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A NOVEL CLASS OF GENERALIZED WEAK ENRICHED CONTRACTIONS AND COMMON FIXED POINT RESULTS FOR ITS SEMIGROUPS

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Abstract. This paper presents a unified framework for generalized weak enriched contraction mappings and extends it to contraction semigroups. Several fixed and common fixed point theorems are established, enhancing the understanding of their structural and convergence properties. Numerical examples verify the validity and efficiency of the proposed approach. The results provide useful tools for convex optimization, variational analysis, and applied nonlinear models.

1. Introduction and preliminaries

Throughout this paper, $\mathbb{R}$ and $\mathbb{N}$ denote the sets of all real numbers and positive integers, respectively. Let (X, d) be a metric space and $T : X \to X$ a self-mapping. The mapping T is called a *Picard operator* if the following properties hold: (i) T has a unique fixed point $q \in X$, and (ii) $\lim_{n\to\infty} T^n x_0 = q$ for all $x_0 \in X$.Moreover, we use $Fix(T)$ to denote the set of all fixed points of T. One of the most classical fixed point results is Banach's Fixed Point Theorem, originally established in [3]. It states that in a complete metric space (X, d), if a self-mapping $T : X \to X$ satisfies the following condition:

$$d(Tx, Ty) \leq ad(x, y) \tag{1.1}$$

for all $x, y \in X$, where $a \in [0, 1)$, then T is a Picard operator. This fundamental principle remains a cornerstone in numerous mathematical and applied disciplines. Over time, many generalizations have been developed, underscoring both its profound importance and elegant simplicity.

In 2004, Berinde [4] introduced the generalized contractive condition in Definition 1.1 and also established the fixed point results for them (see Theorems 1.2 and 1.3).

Definition 1.1 ([4]). Let (X, d) be a metric space. A mapping $T : X \to X$ is called a *weak contraction mapping* if there are constants $\delta \in [0, 1)$ and $L \geq 0$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx). \tag{1.2}$$

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Theorem 1.2 ([4])**.** *Let (X,d) be a complete metric space and $T:X\to X$ be a weak contraction mapping, i.e., a mapping satisfying the condition (1.2) with $\delta\in[0,1)$ and some $L\geq 0$. Then T has a fixed point. Moreover, for each $x_0\in X$, the Picard iteration $\{x_n\}$, which is defined by $x_{n+1}=Tx_n$ for all $n\in\mathbb{N}\cup\{0\}$, converges to a fixed point of T.*

Theorem 1.3 ([4])**.** *Let (X,d) be a complete metric space and $T:X\to X$ be a weak contraction mapping for which there exist constants $\theta\in[0,1)$ and $L_1\geq 0$ such that for all $x,y\in X$,*

$$d(Tx,Ty)\leq \theta d(x,y)+L_1 d(x,Tx). \tag{1.3}$$

Then T is a Picard operator.

The weak contractive condition (1.2) is one of the most well-known generalized forms in fixed point theory. Each of the classical contractive conditions introduced by Banach [3], Kannan [13], Chatterjea [10], and Zamfirescu [30] implies both (1.2) and (1.3). Furthermore, the generalized contraction mapping proposed by Reich [24] and the quasi-contraction mapping introduced by Ćirić [12], with the contractive constant $k\in[0,\frac{1}{2})$, also belong to the class of weak contraction mappings.

Recently, Berinde and Păcurar [5–8] employed the enrichment technique of contractive mappings via Krasnoselskij averaging to propose several new classes of Picard mappings. For instance, the enriched Banach contraction mapping defined in [6] in a normed space is given as follows:

Definition 1.4 ([6])**.** A self-mapping T on a nonempty subset C of a normed space $(X,\|\cdot\|)$ is called an *enriched Banach contraction mapping* if there exist $b\in[0,\infty)$ and $k\in[0,b+1)$ such that for each $x,y\in C$,

$$\|b(x-y)+Tx-Ty\|\leq k\|x-y\|. \tag{1.4}$$

To indicate two constants involved in (1.4), we shall also call T a (b,k)-enriched Banach contraction mapping.

It is easy to see that a Banach contraction mapping on a normed space with a contractive constant $a\in[0,1)$ is a $(0,a)$-enriched Banach contraction mapping; however, the converse does not necessarily hold. For example, let $C=[0,1]\subseteq\mathbb{R}$ equipped with the usual norm, and define $T:C\to C$ by $Tx=1-x$ for all $x\in C$. Then, for any $b\in(0,1)$, T is a $(b,1-b)$-enriched Banach contraction mapping, but it is not a Banach contraction mapping (see [6] for details). In [6], the authors also established the existence and uniqueness of fixed points for enriched Banach contraction mappings and proved a strong convergence theorem using the Krasnoselskij iteration $\{x_n\}$ in C defined by

$$x_{n+1}=(1-\lambda)x_n+\lambda Tx_n \tag{1.5}$$

for all $n\in\mathbb{N}\cup 0$, where $x_0\in C$ and $\lambda\in(0,1]$.

In recent years, several extensions of this concept have been proposed to enhance the applicability of fixed point theory. For example, Shaheryar et al. [25] studied fuzzy enriched contractions in fuzzy Banach spaces with applications to fractals and dynamic market equilibrium, while Yu et al. [29] investigated Suzuki-enriched Kannan-type mappings in convex metric spaces and their use in approximating solutions of Volterra integral equations. Moreover, Nithiarayaphaks and Sintunavarat [22], inspired by [6], introduced a new contractive condition unifying all enriched Banach contractions in the setting of normed spaces as follows:

Definition 1.5 ([22])**.** A self-mapping T on a nonempty subset C of a normed space $(X, \|\cdot\|)$ is called a *weak enriched contraction mapping* if there exist $a, b \in [0, \infty)$ and $w \in [0, a+b+1)$ such that for each $x, y \in C$,

$$\|a(x-y) + Tx - Ty + b(T^2x - T^2y)\| \leq w\|x-y\|. \tag{1.6}$$

It is easily seen that the enriched Banach contraction mapping on a normed space satisfying (1.4) is obtained from (1.6) for $b = 0$. A nontrivial example below has been presented in [22] to support their main results. Additionally, they also introduced the concept of a double averaged mapping, which is defined by

$$T_{\mu_2}^{\mu_1} := (1-\mu_1-\mu_2)I + \mu_1 T + \mu_2 T^2, \tag{1.7}$$

where I is the identity mapping, T is a given self-mapping on a nonempty subset of a normed space, $\mu_1 > 0, \mu_2 \geq 0$ and $\mu_1 + \mu_2 \in (0,1]$. It is easy to verify that $Fix(T) \subseteq Fix\,(T_{\mu_2}^{\mu_1})$. Indeed, suppose that $z \in Fix(T)$ and then

$$T_{\mu_2}^{\mu_1} z = (1-\mu_1-\mu_2)z + \mu_1 Tz + \mu_2 T^2 z = (1-\mu_1-\mu_2)z + \mu_1 z + \mu_2 z = z.$$

Hence, $z \in Fix\,(T_{\mu_2}^{\mu_1})$. In particular, in [22], the authors presented some sufficient conditions for $Fix(T) = Fix\,(T_{\mu_2}^{\mu_1})$ as follows:

Lemma 1.6 ([22])**.** *Let T be a self-mapping on a nonempty closed convex subset C of a normed space $(X, \|\cdot\|)$. Suppose that $T_{\mu_2}^{\mu_1}$ is the double averaged operator of T such that $Fix(T_{\mu_2}^{\mu_1}) \neq \varnothing$ and one of the following assertions holds:*

- (D_1) *for each $c \in [0,1)$ and $z \in Fix(T_{\mu_2}^{\mu_1})$, we have*

$$\|z - Tz\| \leq \|z - (1-c)Tz - cT^2 z\|;$$

- (D_2) *there exists a nonnegative real number $k < 1$ such that for all $x \in C$*

$$\|T_{\mu_2}^{\mu_1} x - Tx\| \leq k\|x - Tx\|; \tag{1.8}$$

- (D_3) *for each $z \in Fix(T_{\mu_2}^{\mu_1})$, there exists a closed convex subset $B \subseteq C$ that contains z such that $T(B) \subseteq B$ and T satisfies (1.8) only on the set B.*

Then $Fix(T) = Fix\,(T_{\mu_2}^{\mu_1})$. Moreover, in the case of the condition (D_3) holds, we obtain the conclusion that $Fix(T|_B) = Fix\,(T_{\mu_2}^{\mu_1}|_B)$.

Furthermore, they presented fixed point results demonstrating that the fixed point of a weak enriched contraction mapping can be approximated via a suitable Kirk-type iterative process $\{x_n\}$ defined by

$$x_{n+1} = (1-\lambda_1-\lambda_2)x_n + \lambda_1 Tx_n + \lambda_2 T^2 x_n, \tag{1.9}$$

for all $n \in \mathbb{N} \cup 0$, where $x_0 \in C$, $\lambda_1 > 0$, $\lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 \in (0,1]$.

On the other hand, common fixed point theory plays a crucial role in nonlinear analysis, offering a unified framework for studying stability, equilibrium states, and the solvability of nonlinear equations. In many applications, attention has been directed to semigroups of nonlinear mappings, which provide a natural setting for dynamic systems and operator equations. To proceed further, we recall the formal definition of a common fixed point for a semigroup of mappings. Let G be an unbounded subset of $[0,\infty)$ satisfying

$$s+t \in G \text{ and if } s>t, \text{ then } s-t \in G, \tag{1.10}$$

for all $s,t \in G$ (for instance, $G=[0,\infty)$, $\mathbb{N}$, or $\mathbb{N}\cup\{0\}$). For a nonempty set X, consider a family of self-mappings $\tau=\{T_t : X \to X \mid t \in G\}$. A point $x \in X$ is said to be a *common fixed point* of τ if $T_t x = x$ for all $T_t \in \tau$, and the set of all such points is denoted by $Fix(\tau)$. This concept has inspired a wide range of studies on different classes of nonlinear semigroups, including nonexpansive semigroups (see [20, 26]), Lipschitzian semigroups (see [9, 19]), and pseudocontraction semigroups (see [1,11]), as well as other related classes in [21,23,27,28,31]. Building upon these foundations, Kesahorm and Sintunavarat proposed several new semigroups such as weak contraction semigroups (see [14, 16]), enriched nonexpansive semigroups (see [15]), enriched Kannan semigroups (see [17]), and enriched Chatterjea-type semigroups (see [18]), and established weak and strong convergence theorems for iterative processes used to approximate their common fixed points in Banach spaces.

Motivated by the above discussions, this paper aims to introduce the concept of generalized weak enriched contraction mappings and to establish new fixed point results corresponding to the proposed contractive conditions within the framework of Banach spaces. Furthermore, building upon these ideas, we define the notion of generalized weak enriched semigroups and derive common fixed point results for such semigroups. Moreover, to illustrate the significance and applicability of the developed theory, several examples are presented that demonstrate the effectiveness of the obtained results.

2. Generalized weak enriched contraction mappings

We begin this section by introducing the concept of a generalized weak enriched contraction mapping, which serves as the foundation for the main theorems presented in this section.

Definition 2.1. A self-mapping T on a nonempty subset C of a normed space $(X,\|\cdot\|)$ is called a *generalized weak enriched contraction mapping* if there are $a,b,c \in [0,\infty)$ and $w \in [0,a+b+1)$ such that for all $x,y \in C$,

$$\begin{aligned} &\|a(x-y)+Tx-Ty+b(T^2x-T^2y)\| \\ &\quad\leq \; w\|x-y\|+c\|a(x-y)+(Tx-y)+b(T^2x-y)\|. \end{aligned} \tag{2.1}$$

It is easy to see that the condition (1.2) in a normed space also meets (2.1) when $a=b=0$, $w=\delta\in[0,1)$, and $c=L\in[0,\infty)$. Moreover, every weak enriched contraction mapping fulfilling (1.6) is a generalized weak enriched contraction mapping with $c=0$. However, the converse does not necessarily hold, as shown in the next example.

