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ASYMPTOTICALLY ALMOST AUTOMORPHIC SOLUTION OF ABSTRACT INTEGRO-DYNAMIC EQUATION

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Dedicated to Prof. Ravi P. Agarwal on his 76th Birthday

Abstract. In this paper, we investigate the existence and uniqueness of solution of an abstract nonlinear integro-dynamic equation in the Banach space of rd-continuous function on a time scale. We apply the Krasnoselskiĭ fixed point theorem and Grönwall-type dynamic inequality to show existence and uniqueness of mild solution of the equation in a bounded interval of the time scale. We also prove the existence and uniqueness of asymptotically almost automorphic solution of the equation using Banach fixed point theorem.

1. Introduction

In 1988, German Mathematician Stefan Hilger introduced the concept of time scale in Mathematics, through his Ph.D. thesis. After that, Hilger published two interesting articles on this topic [22, 23]. Time scale unifies the discrete and continuous calculi, to study them simultaneously rather than separately. For details on time scale calculus, see the monographs [6, 7]. In recent times researchers have been quite actively working on dynamic equations to merge results from both differential and difference equations. Dynamic equations are associated with several real-world phenomena involving discrete as well as continuous variables, for example, we refer to Population Dynamics [25, 42], Optimization Theory [41], Economics [2], production-inventory modeling [3], etc.

Periodic functions has a wide range of applications in real word problems, for example in field of astronomy, physics, biology, in the use of signal processing, control system, electrical engineering etc. But, in many situations, we actually come across phenomena that involves functions which do not have a exact periodic nature but have several periodic natures. We call them almost periodic functions. Almost periodic function was first introduced by Bohr [10]. Usage of this type of function can be observed in several scientific fields, such as electromagnetic theory, plasma physics, engineering, celestial mechanics etc. Almost periodic functions are again generalized to almost automorphic(AA) functions and then generalized to asymptotically almost automorphic(AAA) functions etc. The notion of asymptotically almost automorphic functions also abbreviated as AAA functions, are typically those type of functions which behaves like almost automorphic functions after a certain initial transient phase. This type of function behaves differently at the initial phase and gradually settles down with the behavior of an equivalent almost automorphic

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function. These type of functions are useful for studying long-term behavior of a dynamical system, specially used while dealing with a non-autonomous system or systems which are influenced by some external influences.

In 2011, Y. Li and C. Wang [30] introduced the concept of almost periodic functions on time scales. Then they applied the results to a class of high-order Hopfield neural networks with variable delays. On the other hand, in 2012 , Hamza and Oraby [20], studied the stability of abstract dynamic equation $x^{\Delta}(t) = Ax(t)$, where A is the infinitesimal generator of a C_0-semigroup $T = \{T(s) : s \in \mathsf{T} \subset \mathcal{L}(\mathbb{Y})\}$.

In 2013, Lizama and Mesquita [29] introduced the concept of almost automorphic functions on time scale and presented the first basic results concerning such functions. They also studied the existence and uniqueness of solutions of the following dynamic equation

$$x^{\Delta}(t) = A(t)x(t) + f(t, x(t)),\ t \in \mathsf{T},$$

over Euclidean space $\mathbb{R}^n$. Assuming the equation $x^{\Delta}(t) = A(t)x(t)$ admitting exponential Dichotomy and $f : \mathsf{T} \times X \to X$ satisfying global Lipschitz condition on its second variable, they proved the existence and uniqueness of the almost automorphic solution to the above dynamic equation.

In 2014, Guérékata et al. [16] presented almost automorphic functions of order n. They also studied the existence and uniqueness, global stability of the solution of first order dynamic equation with finite time varying delay.

Using the results of [16, 20, 29, 30], Milcè and Mado proved in 2015 the existence and uniqueness of almost automorphic mild solutions to the following dynamic equation.

$$u^{\Delta}(t) = A(t)u(t) + f\left(t, u(t), \int_0^t \phi(s, u(s))\Delta s\right),\ t \in \mathsf{T}^k,$$

on Euclidian space, $\mathbb{R}^n$. $A : \mathsf{T} \to \mathbb{R}^n$ is a matrix-valued function. ϕ, f both satisfy some Lipschitz conditions.

After that in 2015, Guérékata et al. studied semilinear dynamic equation$x^{\Delta}(t) = Ax(t) + f(t, x(t))$ in Banach space, where A is the infinitesimal generator of a C_0-semigroup of bounded linear operators. They established an almost automorphic mild solution to the equation.

In 2018, Hamza and Oraby [21] studied the stability of nonlinear dynamic equation

$$x^{\Delta}(t) = A(t)x(t) + f(t, x(t)),\ t \in [\tau, \infty)_{\mathsf{T}}$$

Several other aspects are also studied.

In 2016, Guerekata et al. [16] introduced asymptotically almost automorphic functions of order n and applied to dynamic equation on time scale. Following the above results, in 2016, Milce [33] studied the existence of asymptotically almost automorphic solutions for the following integro-dynamic equation

$$x^{\Delta}(t) = Ax(t) + \int_0^t B(t-s)x(s)\Delta s + f(t, x(t))$$

with nonlocal initial condition $x(0) = x_0 + \psi(x)$. A is matrix, B is matrix-valued function, f, ψ are rd-continuous functions satisfying some kind Lipschitz condition.

In 2018, Guerekata et al. [11] studied the existence of asymptotically almost automorphic mild solutions for nonautonomous semilinear evolution equations.
In 2019, Lizama and Mesquita [32] introduced the concept of asymptotically almost automorphic functions of order n on time scale. They established fundamental properties of such functions and investigated the unique solution of IVP associated to the semilinear equation,

$$\begin{aligned} x^{\Delta}(t) &= A(t)x(t) + f(t, x(t)),\ t \in [t_0, \infty)_{\mathsf{T}} \\ x(t_0) &= x_0 \end{aligned}$$

In 2022, Bohner et al. [9] gave some qualitative results for nonlinear integro-dynamic equations via integral inequalities. They considered the equation

$$\begin{aligned} x^{\Delta}(t) + p(t)x^{\sigma}(t) &= \mathcal{F}\left(t, x(t), \int_{t_0}^{t} \mathcal{H}(t, s, x(x))\Delta s\right),\ t \in \mathsf{T}^k, a \le t \le b \\ x(t_0) &= x_0 \in \mathbb{R}^n \end{aligned}$$

where existence, stability, boundedness and dependence of the solution on initial data are discussed. In 2023, Cosme et al. [15], discussed the existence and stability of the bounded solution of the following abstract dynamic equation

$$\begin{aligned} z^{\Delta}(t) &= Az(t) + f(t, z(t)),\ t \in [t_0, \infty)_{\mathsf{T}}, \\ z(t_0) &= z_0 \end{aligned}$$

both mild and classical solutions are discussed.

Motivated by the above we study the following abstract integro-dynamic equation

$$\begin{aligned} y^{\Delta}(s) &= Ay(s) + \mathcal{F}\left(s, y(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, y(\tau))\Delta\tau\right),\ s \in \mathsf{T} \\ y(0) &= y_0, \end{aligned}$$

where A is the infinitesimal generator of a C_0-semigroup of bounded linear operators, $T = \{T(s) : s \in \mathsf{T}\} \subset \mathcal{L}(\mathbb{Y})$.

2. Preliminaries

Definition 2.1. A time scale is any nonempty and closed subset of $\mathbb{R}$.

For a time scale, T, we define, T^{κ} as

$$\mathsf{T}^{\kappa} = \begin{cases} \mathsf{T} \setminus (\rho(\sup \mathsf{T}), \sup \mathsf{T}] & \text{if } \sup \mathsf{T} < \infty \\ \mathsf{T}, & \text{otherwise} \end{cases}$$

And similarly, for $\mathcal{I} = [s_0, S]_{\mathsf{T}} = [s_0, S] \cap \mathsf{T}$, we define $\mathcal{I}^{\kappa}$ as

$$\mathcal{I}^{\kappa} = \begin{cases} \mathcal{I} \setminus (\rho(\sup \mathcal{J}), \sup \mathcal{I}] & \text{if } \sup \mathcal{I} < \infty \\ \mathcal{I}, & \text{otherwise,} \end{cases}$$

Definition 2.2. For time scale, T, the forward and backward jump operators, σ and ρ respectively are defined as follows:

$$\sigma : \mathsf{T} \to \mathbb{R},\ \text{given by } \sigma(t) = \inf\{s \in \mathsf{T} : s \ge t\}$$

and

$$\rho : \mathsf{T} \to \mathbb{R},\ \text{given by } \rho(t) = \sup\{s \in \mathsf{T} : s \le t\}.$$

Definition 2.3. For time scale, T, the forward and backward graininess functions, μ and ν respectively, are defined as:

$$\mu : \mathsf{T} \to \mathbb{R},\ \text{given by } \mu(t) = \sigma(t) - t$$

and

$$\nu : \mathsf{T} \to \mathbb{R},\ \text{given by } \nu(t) = t - \rho(t).$$

Let us mention some existing results related to time scale calculus, some fixed point theorems, C_0-semigroup on time scale and their properties, almost automorphic functions and asymptotically almost automorphic functions and their properties, exponential stability of a function in time scale etc.

