k.$$

Hence, Lemma 3.2 guarantees that$0 < \limsup_{k\to\infty} t_k \leq \mathfrak{s} \liminf_{k\to\infty} s_k$. Therefore, the property $0 < \limsup_{k\to\infty} t_k \leq \mathfrak{s}^{\mathfrak{e}} \liminf_{k\to\infty} s_k$ holds whatever $\mathfrak{e} \in \{1, 3\}$. As T is a type-1 great $\mathcal{G}^1_{\mathfrak{s},\mathfrak{e}}$-contraction (for $\mathfrak{e} = 1$ or $\mathfrak{e} = 3$), then condition (4.2) ensures that $\limsup_{k\to\infty} F(t_k, s_k, r_k, u_k) < 0$. However, this inequality contradicts the fact that, for each $k \in \mathbb{N}$, using (4.3) with $= \mathsf{x}_{n(k)-1}$ and $\mathsf{y} = \mathsf{x}_{m(k)-1}$,

$$\begin{aligned} &F(t_k, s_k, r_k, u_k) \\ &\quad = F\left(\mathsf{d}(\mathsf{x}_{n(k)}, \mathsf{x}_{m(k)}),\, \mathsf{d}(\mathsf{x}_{n(k)-1}, \mathsf{x}_{m(k)-1}),\, \mathsf{d}(\mathsf{x}_{n(k)-1}, \mathsf{x}_{n(k)}),\, \mathsf{d}(\mathsf{x}_{m(k)-1}, \mathsf{x}_{m(k)})\right) \\ &\quad = F\left(\mathsf{d}(T\mathsf{x}_{n(k)-1}, T\mathsf{x}_{m(k)-1}),\, \mathsf{d}(\mathsf{x}_{n(k)-1}, \mathsf{x}_{m(k)-1}),\right. \\ &\qquad\qquad \left.\mathsf{d}(\mathsf{x}_{n(k)-1}, T\mathsf{x}_{n(k)-1}),\, \mathsf{d}(\mathsf{x}_{m(k)-1}, T\mathsf{x}_{m(k)-1})\right) \\ &\quad \geq 0. \end{aligned}$$

This contradiction guarantees that $\{\mathsf{x}_n\}$ is a Cauchy sequence in $(\mathsf{X}, \mathsf{d}, \mathfrak{s})$. As it is complete, then there is $\mathsf{z}_0 \in \mathsf{X}$ such that $\mathsf{x}_n \to \mathsf{z}_0$ as $n \to \infty$. As $\{\mathsf{x}_n\}$ is an infinite sequence, there is $n_0 \in \mathbb{N}$ such that $\mathsf{x}_n \neq \mathsf{z}_0$ and $\mathsf{x}_n \neq T\mathsf{z}_0$ for all $n \geq n_0$. Therefore, (G_1^1) guarantees that $\mathsf{d}(\mathsf{x}_{n+1}, \mathsf{T}\mathsf{z}_0) = \mathsf{d}(\mathsf{T}\mathsf{x}_n, \mathsf{T}\mathsf{z}_0) < \mathsf{d}(\mathsf{x}_n, \mathsf{z}_0)$ for all $n \geq n_0$. Hence, $\mathsf{x}_{n+1} = T\mathsf{x}_n \to T\mathsf{z}_0$ as $n \to \infty$, and the uniqueness of the limit of a convergent sequence in a b-metric space finally guarantees that $T\mathsf{z}_0 = \mathsf{z}_0$, that is, z_0 is a fixed point of T. The uniqueness of the fixed point follows from Remark 4.7. □

The previous result can be particularized in several ways. We illustrate some of them.

Corollary 4.9. *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{s})$ be a complete b-metric space and let $T : \mathsf{X} \to \mathsf{X}$ be a mapping. Suppose that there exists a simulation function $\zeta : [0, \infty)^2 \to \mathbb{R}$ (in the sense of Definition 2.4 or in the sense of Definition 2.5) such that*

$$\zeta\left(\mathsf{d}(T\mathsf{x}, T\mathsf{y}),\, \mathsf{d}(\mathsf{x}, \mathsf{y})\right) \geq 0$$

for all $\mathsf{x}, \mathsf{y} \in \mathsf{X}$ with $\mathsf{x} \neq \mathsf{y}$. Then T is a Picard operator.