Example 2.2. Let $C=[-1,1]\subseteq\mathbb{R}$ be equipped with the usual norm and let $T : C \to C$ be defined by

$$Tx = \begin{cases} -0.5 & \text{if } x \in [-1,0)\,, \\ 1-x & \text{if } x \in [0,1]. \end{cases} \tag{2.2}$$

It is easy to see that $Fix(T) = \{-0.5, 0.5\}$. Additionally, T is not a weak enriched contraction mapping since (1.6) does not holds with $x = -0.5$ and $y = 0.5$. Next, we verify that T satisfies (2.1) with $a = b = 1, c = 2$ and any $w \in [1, a+b+1)$. For each $x, y \in X$, we introduce the notation $LS := |a(x-y)+Tx-Ty+b(T^2x-T^2y)|$ and analyze the following cases.

- Case I: If $x, y \in [-1, 0)$, then

$$LS = |x-y| \leq w\|x-y\| + c\|a(x-y) + (Tx-y) + b(T^2x-y)\|.$$

- Case II: If $x, y \in [0, 1]$, then

$$\begin{aligned} LS &= |(1)(x-y) + (1-x) - (1-y) + (1)(x-y)| \\ &= |x-y| \\ &\leq w\|x-y\| + c\|a(x-y) + (Tx-y) + b(T^2x-y)\|. \end{aligned}$$

- Case III: If $x \in [-1, 0)$ and $y \in [0, 1]$, then

$$\begin{aligned} LS &= |(1)(x-y) + (-0.5) - (1-y) + (1)(-0.5-y)| \\ &= |x-y-2| \\ &\leq |x-y| + 2 \\ &\leq |x-y| + 2\,|3y-x+1| \\ &\leq w\|x-y\| + c\|a(x-y) + (Tx-y) + b(T^2x-y)\|. \end{aligned}$$

- Case IV: If $x \in [0, 1]$ and $y \in [-1, 0)$, then

$$\begin{aligned} LS &= |(1)(x-y) + (1-x) - (-0.5) + (1)(x-(-0.5))| \\ &= |x-y+2| \\ &\leq |x-y| + 2 \\ &\leq |x-y| + 2\,|x-3y+1| \\ &\leq w\|x-y\| + c\|a(x-y) + (Tx-y) + b(T^2x-y)\|. \end{aligned}$$

From all cases, T is a generalized weak enriched contraction mapping.

Next, we will present the existence and uniqueness of a fixed point of generalized weak enriched contraction mappings and prove a strong convergence theorem for the Kirk's iteration used to approximate the fixed points of such mappings by using the concept of the double averaged mapping.

Theorem 2.3. *Let C be a nonempty closed convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ be a generalized weak enriched contraction mapping satisfying (2.1) with $a, b, c \in [0, \infty)$ and $w \in [0, a+b+1)$. Then the following conclusions hold:*

(i) $T_{\mu_2}^{\mu_1}$ has a fixed point, where $\mu_1 := \frac{1}{a+b+1}$ and $\mu_2 := \frac{b}{a+b+1}$;
(ii) for each $x_0 \in C$, the Kirk's iteration $\{x_n\}$, given by the iterative scheme (1.9) with $\lambda_1 := \mu_1$ and $\lambda_2 := \mu_2$, converges to a fixed point of $T_{\mu_2}^{\mu_1}$;
(iii) if one of the conditions $(D_1) - (D_3)$ holds, then $Fix(T) \neq \varnothing$ and for each $x_0 \in C$, the Kirk's iteration $\{x_n\}$, given by the iterative scheme (1.9) with $\lambda_1 := \mu_1$ and $\lambda_2 := \mu_2$, converges to a fixed point of T.

Proof. From the defining of μ_1 and μ_2, we obtain $a = \frac{1-\mu_2}{\mu_1} - 1$ and $b = \frac{\mu_2}{\mu_1}$. From (2.1), for all $x, y \in C$, we have

$$\begin{aligned}&\left\|\left(\frac{1-\mu_2}{\mu_1} - 1\right)(x-y) + Tx - Ty + \left(\frac{\mu_2}{\mu_1}\right)(T^2x - T^2y)\right\| \\ &\leq \quad w\|x-y\| + c\left\|\left(\frac{1-\mu_2}{\mu_1} - 1\right)(x-y) + (Tx-y) + \left(\frac{\mu_2}{\mu_1}\right)(T^2x-y)\right\|,\end{aligned}$$

which yields

$$\begin{aligned}&\left\|(1-\mu_2-\mu_1)(x-y) + \mu_1(Tx-Ty) + \mu_2(T^2x-T^2y)\right\| \\ &\quad \leq \mu_1 w\|x-y\| + c\|(1-\mu_2-\mu_1)(x-y) + \mu_1(Tx-y) + \mu_2(T^2x-y)\|.\end{aligned}$$

From the above inequality, we obtain

$$\|T_{\mu_2}^{\mu_1}x - T_{\mu_2}^{\mu_1}y\| \leq W\|x-y\| + C\|y - T_{\mu_2}^{\mu_1}x\|, \tag{2.3}$$

for all $x, y \in C$, where $W := \mu_1 w \in [0,1)$ and $C := c \geq 0$. This means $T_{\mu_2}^{\mu_1}$ is a weak contraction mapping. Taking $\lambda_1 := \mu_1, \lambda_2 := \mu_2$ in the iteration $\{x_n\}$ defined by (1.9), it is exactly the Picard iteration associated with $T_{\mu_2}^{\mu_1}$, that is, $x_{n+1} = T_{\mu_2}^{\mu_1}x_n$ for all $n \in \mathbb{N} \cup \{0\}$. By Theorem 1.2, we obtain $Fix(T_{\mu_2}^{\mu_1}) \neq \varnothing$ and $\{x_n\}$ converges to some $z_0 \in Fix(T_{\mu_2}^{\mu_1})$. Hence, the claims (i) and (ii) are proven. Finally, the claim (iii) is proved by applying the results from (i) and (ii) together with Lemma 1.6. □