Definition 2.4. [6, Definition 1.58] A function $f : \mathsf{T} \longrightarrow \mathbb{Y}$ is called a rd-continuous, if it is continuous at every right dense point of T, and also ensures the existence of its limits at all left dense points in T. We denote by $C_{rd}(\mathsf{T}, \mathbb{Y})$, the collection of all rd-continuous functions $f : \mathsf{T} \longrightarrow \mathbb{Y}$.
We also denote by $BC_{rd}(\mathsf{T}, \mathbb{Y})$, the collection of all rd-continuous and bounded functions $f : \mathsf{T} \longrightarrow \mathbb{Y}$.

Definition 2.5. A function $f : \mathsf{T} \times \mathbb{Y} \times \mathbb{Y}$ is said to be an rd-continuous function on $\mathsf{T} \times \mathbb{Y} \times \mathbb{Y}$ if $f(s, \cdot, \cdot)$ is continuous on $\mathbb{Y} \times \mathbb{Y}$ for every $s \in \mathsf{T}$ and $f(\cdot, x, y)$ is rd-continuous on T for every $(x, y) \in \mathbb{Y} \times \mathbb{Y}$.
Moreover, if the continuity of $f(s, \cdot, \cdot)$ is uniform for every $s \in \mathsf{T}$, then the function f is called uniformly rd-continuous.

Definition 2.6. [6, Definition 2.25] A function $p : \mathsf{T} \longrightarrow \mathbb{R}$ is called a regressive function if $\forall\ s \in \mathsf{T}^k$, the quantity $1 + \mu(s)p(s)$ is always a nonzero quantity, where $\mu : \mathsf{T} \longrightarrow \mathbb{R}$ is the graininess function on T, defined as $\mu(s) = \sigma(s) - s$.
We denote by $\mathcal{R}(\mathsf{T}, \mathbb{R})$, the collection of all regressive functions $p : \mathsf{T} \longrightarrow \mathbb{R}$.

Definition 2.7. If $p \in \mathcal{R}$, then the generalized exponential function is defined as

$$e_p(s, t) = \exp\left(\int_t^s \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right),\ \text{for } t, s \in \mathsf{T}$$

where the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$, given by

$$\xi_h(z) = \frac{1}{h}\log(1 + zh),$$

where log is the principal logarithm function. For $h = 0$, ξ_o is supposed to be the identical transformation.

For more properties of the generalized exponential function relating to regressive function, refer to [6, 29] etc.

Definition 2.8. A mapping between two normed linear spaces is considered compact if bounded sets are mapped into relatively compact sets.

Theorem 2.9 (Arzelà-Ascoli Theorem ([43, Lemma 4]))**.** *A subset of* $C(\mathsf{T}, \mathbb{R})$ *which is both equicontinuous and bounded is relatively compact.*

Theorem 2.10. *In the space of continuous functions on a compact metric space, a subset is relatively compact if and only if it is bounded and equicontinuous.*

Definition 2.11. An operator $T : \mathbb{X} \to \mathbb{Y}$ is said to be completely continuous if it is continuous and sends a bounded set to a relatively compact set. i,e. T continuous as well as compact

Theorem 2.12 (Krasnoselskiĭ fixed point theorem ([34, Theorem 11.2]))**.** *Let $\mathbb{Y}$ be a Banach space and $B \subset \mathbb{Y}$ be a nonempty, closed and convex subset of $\mathbb{Y}$. Let $F_1, F_2 : B \to \mathbb{Y}$ be such that*

(i) *F_1 is continuous and $F_1(B)$ is relatively compact (F_1 is completely continuous)*
(ii) *F_2 is a contraction.*
(iii) *$F_1(y_1) + F_2(y_2) \in B, \ \forall\, y_1, y_2 \in B$.*

Then $\exists\ \bar{y}$ such that $T_1(\bar{y}) + T_2(\bar{y}) = \bar{y}$.

Theorem 2.13 ([29,31])**.** *If $\varepsilon > 0$, then $e_{\ominus\varepsilon}(t, s) \leq 1$, $t, s \in \mathsf{T}$, $t > s$.*

Lemma 2.14 ([29,31])**.** *Let $\varepsilon > 0$, then for any fixed $s \in \mathsf{T}$ and $s = -\infty$, one has the following:* $\lim_{t\to+\infty} e_{\ominus\varepsilon}(t, s) = 0$.

Lemma 2.15 ([39])**.** *Let $y, f \in C_{rd}(\mathsf{T}, \mathbb{R}^+)$ with f a nondecreasing function and $g, h \in \mathcal{R}^+(\mathsf{T}, \mathbb{R})$ with $g \geq 0, h \geq 0$. If*

$$y(s) \leq f(s) + \int_a^s h(s) \left[y(t) + \int_a^t g(\tau)y(\tau)\Delta\tau \right] \Delta s \text{ for } s \in \mathsf{T}^k$$

then the following two conditions hold:

a. $y(s) \leq f(s)\big[1 + \int_a^s h(s)e_{h+g}(s, a)\Delta s\big]$ for $s \in \mathsf{T}^k$;
b. $y(s) \leq f(s)e_{h+g}(s, a)$ for $s \in \mathsf{T}^k$.

In particular if $f(t) = 0$, then $y(t) = 0$ for $s \in \mathsf{T}^k$

Now we recall some results concerning semigroups of linear operators on time scales.

Definition 2.16 ([20])**.** A time scale T satisfying $a - b \in \mathsf{T}$, for any $a, b \in \mathsf{T}$ with $a > b$ is called a semigroup time scale, usually denoted by $\mathsf{T} \subseteq \mathbb{R}^{\geq 0}$.

Definition 2.17 ([20])**.** Let $\mathbb{Y}$ be a Banach space and T is a time scale containing 0. We say that $T : \mathsf{T} \to \mathcal{L}(\mathbb{Y})$ is strongly continuous if $\|T(s)y - y\| \to 0$ as $s \to 0^+$ for each $y \in \mathbb{Y}$.

Definition 2.18 ([20])**.** Let T be a semigroup time scale containing zero and $\mathcal{L}(\mathbb{Y})$ be the space of all bounded linear operators from $\mathbb{Y}$ into itself. A family $T = \big\{T(t) : t \in \mathsf{T}\big\} \subset \mathcal{L}(\mathbb{Y})$, $T : \mathsf{T} \to \mathcal{L}(\mathbb{Y})$ is a C_0-semigroup if it satisfies the following conditions:

(1) $T(s + t) = T(s)T(t)$, for all $s, t \in \mathsf{T}$(the semigroup property).
(2) $T_0 = T(0) = I$, where I is the identity operator on $\mathbb{Y}$.
(3) $\lim_{s\to0^+} T(s)y = y$, i.e., $T(\cdot)y : \mathsf{T} \to \mathbb{Y}$ is continuous at 0 for each $y \in \mathbb{Y}$.

In addition if $\lim_{t\to0} \|T(t) - I\| = 0$, the T is called uniformly continuous semigroup. Also if we have one more condition * $\|T(s)\|_{\mathsf{T}} \leq 1$, along with the conditions as in Definition 2.18, then we call T to be the contraction semigroup of class (C_0).

Definition 2.19 ([20])**.** A linear operator, A is called the generator of a C_0-semigroup $T = \{T(s) : s \in \mathsf{T}\}$ if

$$Ay = \lim_{s \to 0^+} \frac{T(\mu(s))y - T(s)y}{\mu(s) - s}, y \in D(A),$$

where the domain of A, $D(A)$ is the set of all $y \in \mathbb{Y}$ for which the above limit exists uniformly in s.

The semigroup, $T = \{T(s) : s \in \mathsf{T}\}$ is said to be exponentially stable if, there exists $M \geq 1$ and $\varepsilon > 0$ such that $\|T(s - s_0)\| \leq Ke_{\ominus\varepsilon}(s, s_0)$, for all $s, s_0 \in \mathsf{T}$ with $s > s_0$.

For more details on semigroups on time scales refer to [20].

Definition 2.20 ([26])**.** Let A be a generator of a C_0-semigroup $T = \{T(s) : s \in \mathsf{T}\}$. A function $y : \mathsf{T} \to \mathbb{Y}$ is said to be mild solution of the equation

$$y^{\Delta}(t) = Ay(t) + f(s)$$

if it is rd-continuous and satisfied the integral equation

$$y(t) = T(s - s_0)y_0 + \int_{s_0}^{s} T(s - \sigma(t))f(t)\Delta t.$$

Following the above variation of constant formula we define the following.