Proof. Let consider the function $\digamma_{\zeta} : (0,\infty)^4 \to \mathbb{R}$ defined by Proposition 4.3. Proposition 4.3 ensures that $\digamma_{\zeta} \in \mathcal{G}^1_{1,\mathfrak{e}}$ for $\mathfrak{e} \in \{1,3\}$. By condition (ζ_2), for all $\mathsf{x},\mathsf{y} \in \mathsf{X}$ with $T\mathsf{x} \neq T\mathsf{y}$,

$$0 \le \zeta\left(\mathsf{d}(T\mathsf{x},T\mathsf{y}),\, \mathsf{d}(\mathsf{x},\mathsf{y})\right) < \mathsf{d}(\mathsf{x},\mathsf{y}) - \mathsf{d}(T\mathsf{x},T\mathsf{y}).$$

Hence, T is a type-1 great $\mathcal{G}^1_{\mathfrak{s},\mathfrak{e}}$-contraction with respect to $\digamma_{\zeta}$, and Theorem 4.8 guarantees that T is a Picard operator. □

The case $\mathfrak{s} = 1$ is the following one.

Corollary 4.10. *Let* (X,d) *be a complete metric space and let* $\digamma : (0,\infty)^4 \to \mathbb{R}$ *be a function such that* $\limsup_{k\to\infty} \digamma(t_k,s_k,r_k,u_k) < 0$ *whatever the sequences* $\{t_k\},\{s_k\},\{r_k\},\{u_k\} \subseteq (0,\infty)$ *verifying:*

$$\begin{cases} \bullet\ \{t_k\} \text{ and } \{s_k\} \text{ converge to the same limit } L > 0, \\ \bullet\ t_k < s_k \quad \text{for all } k \in \mathbb{N}, \\ \bullet\ \{r_k\} \text{ and } \{u_k\} \text{ converge to the same limit } L', \text{ and } L' \le L, \\ \bullet\ |\,u_k - r_k\,| \le s_k + t_k \quad \text{for all } k \in \mathbb{N}. \end{cases}$$

If for each $\mathsf{x},\mathsf{y} \in \mathsf{X}$*, a mapping* $T : \mathsf{X} \to \mathsf{X}$ *verifies*

$$\begin{cases} \bullet\ \mathsf{d}(\mathsf{Tx},\mathsf{Ty}) < \mathsf{d}(\mathsf{x},\mathsf{y}) \text{ when } \mathsf{x} \neq \mathsf{y}, \\ \bullet\ \digamma\left(\mathsf{d}(T\mathsf{x},T\mathsf{y}),\, \mathsf{d}(\mathsf{x},\mathsf{y}),\, \mathsf{d}(\mathsf{x},T\mathsf{x}),\, \mathsf{d}(\mathsf{y},T\mathsf{y})\right) \ge 0 \\ \qquad\qquad \text{when } \mathsf{x} \neq T\mathsf{x},\ \mathsf{y} \neq T\mathsf{y} \text{ and } T\mathsf{x} \neq T\mathsf{y}, \end{cases}$$

then, T is a Picard operator.

5. Type-2 great $(\mathfrak{s},\mathfrak{e})$-simulation functions and fixed point theorems for type-2 great $(\mathfrak{s},\mathfrak{e})$-contractions

For a self-mapping T on a b-metric space $(\mathsf{X},\mathsf{d},\mathfrak{s})$, the classical contractive condition

$$(G^1_1) \qquad \mathsf{d}(\mathsf{Tx},\mathsf{Ty}) < \mathsf{d}(\mathsf{x},\mathsf{y}) \text{ for all } \mathsf{x},\mathsf{y} \in \mathsf{X} \text{ with } \mathsf{x} \neq \mathsf{y},$$

though natural in fixed point theory, is often overly restrictive since it enforces continuity of T and guarantees at least one unique fixed point. To relax this, we adopt the following more general assumption:

$$(G^2_1) \qquad \text{for each } \mathsf{x}_0 \in \mathsf{X},\ \lim_{n\to\infty} \mathsf{d}(\mathsf{T}^n\mathsf{x}_0, \mathsf{T}^{n+1}\mathsf{x}_0) \text{ exists.}$$

This modification demands stronger conditions on the associated great simulation functions, making (G^2_2) similar in form but distinct in nature.