Theorem 2.4. *Let C be a nonempty closed convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ be a generalized weak enriched contraction mapping* (2.1) *with $a, b, c \in [0, \infty)$ and $w \in [0, a+b+1)$. If*

$$\begin{aligned}&\|a(x-y) + Tx - Ty + b(T^2x - T^2y)\| \\ &\leq \quad w\|x-y\| + c\|(Tx-x) + b(T^2x-x)\|\end{aligned} \tag{2.4}$$

for all $x, y \in C$, then the following conclusions hold:

- (i) *$T_{\mu_2}^{\mu_1}$ has a unique fixed point, where $\mu_1 := \frac{1}{a+b+1}$ and $\mu_2 := \frac{b}{a+b+1}$;*
- (ii) *for each $x_0 \in C$, the Kirk's iteration $\{x_n\}$, given by the iterative scheme (1.9) with $\lambda_1 := \mu_1$ and $\lambda_2 := \mu_2$, converges to a unique fixed point of $T_{\mu_2}^{\mu_1}$;*
- (iii) *if one of the conditions (D_1)–(D_3), then $Fix(T) = \{z\}$ and for each $x_0 \in C$, the Kirk's iteration $\{x_n\}$, given by the iterative scheme (1.9) with $\lambda_1 := \mu_1$ and $\lambda_2 := \mu_2$, converges to a unique fixed point z.*

Proof. By using the same technique with the proof in Theorem 2.3, we obtain $T_{\mu_2}^{\mu_1}$ is a weak contraction mapping. Moreover, from (2.4), for all $x, y \in C$, we have

$$\begin{aligned}&\left\|\left(\frac{1-\mu_2}{\mu_1} - 1\right)(x-y) + Tx - Ty + \left(\frac{\mu_2}{\mu_1}\right)(T^2x - T^2y)\right\| \\ &\leq \quad w\|x-y\| + c\left\|(Tx-x) + \left(\frac{\mu_2}{\mu_1}\right)(T^2x-x)\right\|,\end{aligned}$$

which yields

$$\begin{aligned}&\|(1-\mu_2-\mu_1)(x-y) + \mu_1(Tx-Ty) + \mu_2(T^2x-T^2y)\| \\ &\leq \quad \mu_1 w\|x-y\| + c\|\mu_1(Tx-x) + \mu_2(T^2x-x)\|\end{aligned}$$

$$= \mu_1 w\|x-y\| + c\|x-((1-\mu_1-\mu_2)x+\mu_1 Tx+\mu_2 T^2x)\|.$$

This implies that

$$\|T^{\mu_1}_{\mu_2}x - T^{\mu_1}_{\mu_2}y\| \le W\|x-y\| + C\|x - T^{\mu_1}_{\mu_2}x\|, \tag{2.5}$$

for all $x, y \in C$, where $W := \mu_1 w \in [0,1)$ and $C := c \ge 0$. By applying Theorem 1.3 and following the same line of reasoning as in the proof of Theorem 2.3, we obtain all the desired results. □

3. Generalized weak enriched semigroups

Based on the ideas presented in Section 2, this section aims to investigate generalized weak enriched semigroups, which are defined as follows:

Definition 3.1. Let C be a nonempty closed convex subset of a uniformly convex Banach space X and G be an unbounded subset of $[0,\infty)$ satisfying the condition (1.10). Then the family $\tau = \{T_t : C \to C \mid t \in G\}$ is called a *generalized weak enriched contraction semigroup* on C if the following conditions are satisfied:

- (W_1) $T_{s+t}x = T_sT_tx$ for all $s, t \in G$ and $x \in C$;
- (W_2) for all $x \in C$, the mapping $G \ni t \mapsto T_tx$ is continuous;
- (W_3) for each $t \in G$, $T_t : C \to C$ is a generalized weak enriched contraction mapping on C, i.e., there are constants $a_t, b_t, c_t \in [0,\infty)$ and $\omega_t \in [0, a_t + b_t + 1)$ such that

$$\begin{aligned} &\|a_t(x-y) + T_tx - T_ty + b_t(T_t^2x - T_t^2y)\| \\ &\quad \le \omega_t\|x-y\| + c_t\|a_t(x-y) + (T_tx - y) + b_t(T_t^2x - y)\| \end{aligned} \tag{3.1}$$

for all $x, y \in C$.

In the case of $a_t = a \in [0,\infty)$, $b_t = b \in [0,\infty), c_t = c \in [0,\infty)$ and $\omega_t = \omega \in [0, a_t + b_t + 1)$ for all $t \in G$, the family $\tau = \{T_t : C \to C \mid t \in G\}$ is called an (a,b,c,ω)-generalized weak enriched contraction semigroup on C. It is easy to see that any (a,b,c,ω)-generalized weak enriched contraction semigroup is a generalized weak enriched contraction semigroup.

Now, we give the following example, which is a generalized weak enriched contraction semigroup.

Example 3.2. Let $X = \mathbb{R}$ be equipped with the usual norm, $C = [-1,1] \subseteq X$ and $\tau = \{T_t : C \to C \mid t \in \mathbb{N}\}$, where T_t is defined for each $t \in \mathbb{N}$ by

$$T_tx = \begin{cases} xe^{-t} & \text{if } x \in [-1,0), \\ -\frac{1}{2}xe^{-t} & \text{if } x \in \left[0, \frac{1}{2}\right), \\ 0 & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases} \tag{3.2}$$

First, we will show that the family τ satisfies (W_1). Let $x \in C$ and let $s, t \in \mathbb{N}$. Then there are three cases as follows:

- Case I: If $x \in [-1,0)$, we get

$$T_{s+t}x = xe^{-(s+t)} = (xe^{-t})e^{-s} = (T_tx)e^{-s} = T_sT_tx.$$

- Case II: Assume that $x \in \left(0, \frac{1}{2}\right)$. From (3.2), we have

$$T_{s+t}x = -\frac{1}{2}xe^{-(s+t)}$$

and

$$T_sT_tx = T_s\left(-\frac{1}{2}xe^{-t}\right) = \left(-\frac{1}{2}xe^{-t}\right)e^{-s} = -\frac{1}{2}xe^{-(s+t)}.$$

- Case III: If $x \in \{0\} \cup \left[\frac{1}{2}, 1\right]$, we get

$$T_{s+t}x = 0 \text{ and } T_sT_tx = T_s0 = \left(-\frac{1}{2}\right)0e^{-s} = 0.$$

Therefore, the family τ satisfies (W_1). Next, it is easy to see that the family τ satisfies (W_2), i.e., for all $x \in C$, the mapping $\mathbb{N} \ni t \mapsto T_tx$ is continuous.