Definition 2.21. Let A be the infinitesimal generator of a C_0-semigroup $\{T(s) : s \in \mathsf{T}\}$. Also assume that $\mathcal{F}$ and $\mathcal{H}$ are functions in $C_{rd}(\mathcal{I} \times \mathbb{Y} \times \mathbb{Y}, \mathbb{Y})$ and $C_{rd}(\mathcal{I} \times \mathcal{I} \times \mathbb{Y}, \mathbb{Y})$, respectively. Then a function, $y \in BC_{rd}(\mathsf{T}, \mathbb{Y})$ is a mild solution of (3.1)-(3.2) if y satisfies the delta integral equation

$$y(s) = T(s - s_0)y_0 + \int_{s_0}^{s} T(s - \sigma(t))\mathcal{F}\left(t, y(t), \int_{s_0}^{t} \mathcal{H}(t, \tau, y(\tau))\Delta\tau\right)\Delta t. \tag{2.1}$$

Now we recall some properties of almost automorphic functions on time scales.

Definition 2.22 ([29, Definition 3.1])**.** A time scale T is called invariant under translations if

$$\Pi = \{\tau \in \mathbb{R} : s \pm \tau \in \mathsf{T}, \forall s \in \mathsf{T}\} \neq 0.$$

Remark 2.23. One can easily verify the fact that a symmetric time scale which has semigroup property and contains zero is also invariant under translation.

Lemma 2.24 ([16])**.** *Let* T *be invariant under translation time scale. Then*

i) $\Pi \subset \mathsf{T} \Longleftrightarrow 0 \in \mathsf{T}$.
ii) $\Pi \cap \mathsf{T} \longleftrightarrow 0 \notin \mathsf{T}$.

In the following, we present preliminary results concerning almost automorphicity and asymptotically almost automorphicity of functions in the time scale perspective. The concept of almost automorphicity is a more general concept of almost periodic function. For more details on such functions refer to [1, 16, 29]. Asymptotically almost automorphic functions are again a generalization of almost automorphic functions, details of which can be found in [32].

Definition 2.25 ([1, 29])**.** Let T be an invariant under translation time scale and $\mathbb{Y}$ be a Banach space. A function $f : \mathsf{T} \to \mathbb{Y}$ is called a almost periodic, if and only if, for every sequence, (s_n) in Π one can extract a subsequence, (τ_n) such that $f(s + \tau_n)$ converges uniformly in T.

Definition 2.26 ([29, Definition 3.15])**.** Let $\mathbb{Y}$ be a Banach space and T be a time scale that is invariant under translation. Then an rd-continuous function $f : \mathsf{T} \to \mathbb{Y}$ is called almost automorphic on T if for every sequence (s_n) on Π, there exists a subsequence $(\tau_n) \subset (s_n)$ such that,

$$\tilde{f}(s) = \lim_{n\to\infty} f(s + \tau_n)$$

is well defined for each $s \in \mathsf{T}$, and

$$\lim_{n\to\infty \tilde{f}(s-\tau_n)} = f(s)$$

for each $s \in \mathsf{T}$.

The space of all almost automorphic functions, $f : \mathsf{T} \to \mathbb{Y}$ is denoted by $AA(\mathsf{T}, \mathbb{Y})$. It is also a well-known result mentioned in [29], that if T is an invariant under translation time scale, then the graininess function $\mu : \mathsf{T} \to \mathbb{R}_+$ is almost automorphic.

Remark 2.27. The space $AA(\mathsf{T}, \mathbb{Y})$ equiped with the norm $\sup_{s\in\mathsf{T}} \|f(s)\|$ is a Banach space.

Definition 2.28 ([29, Definition 3.20])**.** Let $\mathbb{Y}$ be a (real or complex) Banach space and T be a symmetric time scale which is invariant under translation. Then an rd-continuous function $f : \mathsf{T} \times \mathbb{Y} \to \mathbb{Y}$ is called almost automorphic in $s \in \mathsf{T}$ uniformly for $x \in K$, where K is any compact subset of $\mathbb{Y}$, if for every sequence (s_n) on Π, there exists a subsequence $(\tau_n) \subset (s_n)$ such that

$$\tilde{f}(s, y) = \lim_{n\to\infty} f(s + \tau_n, y) \tag{2.2}$$

is well defined for each $s \in \mathsf{T}, y \in \mathbb{Y}$ and

$$\lim_{n\to\infty} \tilde{f}(s - \tau_n, y) = f(s, y) \tag{2.3}$$

for each $s \in \mathsf{T}$ and $y \in \mathbb{Y}$.

We denote by $AA(\mathsf{T} \times \mathbb{Y}, \mathbb{Y})$, the space of all almost automorphic functions $f : \mathsf{T} \times \mathbb{Y} \to \mathbb{Y}$ on time scale T.
Notwithstanding the above definitions, we can define the following.

Definition 2.29. An rd-continuous function $f : \mathsf{T} \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ is said to be an almost automorphic function on $s \in \mathsf{T}$ uniformly for all $x, y \in \mathbb{Y}$, if for every sequence (s_n) on Π, there exists a subsequence $(\tau_n) \subset (s_n)$ such that

$$\lim_{n\to\infty} f(s + \tau_n, x, y) = \tilde{f}(s, x, y) \tag{2.4}$$

is well defined for each $s \in \mathsf{T},\ x, y \in \mathbb{Y}$ and

$$\lim_{n\to\infty} \tilde{f}(s - \tau_n, x, y) = f(s, x, y) \tag{2.5}$$

for each $s \in \mathsf{T}$ and $x, y \in \mathbb{Y}$.

We denote by $AA(\mathsf{T}\times\mathbb{Y}\times\mathbb{Y},\mathbb{Y})$, the space of all almost automorphic functions $f:\mathsf{T}\times\mathbb{Y}\times\mathbb{Y}\to\mathbb{Y}$ on time scale T.

Definition 2.30 ([29]). An rd-continuous function $f:\mathsf{T}^+\times\mathbb{Y}\times\mathbb{Y}\to\mathbb{Y}$ is said to be asymptotically almost automorphic if it can be uniquely decomposed as $f=g+\phi$, where $g\in AA(\mathsf{T}^+\times\mathbb{Y}\times\mathbb{Y},\mathbb{Y})$ and $\phi\in C_{rd}(\mathsf{T}^+\times\mathbb{Y}\times\mathbb{Y},\mathbb{Y})$ such that $\lim_{s\to\infty}\|\phi(s,x,y)\|=0$, for all $x,y\in\mathbb{Y}$.

The set of all functions, $f:\mathsf{T}^+\times\mathbb{Y}\times\mathbb{Y}\to\mathbb{Y}$ which are asymptotically almost automorphic is denoted by $AAA(\mathsf{T}^+\times\mathbb{Y}\times\mathbb{Y},\mathbb{Y})$.
Note: We denote by, $C_{rd_0}(\mathsf{T}^+\times\mathbb{Y}\times\mathbb{Y},\mathbb{Y})$ being the set of all functions, $f\in C_{rd}(\mathsf{T}^+\times\mathbb{Y}\times\mathbb{Y},\mathbb{Y})$ such that $\lim_{s\to\infty}\|\phi(s,x,y)\|=0$, for all $x,y\in\mathbb{Y}$.

Remark 2.31. If $f=g+\phi$ is asymptotically almost automorphic such that g is principal term and ϕ is corrective term, then

$$\|f\|=\sup_{s\in\mathsf{T}}\|g(s)\|_{\mathbb{Y}}+\sup_{t\in\mathsf{T}^+}\|\phi(s)\|_{\mathbb{Y}}$$

defines a norm such that $(AAA(\mathsf{T}^+\times\mathbb{Y}),\|\cdot\|)$ is a Banach space.

3. Main results

In our first approach, we investigate the existence and uniqueness of the mild solution of the following abstract integro-dynamic initial value problem

$$y^{\Delta}(s) = Ay(s)+\mathcal{F}\left(s,y(s),\int_{t_0}^{s}\mathcal{H}(s,\tau,y(\tau))\Delta\tau\right), \tag{3.1}$$

$$y(s_0) = y_0. \tag{3.2}$$

where $s\in\mathcal{I}^k$.

Lemma 3.1. *Let A be the infinitesimal generator of the C_0-semigroup $\{T(s):s\in\mathsf{T}\}$. Also assume that $\mathcal{F}$ and $\mathcal{H}$ are functions in $C_{rd}(\mathcal{I}\times\mathbb{Y}\times\mathbb{Y},\mathbb{Y})$ and $C_{rd}(\mathcal{I}\times\mathcal{I}\times\mathbb{Y},\mathbb{Y})$ respectively. Then y is a mild solution of* (3.1)-(3.2) *iff y satisfies the $\Delta-$integral equation*

$$y(s)=T(s-s_0)y_0+\int_{s_0}^{s}T(s-\sigma(t))\mathcal{F}\left(t,y(t),\int_{s_0}^{t}\mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t. \tag{3.3}$$

Proof. For proof of the lemma, we refer [9, Lemma 3.1]. □

In the remainder of this paper, we will consider T as a symmetric time scale with the semigroup property and contains zero. We denote, $\mathsf{T}^+=\mathsf{T}\cap[0,\infty)$.