Definition 5.1. Given $\mathfrak{s} \in [1,\infty)$ and $\mathfrak{e} \in \{1,3\}$, a function $\digamma : [0,\infty)^4 \to \mathbb{R}$ is called a *type-2 great* $(\mathfrak{s},\mathfrak{e})$*-simulation function* if it satisfies the following property:

($G^2_{\mathfrak{s},\mathfrak{c}}$): If $\{t_k\},\{s_k\}\subseteq(0,\infty)$ and $\{r_k\},\{u_k\}\subseteq[0,\infty)$ are bounded from above sequences such that

$$(5.1)\quad \begin{cases} (5.1.a) & 0<\limsup_{k\to\infty} s_k \leq \mathfrak{s}\left(\liminf_{k\to\infty} t_k\right), \\ (5.1.b) & 0<\limsup_{k\to\infty} t_k \leq \mathfrak{s}^{\mathfrak{c}}\left(\liminf_{k\to\infty} s_k\right), \\ (5.1.c) & \{r_k\} \text{ and } \{u_k\} \text{ converge to the same limit } L, \text{ and} \\ & \qquad L\leq \min\{\liminf_{k\to\infty} t_k, \liminf_{k\to\infty} s_k\}, \\ (5.1.d) & \max\{r_k-\mathfrak{s}^2u_k,\, u_k-\mathfrak{s}^2r_k\}\leq \mathfrak{s}s_k+\mathfrak{s}^2t_k \quad \text{for all } k\in\mathbb{N}, \end{cases}$$

then

$$(5.2)\qquad \limsup_{k\to\infty} F(t_k,s_k,r_k,u_k)<0.$$

We denote by $\mathcal{G}^2_{\mathfrak{s},\mathfrak{c}}$ the family of all type-2 great $(\mathfrak{s},\mathfrak{c})$-simulation functions.

Items 2, 3, 4 and 5 of Remark 4.2 can be here identically stated. However, the following facts must be highlighted.

Remark 5.2. (1) Although type-2 great $(\mathfrak{s},\mathfrak{c})$-simulation functions are defined on $(0,\infty)^4$, a type-2 great $(\mathfrak{s},\mathfrak{c})$-simulation functions must be defined on $[0,\infty)^4$. The extension to null values will be of importance in order to show that the new contractions will have a unique fixed point. Deeply studying the proof of the results, the reader can observe that it is only necessary to define this kind of functions in $(0,\infty)^2\times[0,\infty)^2$.

(2) The condition (4.1.e) is here avoided because we will not have a control on the comparison of the sequences $\{t_k\}$ and $\{s_k\}$.

(3) Here we again know that $\mathcal{G}^2_{\mathfrak{s},3}\subseteq\mathcal{G}^2_{\mathfrak{s},1}$. However, the main result is only proved by employing type-2 great $(\mathfrak{s},3)$-simulation functions because we will not be able to guarantee the condition (5) of Lemma3.2 because $\{t_k\}$ and $\{s_k\}$ cannot be directly compared.

Another difference between type-1 and type-2 great $(\mathfrak{s},\mathfrak{c})$-simulation functions is the fact that we can use $r_0=0$ in the following result.

Proposition 5.3. *If $F\in\mathcal{G}^2_{\mathfrak{s},\mathfrak{c}}$ and $0\leq r_0<t_0$, then $F(t_0,t_0,r_0,r_0)<0$. In particular, $F(t_0,t_0,0,0)<0$ for each $t_0>0$.*

Proof. Letting $t_k=s_k=t_0>0$ and $r_k=u_k=r_0$ for all $k\in\mathbb{N}$. Notice that the condition (5.1.4) holds because, for all $k\in\mathbb{N}$, $\max\{r_k-\mathfrak{s}^2u_k,\, u_k-\mathfrak{s}^2r_k\}=r_0(1-\mathfrak{s}^2)\leq 0<\mathfrak{s}s_k+\mathfrak{s}^2t_k$. □

Definition 5.1 was inspired by simulations functions in the sense of Definition 2.4, as we show now.