Finally, we verify that for each $t \in \mathbb{N}$, T_t is a generalized weak enriched contraction mapping satisfying the condition (3.1) with $a_t = 0.1, b_t = 1, c_t = 1$ and $\omega_t = 2$. For each $x, y \in C$, we employ the notation

$$LS := \|a_t(x-y) + T_tx - T_ty + b_t(T_t^2x - T_t^2y)\|$$

and examine the following cases.

- Case I: If $x, y \in \left[\frac{1}{2}, 1\right]$, we get

$$\begin{aligned} LS &= |0.1(x-y)| \\ &\leq \omega_t\|x-y\| + c_t\|a_t(x-y) + (T_tx - y) + b_t(T_t^2x - y)\|. \end{aligned}$$

- Case II: If $x, y \in [-1, 0)$, we get

$$\begin{aligned} LS &= \left|0.1(x-y) + xe^{-t} - ye^{-t} + xe^{-2t} - ye^{-2t}\right| \\ &= \left|\left(0.1 + e^{-t} + e^{-2t}\right)(x-y)\right| \\ &= \left|0.1 + e^{-t} + e^{-2t}\right| |x-y| \\ &\leq \left(0.1 + \frac{1}{e} + \frac{1}{e^2}\right)|x-y| \\ &\leq 2|x-y| \\ &\leq \omega_t\|x-y\| + c_t\|a_t(x-y) + (T_tx - y) + b_t(T_t^2x - y)\|. \end{aligned}$$

- Case III: If $x, y \in \left[0, \frac{1}{2}\right)$, we get

$$\begin{aligned} LS &= \left|0.1(x-y) + \left(-\frac{1}{2}xe^{-t}\right) - \left(-\frac{1}{2}ye^{-t}\right) + \left(-\frac{1}{2}xe^{-2t}\right) - \left(-\frac{1}{2}ye^{-2t}\right)\right| \\ &= \left|\left(0.1 - \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t}\right)(x-y)\right| \\ &= \left|0.1 - \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t}\right| |x-y| \\ &\leq 2|x-y| \\ &\leq \omega_t\|x-y\| + c_t\|a_t(x-y) + (T_tx - y) + b_t(T_t^2x - y)\|. \end{aligned}$$

- Case IV: If $x \in [-1,0)$ and $y \in \big[0, \frac{1}{2}\big)$, we have

$$\begin{aligned} x &\leq \frac{\big(2.1 - 1.5(e^{-t} + e^{-2t})\big)}{(0.1 + e^{-t} + e^{-2t})} y, \\ \left(0.1 + e^{-t} + e^{-2t}\right) x &\leq \left(2.1 - 1.5(e^{-t} + e^{-2t})\right) y, \\ 1.5(e^{-t} + e^{-2t}) y &\leq 2.1y - \left(0.1 + e^{-t} + e^{-2t}\right) x. \end{aligned}$$

By the above inequality, we have

$$\begin{aligned} LS &= \left|0.1(x-y) + xe^{-t} - \left(-\frac{1}{2}ye^{-t}\right) + xe^{-2t} - \left(-\frac{1}{2}ye^{-2t}\right)\right| \\ &= \left|(0.1 + e^{-t} + e^{-2t})(x-y) + 1.5(e^{-t} + e^{-2t})y\right| \\ &\leq \left|(0.1 + e^{-t} + e^{-2t})(x-y)\right| + \left|1.5(e^{-t} + e^{-2t})y\right| \\ &\leq 2(y-x) + 2.1y - \left(0.1 + e^{-t} + e^{-2t}\right) x \\ &= 2|x-y| + \left|-2.1y + \left(0.1 + e^{-t} + e^{-2t}\right) x\right| \\ &= 2|x-y| + \left|0.1(x-y) + (xe^{-t} - y) + (xe^{-2t} - y)\right| \\ &= \omega_t \|x-y\| + c_t \|a_t(x-y) + (T_t x - y) + b_t(T_t^2 x - y)\|. \end{aligned}$$

- Case V: If $x \in \big[0, \frac{1}{2}\big)$ and $y \in [-1,0)$, then

$$\left(1.9 - e^{-t} - e^{-2t}\right) y \leq \left(1.9 + \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t}\right) x$$

and

$$\left(2.1 + e^{-t} + e^{-2t}\right) y \leq \left(2.1 - \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t}\right) x.$$

This implies that

$$\left(0.1 - \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t}\right) x - \left(0.1 + e^{-t} + e^{-2t}\right) y \quad \leq \quad 2(x-y)$$

and

$$2(y-x) \quad \leq \quad \left(0.1 - \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t}\right) x - \left(0.1 + e^{-t} + e^{-2t}\right) y.$$

These inequlities yield

$$\left|\left(0.1 - \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t}\right) x - \left(0.1 + e^{-t} + e^{-2t}\right) y\right| \quad \leq \quad 2|x-y|.$$

By the above inequality, we have

$$\begin{aligned} LS &= \left|0.1(x-y) + \left(-\frac{1}{2}xe^{-t}\right) - ye^{-t} + \left(-\frac{1}{2}xe^{-2t}\right) - ye^{-2t}\right| \\ &= \left|\left(0.1 - \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t}\right) x - \left(0.1 + e^{-t} + e^{-2t}\right) y\right| \\ &\leq 2|x-y| \\ &\leq \omega_t \|x-y\| + c_t \|a_t(x-y) + (T_t x - y) + b_t(T_t^2 x - y)\|. \end{aligned}$$