Theorem 3.2. *Consider the following hypothesis*

(H$_1$) *The function $\mathcal{F}:\mathcal{I}\times\mathbb{Y}\times\mathbb{Y}\to\mathbb{Y}$ is rd-continuous such that,*

$$\|\mathcal{F}(s,x_1,y_1)-\mathcal{F}(s,x_2,y_2)\|\le L_{\mathcal{F}}(s)\left(\|x_1-x_2\|+\|y_1-y_2\|\right)$$

for all $s\in\mathcal{I}$ and $x_i,y_i\in\mathbb{Y}$, $L_{\mathcal{F}}\in\mathcal{R}^+(\mathcal{I},\mathbb{R}^+)$.

(H$_2$) *The function,* $\mathcal{H}:\mathcal{I}\times\mathcal{I}\times\mathbb{Y}\to\mathbb{Y}$ *be such that* $\mathcal{H}(\cdot,\cdot,y)$ *is rd-continuous on* $\mathcal{I}\times\mathcal{I}$ *for any* $y\in\mathbb{Y}$ *and* $\mathcal{H}(t,s,\cdot)$ *is continuous on* $\mathbb{Y}$ *for all* $s,t\in\mathcal{I}$ *satisfying,*

$$\|\mathcal{H}(t,s,y_1)-\mathcal{H}(t,s,y_2)\|\leq L_{\mathcal{H}}(s)\|y_1-y_2\|\ \forall\ t,s\in\mathcal{I}\ \text{and}\ y_i(i=1,2)\in\mathbb{Y}$$

where $L_{\mathcal{H}}\in\mathcal{R}^+(\mathcal{I},\mathcal{R}^+)$.

(H$_3$) A *is the generator of an exponentially stable* C_0*-semigroup,* $\{T(s):s\in\mathsf{T}\}$.

(H$_4$) $ML^*_{\mathcal{F}}\left(\frac{1+\tilde{\mu}\varepsilon}{\varepsilon}\right)(1+L^*_{\mathcal{H}}(S-s_0))\leq 1$, *where* $L^*_{\mathcal{F}}=\sup_{s\in\mathcal{I}}L_{\mathcal{F}}(s)$ *and* $L^*_{\mathcal{H}}=\sup_{s\in\mathcal{I}}L_{\mathcal{H}}(s)$.

Then (3.1)-(3.2) *has a unique mild solution whenever* $M_{\mathcal{F}}=\sup\left\{\|\mathcal{F}(s,0,\Psi)\|_{\mathbb{Y}};\ s\in\mathcal{I},\ \Psi\in\mathbb{Y}\right\}<\infty$.

Proof. Define a ball, $B_k\subset C_{rd}(\mathcal{I}\times\mathbb{Y})$ as $B_k=\{y\in C_{rd}(\mathcal{I}\times\mathbb{Y}):\|y\|_{\mathbb{Y}}\leq k\}$, where $k=2M(\|y_0\|+M_{\mathcal{F}})$. Let us also define a function, $\mathcal{W}:B_k\to C_{rd}(\mathcal{I}\times\mathbb{Y})$ as

$$\mathcal{W}(y)(s):=T(s-s_0)y_0+\int_{s_0}^{s}T(s-\sigma(t))\mathcal{F}\left(t,y(t),\int_{s_0}^{t}\mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t. \tag{3.4}$$

In order to apply the Krasnoselskiĭ fixed point theorem given by Theorem 2.12, we dichotomize $\mathcal{W}$ as,

$$\mathcal{W}(y)(s)=\mathcal{W}_1(y)(s)+\mathcal{W}_2(y)(s),$$

where

$$\mathcal{W}_1(y)(s):=T(s-s_0)y_0+\int_{s_0}^{s}T(s-\sigma(t))\mathcal{F}\left(t,0,\int_{s_0}^{t}\mathcal{H}(t,\tau,0)\Delta\tau\right)\Delta t. \tag{3.5}$$

and

$$\begin{aligned}\mathcal{W}_2(y)(s):=&\int_{s_0}^{s}T(s-\sigma(t))\left[\mathcal{F}\left(t,y(t),\int_{s_0}^{t}\mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\right.\\&\left.-\mathcal{F}\left(t,0,\int_{s_0}^{t}\mathcal{H}(t,\tau,0)\Delta\tau\right)\right]\Delta t..\end{aligned} \tag{3.6}$$

It is obvious to see that $\mathcal{W}_1$ is continuous, We show that $\mathcal{W}_1$ is completely continuous and $\mathcal{W}_2$ is a contraction.

Step 1: $\mathcal{W}_1:B_k\to C_{rd}(\mathcal{I}\times\mathbb{Y})$ is completely continuous.

For $y\in B_k$ we have

$$\begin{aligned}\|\mathcal{W}_1(y)\|_{\mathbb{Y}}&=\left\|T(s-s_0)y_0+\int_{s_0}^{s}T(s-\sigma(t))\mathcal{F}\left(t,0,\int_{0}^{t}\mathcal{H}(t,\tau,0)\Delta\tau\right)\Delta t\right\|_{\mathbb{Y}}\\&\leq\|T(s-s_0)\|_{\mathbb{Y}}\ \|y_0\|_{\mathbb{Y}}+\int_{s_0}^{s}\left\|T(s-\sigma(t))\mathcal{F}\left(t,0,\int_{s_0}^{t}\mathcal{H}(t,\tau,0)\Delta\tau\right)\right\|_{\mathbb{Y}}\Delta t\\&\leq Me_{\ominus\varepsilon}(s,s_0)\|y_0\|_{\mathbb{Y}}+MM_{\mathcal{F}}\int_{s_0}^{S}e_{\ominus\varepsilon}(s,\sigma(t))\Delta t\\&\leq M(\|y_0\|_{\mathbb{Y}}+M_{\mathcal{F}}(S-s_0))\quad\text{(using Theorem 2.13).}\end{aligned}$$

Hence we see that, $\mathcal{W}_1$ is bounded in B_k.
Next, we test equicontinuity of $\mathcal{W}_1(B_k)$. Let $s_1, s_2 \in \mathcal{I}$ and $y \in B_k$. Then

$$\begin{aligned}
&\|\mathcal{W}_1(y)(s_2) - \mathcal{W}_1(y)(s_1)\|_{\mathbb{Y}} \\
&= \Bigg\| T(s_2 - s_0)y_0 + \int_{s_0}^{s_2} T(s_2 - \sigma(t))\mathcal{F}\left(t, 0, \int_0^t \mathcal{H}(t,\tau,0)\Delta\tau\right)\Delta t \\
&\quad - T(s_1 - s_0)y_0 - \int_{s_0}^{s_1} T(s_1 - \sigma(t))\mathcal{F}\left(t, 0, \int_{s_0}^t \mathcal{H}(t,\tau,0)\Delta\tau\right)\Delta t \Bigg\|_{\mathbb{Y}} \\
&\leq \Bigg\| (T(s_2 - s_0) - T(s_1 - s_0)y_0) \\
&\quad + \int_{s_0}^{s_2} T(s_2 - \sigma(t))\mathcal{F}\left(t, 0, \int_{s_0}^t \mathcal{H}(t,\tau,0)\Delta\tau\right)\Delta t \\
&\quad - \int_{s_0}^{s_1} T(s_1 - \sigma(t))\mathcal{F}\left(t, 0, \int_0^t \mathcal{H}(t,\tau,0)\Delta\tau\right)\Delta t \Bigg\|_{\mathbb{Y}} \\
&\leq \Bigg\| (T(s_2 - s_0) - T(s_1 - s_0)y_0) \\
&\quad + T(s_2 - s_0)\int_{s_0}^{s_1} T(s_0 - \sigma(t))\mathcal{F}\left(t, 0, \int_{s_0}^t \mathcal{H}(t,\tau,0)\Delta\tau\right)\Delta t \\
&\quad + \int_{s_1}^{s_2} T(s_2 - \sigma(t))\mathcal{F}\left(t, 0, \int_{s_0}^t \mathcal{H}(t,\tau,0)\Delta\tau\right)\Delta t \\
&\quad - T(s_1 - s_0)\int_0^{s_1} T(s_0 - \sigma(t))\mathcal{F}\left(t, 0, \int_{s_0}^t \mathcal{H}(t,\tau,0)\Delta\tau\right)\Delta t \Bigg\|_{\mathbb{Y}} \\
&\leq \big\| T(s_2 - s_0) - T(s_1 - s_0) \big\|_{\mathbb{Y}} \Bigg(\|y_0\|_{\mathbb{Y}} + \Bigg\| \int_{s_0}^{s_1} T(s_0 - \sigma(t)) \\
&\qquad\qquad \mathcal{F}\left(t, 0, \int_{s_0}^t \mathcal{H}(t,\tau,0)\Delta\tau\right)\Delta t \Bigg\|_{\mathbb{Y}} \Bigg) \\
&\quad + \Bigg\| \int_{s_1}^{s_2} T(s_2 - \sigma(t))\mathcal{F}\left(t, 0, \int_{s_0}^t \mathcal{H}(t,\tau,0)\Delta\tau\right)\Delta t \Bigg\|_{\mathbb{Y}} \\
&\leq \big\| T(s_2 - s_0) - T(s_1 - s_0) \big\|_{\mathbb{Y}} \left(\|y_0\|_{\mathbb{Y}} + MM_{\mathcal{F}} \int_{s_0}^{s_1} e_{\ominus\varepsilon}(s_0, \sigma(t)) \right) \\
&\quad + M_{\mathcal{F}} \int_{s_1}^{s^2} e_{\ominus\varepsilon}(s_2, \sigma(t))\Delta t \\
&\leq \big\| T(s_2 - s_0) - T(s_1 - s_0) \big\|_{\mathbb{Y}} \left(\|y_0\|_{\mathsf{T}} + MM_{\mathcal{F}} \int_{s_0}^{s_1} e_{\varepsilon}(\sigma(t), s_0) \right) \\
&+ M_{\mathcal{F}}(s_2 - s_1) \quad \text{(By using Theorem 2.13)}.
\end{aligned}$$