Proposition 5.4. *Given a simulation function $\zeta:[0,\infty)^2\to\mathbb{R}$ (in the sense of Definition 2.4), let define $F_\zeta:[0,\infty)^4\to\mathbb{R}$ by*

$$F_\zeta(t,s,r,u)=\zeta(t,s)\quad \text{for all } (t,s,r,u)\in[0,\infty)^4.$$

Then $F_\zeta\in\mathcal{G}^2_{1,\mathfrak{c}}$ for $\mathfrak{c}\in\{1,3\}$.

Proof. The proof is exactly the same of the proof of Proposition 4.3. □

Remark 5.5. Notice that the previous result would be false for simulation functions in the sense of Definition 2.5 because we cannot deduce that $t_n < s_n$ for all $n \in \mathbb{N}$ before applying condition (ζ_3).

Proposition 5.6. *Given a large* $\mathfrak{s}$*-simulation function* $\zeta : [0,\infty)^2 \to \mathbb{R}$ *(in [3, Definition 3.1]), let define* $\widetilde{F}_\zeta : [0,\infty)^4 \to \mathbb{R}$ *by* $\widetilde{F}_\zeta(t,s,r,u) = \zeta(t,s)$ *for all* $(t,s,r,u) \in [0,\infty)^4$*. Then* $\widetilde{F}_\zeta \in \mathcal{G}^2_{\mathfrak{s},1}$.

Proof. The proof is similar to the proof of Proposition 4.3. □

Now, the new contraction that we study in this section is introduced.

Definition 5.7. Given $\mathfrak{c} \in \{1,3\}$, a mapping $T : \mathsf{X} \to \mathsf{X}$ from a b-metric space $(\mathsf{X},\mathsf{d},\mathfrak{s})$ into itself is a *type-2 great* $\mathcal{G}^2_{\mathfrak{s},\mathfrak{c}}$*-contraction* (or simply a *great* $\mathcal{G}^2_{\mathfrak{s},\mathfrak{c}}$*-contraction*) if it verifies the following two conditions:

(G^2_1)**:** for each $\mathsf{x}_0 \in \mathsf{X}$, the limit $\lim_{n\to\infty} \mathsf{d}(\mathsf{T}^n\mathsf{x}_0, \mathsf{T}^{n+1}\mathsf{x}_0)$ exists;

(G^2_2)**:** there is a type-2 great $(\mathfrak{s},\mathfrak{c})$-simulation function $F \in \mathcal{G}^2_{\mathfrak{s},\mathfrak{c}}$ such that, for all $\mathsf{x},\mathsf{y} \in \mathsf{X}$ with $\mathsf{x} \neq \mathsf{y}$,

$$F\left(\mathsf{d}(T\mathsf{x},T\mathsf{y}),\, \mathsf{d}(\mathsf{x},\mathsf{y}),\, \mathsf{d}(\mathsf{x},T\mathsf{x}),\, \mathsf{d}(\mathsf{y},T\mathsf{y})\right) \geq 0. \tag{5.3}$$

The following result follows directly from Proposition 5.3, and thus the proof is omitted.

Lemma 5.8. *If a great* $\mathcal{G}^2_{\mathfrak{s},\mathfrak{c}}$*-contraction has a fixed point, then it is unique.*

In the following result, which is only proved for $\mathfrak{c} = 3$, we need to assume the continuity of the contraction.

Theorem 5.9. *Each continuous type-2 great* $\mathcal{G}^2_{\mathfrak{s},3}$*-contraction from a complete b-metric space with coefficient* $\mathfrak{s}$ *into itself is a Picard operator.*

Proof. Let $(\mathsf{X},\mathsf{d},\mathfrak{s})$ be a complete b-metric space and let $T : \mathsf{X} \to \mathsf{X}$ be a mapping satisfying the condition (5.3) associated to a great $(\mathfrak{s},3)$-simulation function $F : [0,\infty)^4 \to \mathbb{R}$. Using exactly the same arguments of the proof of Theorem 4.8, we can prove that any Picard sequence $\{\mathsf{x}_n = \mathsf{T}^n\mathsf{x}_0\}$ converges. Notice that condition (G^2_1) guarantees that the sequence $\{\mathsf{d}(\mathsf{T}^n\mathsf{x}_0, \mathsf{T}^{n+1}\mathsf{x}_0)\}$ converges, and later we prove that its limit is zero. Also, we cannot demonstrate that $t_k < s_k$ for all $k \in \mathbb{N}$, but this condition is not required to apply the property $(G^2_{\mathfrak{s},3})$ to get a contradiction (here we are assuming that $\mathfrak{c} = 3$, so the property $(G^2_{\mathfrak{s},1})$ cannot be applied).