- Case VI: If $x \in [-1,0)$ and $y \in \left[\frac{1}{2},1\right]$, we get

$$\begin{aligned} x &\leq \frac{\left(2.1 - e^{-t} - e^{-2t}\right)}{\left(0.1 + e^{-t} + e^{-2t}\right)} y, \\ \left(0.1 + e^{-t} + e^{-2t}\right) x &\leq \left(2.1 - e^{-t} - e^{-2t}\right) y, \\ \left(e^{-t} + e^{-2t}\right) y &\leq 2.1y - \left(0.1 + e^{-t} + e^{-2t}\right) x. \end{aligned}$$

By the above inequality, we have

$$\begin{aligned} LS &= \left|0.1(x-y) + xe^{-t} + xe^{-2t}\right| \\ &= \left|(0.1 + e^{-t} + e^{-2t})(x-y) + (e^{-t} + e^{-2t})y\right| \\ &\leq \left|(0.1 + e^{-t} + e^{-2t})(x-y)\right| + \left|(e^{-t} + e^{-2t})y\right| \\ &\leq 2(y-x) + (2.1y - \left(0.1 + e^{-t} + e^{-2t}\right) x) \\ &= 2|x-y| + \left|-2.1y + \left(0.1 + e^{-t} + e^{-2t}\right) x\right| \\ &= 2|x-y| + \left|0.1(x-y) + (xe^{-t} - y) + (xe^{-2t} - y)\right| \\ &= \omega_t\|x-y\| + c_t\|a_t(x-y) + (T_t x - y) + b_t(T_t^2 x - y)\|. \end{aligned}$$

- Case VII: If $x \in \left[\frac{1}{2},1\right]$ and $y \in [-1,0)$, we get

$$\begin{aligned} (1.9 - e^{-t} - e^{-2t})y &\leq 1.9x, \\ -ye^{-t} - ye^{-2t} &\leq 1.9(x-y). \end{aligned}$$

This implies that

$$\begin{aligned} LS &= \left|0.1(x-y) - ye^{-t} - ye^{-2t}\right| \\ &= 0.1(x-y) - ye^{-t} - ye^{-2t} \\ &= 0.1(x-y) + (-ye^{-t} - ye^{-2t}) \\ &\leq 0.1(x-y) + 1.9(x-y) \\ &= 2|x-y| \\ &\leq 2|x-y| + |0.1(x-y) + (0-y) + (0-y)| \\ &= \omega_t\|x-y\| + c_t\|a_t(x-y) + (T_t x - y) + b_t(T_t^2 x - y)\|. \end{aligned}$$

- Case VIII: If $x \in \left[0,\frac{1}{2}\right)$ and $y \in \left[\frac{1}{2},1\right]$, we get

$$\begin{aligned} 0.1x &\leq 2.1y, \\ 0.1x - (0.1 - \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t})x &\leq 2.1y - (0.1 - \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t})x, \\ \frac{1}{2}x\left(e^{-t} + e^{-2t}\right) &\leq 2.1y - (0.1 - \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t})x. \end{aligned}$$

By the above inequality, we have

$$\begin{aligned} &\|a_t(x-y) + T_t x - T_t y + b_t(T_t^2 x - T_t^2 y)\| \\ &= \left|0.1(x-y) + \left(-\frac{1}{2}xe^{-t}\right) + \left(-\frac{1}{2}xe^{-2t}\right)\right| \\ &= \left|0.1(x-y) + \left(-\frac{1}{2}x\left(e^{-t} + e^{-2t}\right)\right)\right| \end{aligned}$$

$$\begin{aligned}
&\leq 0.1|x-y| + \left|\frac{1}{2}x\left(e^{-t}+e^{-2t}\right)\right| \\
&\leq 2|x-y| + \left|2.1y-(0.1-\frac{1}{2}e^{-t}-\frac{1}{2}e^{-2t})x\right| \\
&= 2|x-y| + \left|0.1(x-y)+\left(-\frac{1}{2}xe^{-t}-y\right)+\left(-\frac{1}{2}xe^{-2t}-y\right)\right| \\
&= \omega_t\|x-y\| + c_t\|a_t(x-y)+(T_tx-y)+b_t(T_t^2x-y)\|.
\end{aligned}$$

- Case IX: If $x \in \left[\frac{1}{2}, 1\right]$ and $y \in \left[0, \frac{1}{2}\right)$, we distinguish two possibilities:
  (i) Assume that $0.1x - 2.1y < 0$. Then we get

$$\left(\frac{1}{2}\left(e^{-t}+e^{-2t}\right)-0.2\right)y \leq 1.8x$$

and so

$$\frac{1}{2}\left(e^{-t}+e^{-2t}\right)y \leq 1.8x+0.2y = 1.9(x-y)+2.1y-0.1x.$$

By the above inequality, we have

$$\begin{aligned}
LS &= \left|0.1(x-y)+\frac{1}{2}ye^{-t}+\frac{1}{2}ye^{-2t}\right| \\
&= 0.1(x-y)+\frac{1}{2}ye^{-t}+\frac{1}{2}ye^{-2t} \\
&\leq 0.1(x-y)+1.9(x-y)+2.1y-0.1x \\
&\leq 2|x-y|+|2.1y-0.1x| \\
&= 2|x-y|+|0.1(x-y)+(0-y)+(0-y)| \\
&= \omega_t\|x-y\| + c_t\|a_t(x-y)+(T_tx-y)+b_t(T_t^2x-y)\|.
\end{aligned}$$

(ii) Assume that $0.1x - 2.1y \geq 0$. It yields that $1.9(x-y) \geq 3.8y$. Then we have

$$\begin{aligned}
LS &= \left|0.1(x-y)+\frac{1}{2}ye^{-t}+\frac{1}{2}ye^{-2t}\right| \\
&= 0.1(x-y)+\frac{1}{2}ye^{-t}+\frac{1}{2}ye^{-2t} \\
&\leq 0.1(x-y)+\frac{1}{2}y(e^{-1}+e^{-2}) \\
&\leq 0.1(x-y)+3.8y \\
&\leq 0.1(x-y)+1.9(x-y) \\
&= 2(x-y) \\
&\leq \omega_t\|x-y\| + c_t\|a_t(x-y)+(T_tx-y)+b_t(T_t^2x-y)\|.
\end{aligned}$$

Thus, for all possible cases, it has been proven that for each $t \in \mathbb{N}$, T_t is a generalized weak enriched contraction on C satisfying the condition (3.1). Therefore, the family $\tau = \{T_t : C \to C \mid t \in \mathbb{N}\}$ is a generalized weak enriched contraction semigroup on C.