We will have a similar inequality when we take $s_1 > s_2$. Since T represents a C_0-semigroup, T is continuous, and hence the first part of the above inequality tends to zero as s_2 tends to s_1. Thus it is obvious that the right-hand side of the above inequality tends to zero as s_2 tends to s_1, thus affirms the equicontinuity of $\mathcal{W}_1(B_k)$ by compact mapping theorem.

Also, since $\mathcal{W}_1(B_k)$ is both equicontinuous and bounded, by Arzela Ascoli Theorem, W_1 is compact. Since every compact operator is also completely continuous and subsequently $\mathcal{W}_1$ is completely continuous.

Step 2: we show that $\mathcal{W}_2 : B_k \to C_{rd}(\mathcal{I} \times \mathbb{Y})$ is a contraction. Let $x, y \in B_k$, then we have

$$\begin{aligned}
&\|\mathcal{W}[y](s) - \mathcal{W}[x](s)\|_{\mathbb{Y}} \\
&\quad \leq \int_{s_0}^{s} |T(s-\sigma(t))| \left\| \mathcal{F}\left(t, y(t), \int_{s_0}^{t} \mathcal{H}(t,\tau,y(\tau))\Delta\tau\right) \right. \\
&\qquad\qquad\qquad \left. - \mathcal{F}\left(t, x, \int_{s_0}^{t} \mathcal{H}(t,\tau,x(\tau))\Delta\tau\right) \right\|_{\mathbb{Y}} \Delta t. \\
&\quad \leq M \int_{s_0}^{s} e_{\ominus\varepsilon}(s,\sigma(t)) L_{\mathcal{F}}(t) \left(\|y(t)-x(t)\|_{\mathbb{Y}} + \int_{s_0}^{t} L_{\mathcal{H}}(\tau)\|y(\tau)-x(\tau)\|_{\mathbb{Y}}\Delta\tau \right) \Delta t \\
&\quad \leq M \int_{s_0}^{s} e_{\ominus\varepsilon}(s,\sigma(t)) L_{\mathcal{F}}(t) \left(1 + L_{\mathcal{H}}^{*}(t-s_0)\right) \|y(t)-x(t)\|_{\mathbb{Y}} \Delta t \\
&\quad \leq M L_{\mathcal{F}}^{*} \left(1 + L_{\mathcal{H}}^{*}(S-s_0)\right) \|y-x\|_{\mathbb{Y}} \int_{s_0}^{s} e_{\ominus\varepsilon}(s,\sigma(t))\Delta t \\
&\quad \leq \; M \left(\frac{1+\tilde{\mu}}{\varepsilon}\varepsilon \right) L_{\mathcal{F}}^{*}(1 + L_{\mathcal{H}}^{*}(S-s_0))\|y-x\|_{\mathbb{Y}}
\end{aligned}$$

Hence, by the conditions (H_4), we see that $\mathcal{W}_2$ is a contraction.

Step 3: For $x, y \in B_k$, $\mathcal{W}_1(x)(s) + \mathcal{W}_1(y)(s) \in B_k$, $\forall\ s \in \mathcal{I}^r$. We have

$$\begin{aligned}
&\left\|\mathcal{W}_1[y](s) + \mathcal{W}_2[x](s)\right\|_{\mathbb{Y}} \\
&\quad = \left\| T(s-s_0)y_0 + \int_{s_0}^{s} T(s-\sigma(t))\mathcal{F}\left(t, 0, \int_{s_0}^{t} \mathcal{H}(t,\tau,0)\Delta\tau\right)\Delta t \right. \\
&\qquad + \int_{s_0}^{s} T(s-\sigma(t)) \left[\mathcal{F}\left(t, y(t), \int_{s_0}^{t} \mathcal{H}(t,\tau,y(\tau))\,\Delta\tau\right) \right. \\
&\qquad\quad \left.\left. - \mathcal{F}\left(t, 0, \int_{s_0}^{t} \mathcal{H}(t,\tau,0)\Delta\tau\right) \right] \Delta t \right\|_{\mathbb{Y}} \\
&\quad \leq \|T(s-s_0)y_0\|_{\mathbb{Y}} + \left\| \int_{s_0}^{s} T(s-\sigma(t))\mathcal{F}\left(t, 0, \int_{s_0}^{t} \mathcal{H}(t,\tau,0)\Delta\tau\right)\Delta t \right\|_{\mathbb{Y}} \\
&\qquad + \int_{s_0}^{s} \|T(s-\sigma(t))\|_{\mathbb{Y}} \left\| \mathcal{F}\left(t, y(t), \int_{s_0}^{t} \mathcal{H}(t,\tau,y(\tau))\,\Delta\tau\right) \right.
\end{aligned}$$

$$- \mathcal{F}\left(t, 0, \int_{s_0}^{t} \mathcal{H}(t,\tau,0)\Delta\tau\right)\Bigg\|_{\mathbb{Y}} \Delta t$$
$$\le M\|y_0\|_{\mathbb{Y}} + MM_{\mathcal{F}} + (S-s_0)(L^*_{\mathcal{F}}(1+L^*_{\mathcal{H}}(S-s_0))\|x\|_{\mathbb{Y}}$$
$$= M(\|y_0\|_{\mathbb{Y}} + M_{\mathcal{F}}) + M(S-s_0)(L^*_{\mathcal{F}}(1+L^*_{\mathcal{H}}(S-s_0))k$$
$$\le k.$$

With the help of **Step 1, Step 2, Step 3**, we can confirm Theorem 2.12 that $\exists$ $\bar{y} \in B_k \subset C_{rd}(\mathcal{I}, \mathbb{Y})$ such that $\mathcal{W}_1(\bar{y}) + \mathcal{W}_2(\bar{y}) = \bar{y}$.

Step 4: We claim that $\bar{y}$ is unique. If possible let us suppose that y_1, y_2 are two distinct solutions of the IVP, then for any $s \in \mathcal{I}$, using (H_1)-(H_3), we get

$$\|y(s) - x(s)\|_{\mathbb{Y}}$$
$$\le \int_{s_0}^{s} \|T(s-\sigma(t))\|_{\mathbb{Y}} \left\|\mathcal{F}\left(t, y(t), \int_{s_0}^{t} \mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\right.$$
$$\left. - \mathcal{F}\left(t, x, \int_{s_0}^{t} \mathcal{H}(t,\tau,x(\tau))\Delta\tau\right)\right\|_{\mathbb{Y}} \Delta t.$$
$$\le M \int_{s_0}^{s} e_{\ominus\varepsilon}(s,\sigma(t)) L_{\mathcal{F}}(t)\left(\|y(t)-x(t)\|_{\mathbb{Y}} + \int_{s_0}^{t} L_{\mathcal{H}}(\tau)\|y(\tau)-x(\tau)\|_{\mathbb{Y}}\Delta\tau\right)\Delta t$$
$$\le \int_{s_0}^{s} M\frac{(1+\tilde{\mu}\varepsilon)}{\varepsilon} L_{\mathcal{F}}(t)\left(\|y(t)-x(t)\|_{\mathbb{Y}} + \int_{s_0}^{t} L_{\mathcal{H}}(\tau)\|y(\tau)-x(\tau)\|_{\mathbb{Y}}\Delta\tau\right)\Delta t.$$

Using the Lemma 2.15, from the above inequality we get $\|y(s)-x(s)\|_{\mathbb{Y}} \le 0$, which gives $y = x$. This completes the proof. ☐

Inspired by the definition of bi-almost automorphic function in $\mathbb{R}$ as in [40], we define the following:

Definition 3.3. (Bi-almost automorphic function) A function, $f(s,t) : \mathsf{T} \times \mathsf{T} \to \mathbb{Y}$ which is rd-continuous with respect to both its variables, is called bi-almost automorphic if for every sequence (s_n) on Π, there exists a subsequence $(\tau_n) \subset (s_n)$ such that

$$\tilde{f}(s,t) = \lim_{n\to\infty} f(s+\tau_n, t+\tau_n) \tag{3.7}$$

is well defined for each $s, t \in \mathsf{T}$ and

$$\lim_{n\to\infty} \tilde{f}(s-\tau_n, t-\tau_n) = f(s,t) \tag{3.8}$$

for each $s, t \in \mathsf{T}$.