Having in mind that there is $\mathsf{z}_0 \in \mathsf{X}$ such that $\mathsf{x}_n \to \mathsf{z}_0$, the continuity of T guarantees that $\mathsf{x}_{n+1} = T\mathsf{x}_n \to T\mathsf{z}_0$, and the uniqueness of the limit of a convergent sequence in a b-metric space finally proves that $T\mathsf{z}_0 = \mathsf{z}_0$, that is, z_0 is a fixed point of T. The uniqueness of the fixed point follows from Lemma 5.8. □

There are several simple ways to guarantee that the mapping T is continuous. For instance, the following ones.

Corollary 5.10. *Each type-2 great* $\mathcal{G}^2_{\mathfrak{s},3}$*-contraction* $T : \mathsf{X} \to \mathsf{X}$ *from a complete b-metric* $(\mathsf{X},\mathsf{d},\mathfrak{s})$ *into itself space is a Picard operator (in particular, it has a unique fixed point) if it satisfies at least one of the following properties:*

- $\mathsf{d}(\mathsf{Tx},\mathsf{Ty}) \leq \lambda\,\mathsf{d}(\mathsf{x},\mathsf{y})$ *for all* $\mathsf{x},\mathsf{y} \in \mathsf{X}$ *with* $\mathsf{x} \neq \mathsf{y}$*, where* $\lambda \in [0,\infty)$*;*
- $\mathsf{d}(\mathsf{Tx},\mathsf{Ty}) \leq \sum_{n=1}^{N} \lambda_n\,\mathsf{d}(\mathsf{x},\mathsf{y})^n$ *for each* $\mathsf{x},\mathsf{y} \in \mathsf{X}$ *with* $\mathsf{x} \neq \mathsf{y}$*, where* $\lambda_1,\lambda_2,\ldots,\lambda_N \in [0,\infty)$*;*
- $\mathsf{d}(\mathsf{Tx},\mathsf{Ty}) \leq f(\mathsf{d}(\mathsf{x},\mathsf{y}))$ *for each* $\mathsf{x},\mathsf{y} \in \mathsf{X}$ *with* $\mathsf{x} \neq \mathsf{y}$*, where* $f:[0,\infty)\to[0,\infty)$ *satisfies* $\lim_{t\to 0^+} f(t) = 0$*.*

Proof. In all cases, T is continuous and so Theorem 5.9 is applicable. □

The following example show an illustrative framework in which Theorem 4.7 is not applicable, but Theorem 5.9 is so.

Example 5.11. Let $\mathsf{X} = [0,2] \cup \{7, 7.6\}$ be endowed with $\mathsf{d}(\mathsf{x},\mathsf{y}) = (\mathsf{x}-\mathsf{y})^2$ for all $\mathsf{x},\mathsf{y} \in \mathsf{X}$. Then d is a b-metric with coefficient $\mathfrak{s} = 2$. Let define $T : \mathsf{X} \to \mathsf{X}$ as follows:

$$T\mathsf{x} = \begin{cases} \dfrac{\mathsf{x}}{3}, & \text{if } \mathsf{x} \in [0,2], \\ 0, & \text{if } \mathsf{x} = 7, \\ 2, & \text{if } \mathsf{x} = 7.6. \end{cases}$$

If $\mathsf{x}_0 = 7$ and $\mathsf{y}_0 = 7.6$, then $\mathsf{d}(\mathsf{x}_0,\mathsf{y}_0) = 0.6^2 = 0.36$ and $\mathsf{d}(T\mathsf{x}_0, T\mathsf{y}_0) = \mathsf{d}(0,2) = 2^2 = 4$. Since $\mathsf{d}(T\mathsf{x}_0, T\mathsf{y}_0) > \mathsf{d}(\mathsf{x}_0,\mathsf{y}_0)$, then T is not a great $\mathcal{G}^1_{2,\mathfrak{e}}$-contraction mapping (condition (G^1_1) is not fulfilled), so Theorem 4.7 is not applicable to T. Nevertheless, let show that Theorem 5.9 is applicable because T is a continuous great $\mathcal{G}^2_{2,3}$-contraction and, consequently, Theorem 5.9 guarantees that T is a Picard operator (in particular, it has a unique fixed point).