Remark 3.3. Example 3.2 shows that for all $t \in \mathbb{N}$, T_t is a generalized weak enriched contraction mapping on C, but T_1 does not satisfy a weak enriched contractive condition (1.6) with $x = 0.49$ and $y = 0.5$.

Next, we introduce the new iterative process based on the Kirk's iterative processes for approximating the common fixed point of generalized weak enriched contraction semigroups in Banach spaces. Under the setting where C denotes a nonempty closed convex subset of a Banach space X, and G is an unbounded subset of $[0,\infty)$ satisfying condition (1.10), let $\tau = \{T_t : C \to C \mid t \in G\}$ be a generalized weak enriched contraction semigroup on C. The proposed iterative process $\{x_n\} \subseteq C$ is defined by

$$x_{n+1} = (1 - \lambda_{t_n} - \hat{\lambda}_{t_n})x_n + \lambda_{t_n} T_{t_n} x_n + \hat{\lambda}_{t_n} T_{t_n}^2 x_n, \tag{3.3}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $x_0 \in C$ and $\{t_n\}_{n\in\mathbb{N}\cup\{0\}} \subseteq G$ such that $\lambda_{t_n} \in (0,\infty)$ and $\hat{\lambda}_{t_n} \in [0,\infty)$ satisfy $\lambda_{t_n} + \hat{\lambda}_{t_n} \in (0,1]$ for all $n \in \mathbb{N} \cup \{0\}$.

Theorem 3.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, G be an unbounded subset of $[0,\infty)$ satisfying the condition (1.10) and $\tau = \{T_t : C \to C \mid t \in G\}$ be a (a,b,c,ω)-generalized weak enriched contraction semigroup satisfying the following condition:*

- *for each $s,t \in G$ and for each $x,y \in C$, we have*

$$\begin{aligned} &\|a(x-y) + T_s x - T_t y + b(T_s^2 x - T_t^2 y)\| \\ &\quad \leq \ \omega\|x-y\| + c\|(T_s x - x) + b(T_s^2 x - x)\| \end{aligned} \tag{3.4}$$

Then the following assertions hold:

(*i*) *$Fix(\overline{\tau}) = \{z\}$ for some $z \in C$, where $\overline{\tau} := \left\{T_t|_{\hat{\mu}_t}^{\mu_t} \mid T_t \in \tau\right\}$ and $T_t|_{\hat{\mu}_t}^{\mu_t}$ represents the double averaged mapping of T_t with parameters $\mu_t := \frac{1}{a+b+1} \in (0,1]$ and $\hat{\mu}_t := \frac{b}{a+b+1} \in [0,1]$ for all $t \in G$.*

(*ii*) *for each $x_0 \in C$ and $\{t_n\}_{n\in\mathbb{N}\cup\{0\}} \subseteq G$, a sequence $\{x_n\}$ defined by the iterative scheme (3.3), where $\lambda_{t_n} := \mu_{t_n}$ and $\hat{\lambda}_{t_n} := \hat{\mu}_{t_n}$ for all $n \in \mathbb{N} \cup \{0\}$, converges to a unique common fixed point of $\overline{\tau}$;*

(*iii*) *if for each $t \in G$, one of the conditions $(D_1) - (D_3)$ holds, then for each $x_0 \in C$ and $\{t_n\}_{n\in\mathbb{N}\cup\{0\}} \subseteq G$, a sequence $\{x_n\}$ defined by the iterative scheme (3.3), where $\lambda_{t_n} := \mu_{t_n}$ and $\hat{\lambda}_{t_n} := \hat{\mu}_{t_n}$ for all $n \in \mathbb{N} \cup \{0\}$, converges to a unique common fixed point of τ.*

Proof. For each $t \in G$, we obtain T_t is a generalized weak enriched contraction mapping satisfying (2.1) with $a,b,c \in [0,\infty)$ and $w \in [0, a+b+1)$. Moreover, from (3.4), we get

$$\begin{aligned} &\|a(x-y) + T_t x - T_t y + b(T_t^2 x - T_t^2 y)\| \\ &\quad \leq \ \omega\|x-y\| + c\|(T_t x - x) + b(T_t^2 x - x)\| \end{aligned} \tag{3.5}$$

for all $x,y \in C$. From Theorem 2.4, for each $t \in G$, we obtain $T_t|_{\hat{\mu}_t}^{\mu_t}$, which is represented the double averaged mapping of T_t has a unique fixed point, where

$\mu_t := \frac{1}{a+b+1} \in (0,1]$ and $\hat{\mu}_t := \frac{b}{a+b+1} \in [0,1]$ for all $t \in G$. Moreover, for each $t \in G$, from (3.4), we get

$$\|T_t|^{\mu_t}_{\hat{\mu}_t} x - T_t|^{\mu_t}_{\hat{\mu}_t} y\| \leq W\|x-y\| + C\|T_t|^{\mu_t}_{\hat{\mu}_t} x - x\| \tag{3.6}$$

for all $x, y \in C$, where $W := \mu_1 w \in [0,1)$ and $C := c \geq 0$.