By $bAA(\mathsf{T}\times\mathsf{T}, \mathbb{Y})$, we denote the set of all those bi-almost automorphic functions.

Remark 3.4. The notion of bi-almost automorphicity is the generalization of the function $f(s,t)$ having the same period with respect to both of its variables. i,e. $f(s+P, t+P) = f(s,t)$ $\forall s, t \in \mathsf{T}$ for some $P \in \mathbb{R} - \{0\}$.

Definition 3.5. A function $f(s,t,y) : \mathsf{T} \times \mathsf{T} \times \mathbb{Y} \to \mathbb{Y}$ which is rd-continuous in its first and second variable, is called bi-almost automorphic if for every sequence (s_n)

on Π, there exists a subsequence $(\tau_n) \subset (s_n)$ such that

$$\tilde{f}(s,t,y) = \lim_{n\to\infty} f(s+\tau_n, t+\tau_n, y) \tag{3.9}$$

is well defined for each $s,t \in \mathsf{T}$ uniformly in $\mathbb{Y}$ and

$$\lim_{n\to\infty} \tilde{f}(s-\tau_n, t-\tau_n, y) = f(s,t,y) \tag{3.10}$$

for each $s,t \in \mathsf{T}$ uniformly in $\mathbb{Y}$.

By $bAA(\mathsf{T}\times\mathsf{T}\times\mathbb{Y},\mathbb{Y})$, we denote the set of all those bi-almost automorphic functions.

Definition 3.6 (**Bi-asymptotically almost automorphic function**). A function $f : \mathsf{T}\times\mathsf{T} \to \mathbb{Y}$, which is rd-continuous with respect to both of its variables, is said to be bi-asymptotically almost automorphic if the function $f(s,t)$ has a unique decomposition, $f(s,t) = g(s,t) + h(s,t)$ with $g \in bAA(\mathsf{T}\times\mathsf{T},\mathbb{Y})$ and $h \in C_{rd_0}(\mathsf{T}^+\times\mathsf{T}^+)$, i,e. h is rd-continuus with respect to both the variables and $\lim_{(s,t)\to(\infty,\infty)} h(s,t) = 0$.

By $bAAA(\mathsf{T}\times\mathsf{T},\mathbb{Y})$, we denote the set of all those bi-asymptotically almost automorphic functions.

The above definition can be extended to a function, $f : \mathsf{T}\times\mathsf{T}\times\mathbb{Y} \to \mathbb{Y}$ as follows:

Definition 3.7. A function $f : \mathsf{T}\times\mathsf{T}\times\mathbb{Y} \to \mathbb{Y}$, which is rd-continuous with respect to first and second variables, is said to be bi-asymptotically almost automorphic if the function f has a unique decomposition, $f(s,t,y) = g(s,t,y) + h(s,t,y)$ with $g \in bAA(\mathsf{T}\times\mathsf{T}\times\mathbb{Y},\mathbb{Y})$ and $h \in C_{rd_0}(\mathsf{T}^+\times\mathsf{T}^+\times\mathbb{Y},\mathbb{Y})$, i,e. h is rd-continuous with respect to both the variables and $\lim_{(s,t)\to(\infty,\infty)} \|h(s,t,y)\| = 0$ uniformly for any $y \in \mathbb{Y}$.

Definition 3.8. We define $C_{rd}^{loc}(\mathsf{T}^+\times\mathbb{Y}\times\mathbb{Y},\mathbb{Y})$, to be the set of all functions, $f : \mathsf{T}^+\times\mathbb{Y}\times\mathbb{Y} \to \mathbb{Y}$ which are rd-continuous on $\mathsf{T}^+\times\mathbb{Y}\times\mathbb{Y}$ and are locally Lipschitz as follows:

$$\|f(s,x_1,y_1) - f(s,x_2,y_2)\| \le L_f(s)\left(\|x_1-x_2\| + \|y_1-y_2\|\right)$$

for all $s \in \mathsf{T}^+$ and $x_i, y_i \in \mathbb{Y}, i = 1,2,$ where $L_f \in BC_{rd}(\mathsf{T}^+,\mathsf{R}^+)$

Definition 3.9. Let $\mathcal{T} \subset \mathsf{T}$. We define $C_{rd}^{loc}(\mathcal{T}\times\mathcal{T}\times\mathbb{Y},\mathbb{Y})$, to be the set of all functions, $f : \mathsf{T}^+\times\mathcal{T}\times\mathbb{Y} \to \mathbb{Y}$ which are rd-continuous on $\mathcal{T}\times\mathsf{T}^+\times\mathbb{Y}$ and are locally Lipschitz as follows:

$$\|f(s,t,y_1(t)) - f(s,t,y_2(t))\|_{\mathbb{Y}} \le L_f(t)\|y_1 - y_2\|_{\mathbb{Y}}$$

for all $s,t \in \mathcal{T}$ and $y_i \in \mathbb{Y}, i = 1,2,$ where $L_f \in BC_{rd}(\mathcal{T},\mathsf{R}^+)$

In the following we establish results concerning the existence and uniqueness of the bounded, asymptotically almost automorphic solution to the given IVP; (3.1)-(3.2) such that $s \in \mathsf{T}$.

Let $y \in BC_{rd}(\mathsf{T},\mathbb{Y})$ and consider the following hypotheses:

(H_1') The function $\mathcal{F}(s,y,z)(= \mathcal{P}(s,y,z) + \mathcal{Q}(s,y,z)) \in AAA(\mathsf{T}^+\times\mathbb{Y}\times\mathbb{Y},\mathbb{Y})$ be such that, $\mathcal{F} \in C_{rd}^{loc}(\mathsf{T}^+\times\mathbb{Y}\times\mathbb{Y},\mathbb{Y})$ with $L_{\mathcal{F}} \in BC_{rd}(\mathsf{T}^+,\mathbb{R}^+)$ and $\mathcal{P} \in C_{rd}^{loc}(\mathsf{T}\times\mathbb{Y}\times\mathbb{Y},\mathbb{Y})$ with $L_{\mathcal{P}} \in BC_{rd}(\mathsf{T},\mathsf{R}^+)$.

(H_2') The function, $\mathcal{H}(s,\tau,y(\tau))(=\mathcal{J}(s,\tau,y(\tau))+\mathcal{K}(s,\tau,y(\tau)))\in bAAA(\mathsf{T}^+\times\mathsf{T}^+\times\mathbb{Y},\mathbb{Y})$ be such that $\mathcal{H}\in C_{rd}^{loc}(\mathsf{T}^+\times\mathsf{T}^+\times\mathbb{Y},\mathbb{Y})$ with $L_{\mathcal{H}}\in BC_{rd}(\mathsf{T}^+,\mathbb{R}^+)$ and $\mathcal{J}\in C_{rd}^{loc}(\mathsf{T}\times\mathbb{Y}\times\mathbb{Y},\mathbb{Y})$ with $L_{\mathcal{J}}\in BC_{rd}(\mathsf{T},\mathbb{R}^+)$.

(H_3') A generates an exponentially stable C_0-semigroup, $T=\{T(s):s\in\mathsf{T}^+\}$.

(H_4') There exists $r>0$ such that $\frac{\varepsilon r}{M}-r(1+\tilde{\mu}\varepsilon)\left(L_{\mathcal{F}}^*+L_{\mathsf{H}}^{1\,*}\right)>(1+\tilde{\mu}\varepsilon)M_{\mathcal{F}}$, where $M_{\mathcal{F}}=\sup\left\{\|\mathcal{F}(s,0,z)\|_{\mathbb{Y}};\ s\in\mathsf{T}^+,\ z\in\mathbb{Y}\right\}$, where, $L_{\mathcal{F}}^*=\sup_{t\in\mathsf{T}^+}L_{\mathcal{F}}(t)$, $L_{\mathcal{H}}^{1\,*}=\sup_{t\in\mathsf{T}^+}L_{\mathcal{H}}^1(s)$.