First, we can see that the mapping T verifies the inequality

$$\mathsf{d}(\mathsf{Tx},\mathsf{Ty}) \leq 10\,\mathsf{d}(\mathsf{x},\mathsf{y}) \quad \text{for each } \mathsf{x},\mathsf{y} \in \mathsf{X}.$$

Hence, it is continuous. Next, the condition (G^2_1) holds because $T^n\mathsf{x} \in [0,2]$ for all $n \geq 1$, and $T^n\mathsf{x} = (T\mathsf{x})/3^{n-1}$ for all $n \geq 2$. To prove the hypothesis (G^2_2), let $\digamma : [0,\infty)^4 \to \mathbb{R}$ be the mapping defined, for each $t,s,r,u \in [0,\infty)$, as:

$$\digamma(t,s,r,u) = \begin{cases} (\,2\pi s - 5\pi t\,)\,(\,1 + |r-u|^3\,), & \text{if } 0 < t < 10s \text{ and } 0 < s < 2.5t, \\ 0, & \text{otherwise.} \end{cases}$$

In the rest of example, we will prove that $\digamma$ is a great $\mathcal{G}^2_{2,3}$-simulation function and T is a great $\mathcal{G}^2_{2,3}$-contraction with respect to $\digamma$. Let show that $\digamma$ is a great $\mathcal{G}^2_{2,3}$-simulation function. Since $\mathfrak{s} = 2$ and $\mathfrak{e} = 3$, let $\{t_n\},\{s_n\} \subseteq (0,\infty)$ and $\{r_n\},\{u_n\} \subseteq [0,\infty)$ be bounded from above sequences verifying:

$$\begin{cases} 0 < \limsup_{k\to\infty} s_k \leq \mathfrak{s}\,\liminf_{k\to\infty} t_k = 2\,\liminf_{k\to\infty} t_k, \\[2ex] 0 < \limsup_{k\to\infty} t_k \leq \mathfrak{s}^{\mathfrak{e}}\,\liminf_{k\to\infty}\, s_k = 8\,\liminf_{k\to\infty} s_k, \\[2ex] \{r_k\} \text{ and } \{u_k\} \text{ converge to the same limit } L. \end{cases}$$

By the first two conditions, there is $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$0.9\left(\liminf_{k\to\infty}\, t_k\right) \leq t_n \leq 1.1\left(\limsup_{k\to\infty}\, t_k\right)$$

and

$$0.9\left(\liminf_{k\to\infty} s_k\right) \le s_n \le 1.1\left(\limsup_{k\to\infty} s_k\right).$$

Therefore, for all $n \ge n_0$,

$$\begin{aligned} 0 < t_n &\le 1.1\left(\limsup_{k\to\infty} t_k\right) \le 1.1(8)\left(\liminf_{k\to\infty} s_k\right) \\ &= \frac{88}{9}\left(0.9 \liminf_{k\to\infty} s_k\right) \le \frac{88}{9} s_n < 10 s_n, \end{aligned}$$

and also

$$\begin{aligned} 0 < s_n &\le 1.1\left(\limsup_{k\to\infty} s_k\right) \le 1.1(2)\left(\liminf_{k\to\infty} t_k\right) \\ &= \frac{22}{9}\cdot\left(0.9 \liminf_{k\to\infty} t_k\right) \le \frac{22}{9} t_n. \end{aligned} \tag{5.4}$$

Since $0 < t_n < 10 s_n$ and $0 < s_n \le \frac{22}{9} t_n < 2.5\, t_n$ for all $n \ge n_0$, we obtain

$$\digamma(t_n, s_n, r_n, u_n) = (2\pi s_n - 5\pi t_n)(1 + |r_n - u_n|^3)$$

for all $n \ge n_0$. It follows from $\{r_k\}$ and $\{u_k\}$ converge to the same limit L that $\lim_{n\to\infty}(1 + |r_n - u_n|^3) = 1$. By (5.4), we get

$$2\pi s_n - 5\pi t_n = 2\pi\left(s_n - \frac{22}{9} t_n\right) - \frac{\pi}{9} t_n \le -\frac{\pi}{18} t_n.$$