Next, we will verify $Fix(\overline{\tau}) \neq \emptyset$. Let $s, t \in G$. Suppose that $z_s \in Fix\left(T_s|^{\mu_s}_{\hat{\mu_s}}\right)$, $z_t \in Fix\left(T_t|^{\mu_t}_{\hat{\mu_t}}\right)$ and $z_s \neq z_t$. Then we have

$$\begin{aligned} \|(a+b+1)(z_s - z_t)\| &= \|(a+b+1)(T_s|^{\mu_t}_{\hat{\mu_t}} z_s - T_t|^{\mu_t}_{\hat{\mu_t}} z_t)\| \\ &= \|a(z_s - z_t) + T_s z_s - T_t z_t + b(T_s^2 z_s - T_t^2 z_t)\| \\ &\leq \omega_t\|z_s - z_t\| + c\|(T_s z_s - z_s) + b(T_s^2 z_s - z_s)\| \end{aligned}$$

and so

$$(a+b+1)\|z_s - z_t\| \leq \omega\|z_s - z_t\| < (a+b+1)\|z_s - z_t\|,$$

which is a contradiction. This implies that $z_s = z_t$. This means that $Fix(\overline{\tau}) = \{z\}$ for some $z \in C$. This shows that (i) holds. Moreover, statements (ii) and (iii) follow directly from statements (ii) and (iii) of Theorem 2.4. □

References

[1] R. P. Agarwal, X. Qin and S. M. Kang, *Strong convergence theorems for strongly continuous semigroups of pseudocontractions*, Appl. Math. Lett. **24** (2011), 1845–1848.

[2] J.-B. Baillon, *On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces*, Houston J. Math. **4** (1978), 1–9.

[3] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application auxéquations intégrales*, Fund. math. **3** (1922), 133–181.

[4] V. Berinde, *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Anal. Forum **9** (2004), 43–54.

[5] V. Berinde, *Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces*, Carpathian J. Math. **35** (2019), 293–304.

[6] V. Berinde and M. Păcurar, *Approximating fixed points of enriched contractions in Banach spaces*, J. Fixed Point Theory Appl. **22** (2020), 1–10.

[7] V. Berinde and M. Păcurar, *Approximating fixed points of enriched Chatterjea contractions by Krasnoselskij iterative algorithm in Banach spaces*, J. Fixed Point Theory Appl. **23** (2021), 1–16.

[8] V. Berinde and M. Păcurar, *Kannan's fixed point approximation for solving split feasibility and variational inequality problems*, J. Comput. Appl. Math. **386** (2021): 113217.

[9] L. C. Ceng, H. K. Xu and J. C. Yao, *Uniformly normal structure and uniformly Lipschitzian semigroups*, Nonlinear Anal. **73** (2010), 3742–3750.

[10] S. K. Chatterjea, *Fixed Point Theorems*, C. R. Acad. Bulgare Sci. **25** (1972), 727–730.

[11] S. Y. Cho and S. M. Kang, *Approximation of fixed points of pseudocontraction semigroups based on a viscosity iterative process*, Appl. Math. Lett. **24** (2011), 224–228.

[12] L. B. Ćirić, *On contraction type mappings*, Math. Balkanica **1** (1971), 52–57.

[13] R. Kannan, *Some results on fixed points*, Bull. Cal. Math. Soc. **60** (1968), 71–76.

[14] T. Kesahorm and W. Sintunavarat, *Existence and convergence theorems concerning common fixed points of nonlinear semigroups of weak contractions*, J. Fixed Point Theory Appl. **22** (2020), 1–19.

[15] T. Kesahorm and W. Sintunavarat, *On novel common fixed point results for enriched nonexpansive semigroups*, Thai J. Math. **18** (2020), 1549–1563.

[16] T. Kesahorm and W. Sintunavarat, *On convergence analysis of the inertial Mann iterative process for weak contraction semigroups*, J. Nonlinear Funct. Anal. **2022** (2022): 14
[17] T. Kesahorm and W. Sintunavarat, *On numerical approximation of common fixed points for enriched Kannan semigroups and its experiment*, J. Interdisc. Math. **25** (2022), 15–43.
[18] T. Kesahorm and W. Sintunavarat, *On a novel semigroup of enriched Chatterjea-type mappings in Banach spaces with a numerical experiment*, Nonlinear Anal. Model. Control **30** (2025), 1–23.
[19] W. Kozlowski, *Monotone Lipschitzian semigroups in Banach spaces*, J. Aust. Math. Soc. **105** (2018), 417–428.
[20] S. Li, L. Li and Y. Su, *General iterative methods for a one-parameter nonexpansive semigroup in Hilbert space*, Nonlinear Anal. **70** (2009), 3065–3071.
[21] M. Najibufahmi and A. Zulijanto, *A fixed point theorem for generalized Lipschitzian semigroups in Hilbert spaces*, Thai J. Math. **17** (2019), 639–648.
[22] W. Nithiarayaphaks and W. Sintunavarat, *On approximating fixed points of weak enriched contraction mappings via Kirk's iterative algorithm in Banach spaces*, Carpathian J. Math. **39** (2023), 423–432.
[23] S. Panja, M. Saha and R. K. Bisht, *Existence of common fixed points of non-linear semigroups of enriched Kannan contractive mappings*, Carpathian J. Math. **38** (2022), 169–178.
[24] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull. **14** (1971), 121–124.
[25] M. Shaheryar, F. Ud Din, A. Hussain and H. Alsulami, *Fixed point results for fuzzy enriched contraction in fuzzy Banach spaces with applications to fractals and dynamic market equilibrium*, Fractal Fract. **8** (2024): 609.
[26] T. Suzuki and W. Takahashi, *Strong convergence of Mann's type sequences for one-parameter nonexpansive semigroups in general Banach spaces*, J. Nonlinear Convex Anal. **5** (2004), 209–216.
[27] H. K. Xu, *A strong convergence theorem for contraction semigroups in Banach spaces*, Bull. Aust. Math. Soc. **72** (2005), 371–379.
[28] J. Yao and L. Zeng, *Fixed point theorem for asymptotically regular semigroups in metric spaces with uniform normal structure*, J. Nonlinear Convex Anal. **8** (2007), 153–163.
[29] Y. Yu, C. Li and D. Ji, *Fixed point theorems for enriched Kannan-type mappings and application*, AIMS Mathematics **9** (2024), 21580–21595.
[30] T. Zamfirescu, *Strong convergence of a proximal point algorithm with general errors*, Archiv der Mathematik **23** (1972), 292–298.
[31] S. S. Zhang, *Convergence theorem of common fixed points for Lipschitzian pseudo-contraction semi-groups in Banach spaces*, Appl. Math. Mech. **30** (2009), 145–152.

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