Lemma 3.10. *Let A be the infinitesimal generator of the C_0-semigroup $T=\{T(s):s\in\mathsf{T}^+\}$. Also assume that $\mathcal{F}$ and $\mathcal{H}$ are functions in $C_{rd}(\mathsf{T}\times\mathbb{Y}\times\mathbb{Y},\mathbb{Y})$ and $C_{rd}(\mathsf{T}\times\mathsf{T}\times\mathbb{Y},\mathbb{Y})$, respectively. Then $y\in BC_{rd}(\mathsf{T},\mathbb{Y})$ is a mild solution of* (3.1)-(3.2)*, with $s\in\mathsf{T}$, if and only if, y satisfies the following improper $\Delta-$integral*

$$y(s)=\int_{-\infty}^{s}T(s-\sigma(t)\mathcal{F}\left(t,y(t),\int_{-\infty}^{t}\mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t. \tag{3.11}$$

Proof. If y is mild solution of (3.1)-(3.2) then by Definition 2.21 we have

$$y(s)=T(s-s_o)y_0+\int_{s_0}^{s}T(s-\sigma(t))\mathcal{F}\left(t,y(t),\int_{s_0}^{t}\mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t. \tag{3.12}$$

Since T is exponentially stable, so as a result we get

$$\|T(s-s_0)y_0\|=Me_{\ominus\varepsilon}(s,s_0)\|y_0\|. \tag{3.13}$$

Again, since $y_0=y(s_0)$ and $y\in BC_{rd}(\mathsf{T},\mathbb{Y})$, there exists $m>0$ such that $\|y\|_{\mathbb{Y}}\le m$ and hence from (3.13), we get

$$\|T(s-s_0)y_0\|=Mme_{\ominus\varepsilon}(s,s_0),\ s\ge 0. \tag{3.14}$$

Taking $\lim s_0\to-\infty$, we can see from (3.13) that

$$\lim_{s_0\to-\infty}\|T(s-s_0)y_0\|=0. \tag{3.15}$$

Now taking $\lim s_0\to-\infty$ in equation (3.12), we obtain

$$y(s)=\int_{-\infty}^{s}T(s-\sigma(t))\mathcal{F}\left(t,y(t),\int_{-\infty}^{t}\mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t. \tag{3.16}$$

Now we check for convergence of

$$\int_{-\infty}^{s}T(s-\sigma(t))\mathcal{F}\left(t,y(t),\int_{-\infty}^{t}\mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t. \tag{3.17}$$

Let us consider the following

$$F_1=\mathcal{F}\left(t,0,\int_{s_0}^{t}\mathcal{H}(t,\tau,0)\Delta\tau\right)$$
$$F_2=\mathcal{F}\left(t,y(t),\int_{s_0}^{t}\mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)-\mathcal{F}\left(t,0,\int_{s_0}^{t}\mathcal{H}(t,\tau,0)\Delta\tau\right)$$

such that $\mathcal{F}=F_1+F_2$.

Now

$$\begin{aligned}&\left\|\int_{-\infty}^{s} T(s-\sigma(t))\mathcal{F}\left(t, y(t), \int_{-\infty}^{t} \mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t\right\|\\&= \int_{-\infty}^{s} \|T(s-\sigma(t))(F_1+F_2)\|\Delta t\\&\le M\int_{-\infty}^{s} e_{\ominus\varepsilon}(s,\sigma(t))\|F_1+F_2\|\Delta t\\&\le M\left\{\int_{-\infty}^{s} e_{\ominus\varepsilon}(s,\sigma(t))\|F_1\|\Delta t+\int_{-\infty}^{s} e_{\ominus\varepsilon}(s,\sigma(t))\|F_2\|\Delta t\right\}\\&\le M\left\{M_{\mathcal{F}}\int_{-\infty}^{s}\frac{1+\mu(t)\varepsilon}{\varepsilon}\left(-(\ominus\varepsilon)e_{\ominus}(s,\sigma(t))\right)\Delta t\right\}.\end{aligned}$$

We have

$$\begin{aligned}\int_{-\infty}^{s} e_{\ominus\varepsilon}(s,\sigma(t))\|F_1\|\Delta t &= \int_{-\infty}^{s}\frac{1+\mu(t)\varepsilon}{\varepsilon}\left(-(\ominus\varepsilon)e_{\ominus}(s,\sigma(t))\right)\|F_1\|_{\mathbb{Y}}\Delta t\\&\le \frac{M_{\mathcal{F}}(1+\tilde{\mu}\varepsilon)}{\varepsilon}\int_{-\infty}^{s}\left(-(\ominus\varepsilon)e_{\ominus}(s,\sigma(t))\right)\Delta t\\&\le \frac{M_{\mathcal{F}}(1+\tilde{\mu}\varepsilon)}{\varepsilon}\left(e_{\ominus}(s,s)-e_{\ominus}(s,-\infty)\right)\Delta t\\&= \frac{M_{\mathcal{F}}(1+\tilde{\mu}\varepsilon)}{\varepsilon}.\end{aligned}\tag{3.18}$$

Also

$$\begin{aligned}&\int_{-\infty}^{s} e_{\ominus\varepsilon}(s,\sigma(t))\|F_2\|\Delta t\\&= \int_{-\infty}^{s} e_{\ominus}(s,\sigma(t))\left\|\mathcal{F}\left(t,y(t),\int_0^t \mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\right.\\&\qquad\left.-\mathcal{F}\left(t,0,\int_0^t\mathcal{H}(t,\tau,0)\Delta\tau\right)\right\|_{\mathbb{Y}}\Delta t\\&\le \int_{-\infty}^{s} e_{\ominus}(s,\sigma(t))\left(L_{\mathcal{F}}\left(\|y(t)\|+\int_{-\infty}^{s} L_{\mathcal{H}}(\tau)\|y(\tau)\|\right)\right)\\&\le mL_{\mathcal{F}}\left(1+L'_{\mathcal{H}}\right)\int_{-\infty}^{s} e_{\ominus}(s,\sigma(t))\\&= mL_{\mathcal{F}}\left(1+L'_{\mathcal{H}}\right)\frac{(1+\tilde{\mu}\varepsilon)}{\varepsilon}.\end{aligned}\tag{3.19}$$

Using results given by equation (3.18) and equation (3.19), from (3.18), we get

$$\begin{aligned}&\left\|\int_{-\infty}^{s} T(s-\sigma(t))\mathcal{F}\left(t,y(t),\int_{-\infty}^{t}\mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t\right\|\\&\qquad\qquad\le M\frac{(1+\tilde{\mu}\varepsilon)}{\varepsilon}(M_{\mathcal{F}}+mL_{\mathcal{F}}(1+L'_{\mathcal{H}})\end{aligned}$$

which shows that $\| \int_{-\infty}^{s} T(s-\sigma(t))\mathcal{F}\Big(t, y(t), \int_{-\infty}^{t} \mathcal{H}(t,\tau,y(\tau))\Delta\tau\Big)\Delta t\|$ is convergent.

Now

$$y_0 = y(s_0) = \int_{-\infty}^{s_0} T(s_0-\sigma(t))\mathcal{F}\left(t, y(t), \int_{-\infty}^{t} \mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t$$

and

$$\begin{aligned}
y(s) &= \int_{-\infty}^{s} T(s-\sigma(t))\mathcal{F}\left(t, y(t), \int_{-\infty}^{t} \mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t \\
&= \int_{-\infty}^{0} T(s-\sigma(t))\mathcal{F}\left(t, y(t), \int_{-\infty}^{t} \mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t \\
&\quad + \int_{0}^{s} T(s-\sigma(t))\mathcal{F}\left(t, y(t), \int_{-\infty}^{t} \mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t \\
&= T(s-s_0)\int_{-\infty}^{s_0} T(s_0-\sigma(t))\mathcal{F}\left(t, y(t), \int_{-\infty}^{t} \mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t \\
&\quad + \int_{s_0}^{s} T(s-\sigma(t))\mathcal{F}\left(t, y(t), \int_{-\infty}^{t} \mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t \\
&= T(s-s_0)y(s_0) + \int_{0}^{s} T(s-\sigma(t))\mathcal{F}\left(t, y(t), \int_{-\infty}^{t} \mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t.
\end{aligned}$$

By the above discussion, we can confirm that y given by (3.11) is in fact a mild solution to the initial value problem given by (3.1)-(3.2). ☐

Proposition 3.11. *Let $f \in AA(\mathsf{T} \times \mathbb{Y} \times \mathbb{Y}, \mathbb{Y})$ be such*

$$\|f(s,x_1,y_1) - f(s,x_2,y_2)\| \leq L_f(s)(\|x_1-x_2\| + \|y_1-y_2\|$$

uniformly for $s \in \mathsf{T}$ and $x_i, y_i \in \mathbb{Y}$, $i = 1,2$, where, $L_f \in BC_{rd}(\mathsf{T}, \mathbb{R}^+)$. Then for any $x, y \in AA(\mathsf{T}, \mathbb{Y})$, the function $\Psi : \mathsf{T} \to \mathbb{Y}$, given by $\Psi(s) = f(\cdot, x, y)$ is almost automorphic.

Proof. Let $(\tau_n')_{n\in\mathbb{N}}$ be a sequence in Π. since x, y and f are almost automorphic functions, we can get a subsequence $(\tau_n)_{n\in\mathbb{N}}$ of $(\tau_n')_{n\in\mathbb{N}}$ such that

(1) $\lim_{n\to\infty} x(s+\tau_n) = \tilde{x}(s)$ exists for each $s \in \mathsf{T}$.

(2) $\lim_{n\to\infty} \tilde{x}(s-\tau_n) = x(s)$ exists for each $s \in \mathsf{T}$.

(3) $\lim_{n\to\infty} y(s+\tau_n) = \tilde{y}(s)$ exists for each $s \in \mathsf{T}$.