This implies that

$$\begin{aligned} \limsup_{n\to\infty} \digamma(t_n, s_n, r_n, u_n) &= \limsup_{n\to\infty} \left[(2\pi s_n - 5\pi t_n)(1 + |r_n - u_n|^3)\right] \\ &= \limsup_{n\to\infty} (2\pi s_n - 5\pi t_n) \\ &\le \limsup_{n\to\infty} \left(-\frac{\pi}{18} t_n\right) \\ &= -\frac{\pi}{18} \liminf_{n\to\infty} t_n \\ &< 0. \end{aligned}$$

Therefore, $\digamma$ is a great $\mathcal{G}_{2,3}^2$-simulation function.

Next we prove that T is a great $\mathcal{G}_{2,3}^2$-contraction with respect to $\digamma$. Let $\mathsf{x}, \mathsf{y} \in \mathsf{X}$ be such that $\mathsf{x} \ne \mathsf{y}$. We are going to show that it is impossible to satisfy, at the same time,

$$0 < \mathsf{d}(T\mathsf{x}, T\mathsf{y}) < 10\,\mathsf{d}(\mathsf{x}, \mathsf{y}) \quad \text{and} \quad 0 < \mathsf{d}(\mathsf{x}, \mathsf{y}) < 2.5\,\mathsf{d}(T\mathsf{x}, T\mathsf{y}). \tag{5.5}$$

Hence, $\digamma\left(\mathsf{d}(T\mathsf{x}, T\mathsf{y}), \mathsf{d}(\mathsf{x}, \mathsf{y}), \mathsf{d}(\mathsf{x}, T\mathsf{x}), \mathsf{d}(\mathsf{y}, T\mathsf{y})\right) \ge 0$ (that is, T is a great $\mathcal{G}_{2,3}^2$-contraction with respect to $\digamma$). We consider the following cases.

- If $\mathsf{x}, \mathsf{y} \in [0, 2]$, then $\mathsf{d}(\mathsf{x}, \mathsf{y}) = (\mathsf{x} - \mathsf{y})^2$ and $\mathsf{d}(T\mathsf{x}, T\mathsf{y}) = \mathsf{d}(\mathsf{x}/3, \mathsf{y}/3) = (\mathsf{x} - \mathsf{y})^2/9$. Therefore,

$$0 < 2.5\,\mathsf{d}(T\mathsf{x}, T\mathsf{y}) = 2.5\,\frac{(\mathsf{x} - \mathsf{y})^2}{9} < (\mathsf{x} - \mathsf{y})^2 = \mathsf{d}(\mathsf{x}, \mathsf{y})$$

  and so the second inequality in (5.5) is false.

- If $\mathsf{x} \in [0,2]$ and $\mathsf{y} \in \{7, 7.6\}$, then $\mathsf{d}(\mathsf{x},\mathsf{y}) = (\mathsf{x}-\mathsf{y})^2 \geq (7-2)^2 = 25$. As $T\mathsf{x}, T\mathsf{y} \in [0,2]$, then $\mathsf{d}(T\mathsf{x}, T\mathsf{y}) \leq 4$. Hence
$$2.5\,\mathsf{d}(T\mathsf{x}, T\mathsf{y}) \leq 2.5 \cdot 4 = 10 < 25 \leq \mathsf{d}(\mathsf{x},\mathsf{y}),$$
so the second inequality in (5.5) does not hold.
- If $\mathsf{x} \in \{7, 7.6\}$ and $\mathsf{y} \in [0,2]$, the same conclusion holds because d is symmetric.
- If $\mathsf{x}, \mathsf{y} \in \{7, 7.6\}$, then either $\mathsf{x} = 7$ and $\mathsf{y} = 7.6$, or vice-versa (because $\mathsf{x} \neq \mathsf{y}$). Hence, $10\,\mathsf{d}(\mathsf{x},\mathsf{y}) = 10(0.36) = 3.6 < 4 = \mathsf{d}(0,2) = \mathsf{d}(T\mathsf{x}, T\mathsf{y})$, which means that the first condition in (5.5) does not hold.

As a result, T is a great $\mathcal{G}^2_{2,3}$-contraction with respect to $\digamma$.

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