(4) $\lim_{n\to\infty} \tilde{y}(s-\tau_n) = y(s)$ exists for each $s \in \mathsf{T}$.

(5) $\lim_{n\to\infty} f(s+\tau_n, x, y) = \tilde{f}(s, x, y)$ exists for each $s \in \mathsf{T}$.

(6) $\lim_{n\to\infty} \tilde{f}(s-\tau_n, x, y) = f(s, x, y)$ exists for each $s \in \mathsf{T}$.

Let $\tilde{\Psi}(s) = \tilde{f}(\cdot, \tilde{x}(s), \tilde{y}(s))$. We have

$$\begin{aligned} \|\Psi(s+\tau_n) - \tilde{\Psi}(s)\| &= \|f(s+\tau_n, x(s+\tau_n), y(s+\tau_n)) - \tilde{f}(\cdot, \tilde{x}(s), \tilde{y}(s))\| \\ &\leq \|f(s+\tau_n, x(s+\tau_n), y(s+\tau_n)) \\ &\qquad - f(s+\tau_n, \tilde{x}(s), \tilde{y}(s))\| \\ &+ \quad \|f(s+\tau_n, \tilde{x}(s), \tilde{y}(s)) - \tilde{f}(\cdot, \tilde{x}(s), \tilde{y}(s))\|. \end{aligned} \tag{3.20}$$

According to the given assumptions, we have

$$\begin{aligned} &\|f(s+\tau_n, x(s+\tau_n), y(s+\tau_n)) - f(s+\tau_n, \tilde{x}(s), \tilde{y}(s))\| \\ &\qquad \leq L_f(s+\tau_n)(\|x(s+\tau_n) - \tilde{x}(s)\| + \|y(s+\tau_n) - \tilde{x}(s)\|). \end{aligned}$$

Since, $L_f \in BC_{rd}(\mathsf{T}, \mathbb{Y})$ and using (1) and (3) from above we get

$$\lim_{n\to\infty} \|f(s+\tau_n, x(s+\tau_n), y(s+\tau_n)) - \tilde{f}(s+\tau_n, \tilde{x}(s), \tilde{y}(s))\| = 0. \tag{3.21}$$

Also by (5), we have

$$\lim_{n\to\infty} \|f(s+\tau_n, \tilde{x}(s), \tilde{y}(s)) - \tilde{f}(\cdot, \tilde{x}(s), \tilde{y}(s))\| = 0. \tag{3.22}$$

So by using equations (3.21) and (3.22), we get from equation (3.20)

$$\lim_{n\to\infty} \Psi(s+\tau_n) = \tilde{\Psi}(s) \text{ for each } s \in \mathsf{T}.$$

Using a similar argument as above we can also prove that

$$\lim_{n\to\infty} \tilde{\Psi}(s-\tau_n) = \Psi(s) \text{ for each } s \in \mathsf{T}.$$

This proves that $\Psi \in AA(\mathsf{T}, \mathbb{Y})$. □

Proposition 3.12. *If $\mathcal{F} \in AAA(\mathsf{T}^+ \times \mathbb{Y} \times \mathbb{Y}, \mathbb{Y})$ and satisfies (H_2'), then for $x, y \in AAA(\mathsf{T}^+, \mathbb{Y})$, the function $\Gamma : \mathsf{T}^+ \to \mathbb{Y}$, given by $\Gamma(s) = \mathcal{F}(s, x(s), y(s))$ is also asymptotically almost automorphic.*

Proof. By (H_2') $\mathcal{F} \in AAA(\mathsf{T}^+ \times \mathbb{Y} \times \mathbb{Y}, \mathbb{Y})$, and

$$\mathcal{F}(s, x(s), y(s)) = \mathcal{P}(s, x(s), y(s) + \mathcal{Q}(s, x(s), y(s)),$$

where $\mathcal{P} \in AA(\mathsf{T} \times \mathbb{Y} \times \mathbb{Y}, \mathbb{Y})$ and $\mathcal{Q} \in C_{rd_0}(\mathsf{T}^+ \times \mathbb{Y} \times \mathbb{Y}, \mathbb{Y})$.

Again for $y, z \in AAA(\mathsf{T}^+, \mathbb{Y})$, we have $y(s) = u(s) + w(s)$ and $z(s) = v(s) + x(s)$, where $u, v \in AA(\mathsf{T}, \mathbb{Y})$ and $w, x \in C_{rd_0}(\mathsf{T}^+, \mathbb{Y})$.

Now,

$$\begin{aligned} \mathcal{F}(s, y(s), z(s)) &= \mathcal{F}(s, u(s), v(s)) + [\mathcal{F}(s, y(s), z(s)) - \mathcal{F}(s, u(s), v(s))] \\ &= [\mathcal{F}(s, y(s), z(s)) - \mathcal{F}(s, u(s), v(s))] \\ &\quad + \mathcal{P}(s, u(s), v(s)) + \mathcal{Q}(s, u(s), v(s)) \end{aligned}$$

Let $\mathcal{F}_1(s) = \mathcal{F}(s, y(s), z(s)) - \mathcal{F}(s, u(s), v(s))$, $\mathcal{F}_2(s) = \mathcal{P}(s, u(s), v(s))$ and $\mathcal{F}_3(s) = \mathcal{Q}(s, u(s), v(s))$.

Note that, for $w, x \in C_{rd_0}(\mathsf{T}^+, \mathbb{Y})$, we have

$$\begin{cases} \lim_{s\to\infty} \|x(s) - u(s)\|_{\mathbb{Y}} = \lim_{s\to\infty} \|w(s)\|_{\mathbb{Y}} = 0 \\ \lim_{s\to\infty} \|y(s) - v(s)\|_{\mathbb{Y}} = \lim_{s\to\infty} \|z(s)\|_{\mathbb{Y}} = 0 \end{cases} \tag{3.23}$$

Now using (H_2'), we get

$$\begin{aligned}
\|\mathcal{F}_1(s)\|_{\mathbb{Y}} &= \|\mathcal{F}(s,y(s),z(s)) - \mathcal{F}(s,u(s),v(s))\|\|_{\mathbb{Y}} \\
&\leq L_{\mathcal{F}}(s)(\|y(s)-u(s)\|_{\mathbb{Y}} + \|z(s)-v(s)\|_{\mathbb{Y}}) \\
\Rightarrow \lim_{s\to\infty} \|\mathcal{F}_1(s)\|_{\mathbb{Y}} &= \lim_{s\to\infty} L_{\mathcal{F}}(s)(\|y(s)-u(s)\|_{\mathbb{Y}} + \|z(s)-v(s)\|_{\mathbb{Y}}) \\
&= \lim_{s\to\infty}(L_{\mathcal{F}}(s))(\lim_{s\to\infty}\|y(s)-u(s)\|_{\mathbb{Y}} + \lim_{s\to\infty}\|z(s)-v(s)\|_{\mathbb{Y}}) \\
&= 0 \text{ (as } L_{\mathcal{F}} \subset BC_{rd}(\mathsf{T},\mathsf{R}^+)).
\end{aligned}$$

Hence, $\mathcal{F}_1 \in C_{rd_0}(\mathsf{T}^+,\mathbb{Y})$.
Also for $u,v \in AA(\mathsf{T},\mathbb{Y})$ with the help of **Proposition 3.11** we get, $\mathcal{F}_2 \in AA(\mathsf{T},\mathbb{Y})$.
Lastly, for $\mathcal{Q} \in C_{rd_0}(\mathsf{T}^+,\mathbb{Y})$ we have $\lim_{s\to\infty}\mathcal{F}_3(s) = 0$.
So from the above discussion, we establish that $\Gamma \in AAA(\mathsf{T}^+,\mathbb{Y})$. □

Proposition 3.13. *If $\mathcal{J} \in bAA(\mathsf{T}\times\mathsf{T}\times\mathbb{Y},\mathbb{Y})$ be a function as mentioned in (H_2') then for any $y \in AA(\mathsf{T},\mathbb{Y})$ the function*

$$\Phi(s) := \int_{-\infty}^{s} \mathcal{J}(s,\tau,y(\tau))\Delta\tau$$

is almost automorphic.

Proof. Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence in Π. Since $y \in AA(\mathsf{T},\mathbb{Y})$ and $\mathcal{J} \in bAA(\mathsf{T}\times\mathsf{T}\times\mathbb{Y},\mathbb{Y})$, we have a subsequence $(s_n) \subset (\tau_n)_{n\in\mathbb{N}}$ such that

1) $\lim_{n\to\infty} y(s+\tau_n) = \tilde{y}(s)$ exists for each $s \in \mathsf{T}$.
2) $\lim_{n\to\infty} \tilde{y}(s-\tau_n) = y(s)$ exists for each $s \in \mathsf{T}$.
3) $\lim_{n\to\infty} \mathcal{J}(s+\tau_n, t+\tau_n, y) = \tilde{\mathcal{J}}(s,t,y)$ exists for each $s,t \in \mathsf{T}$.
4) $\lim_{n\to\infty} \tilde{\mathcal{J}}(s-\tau_n, t-\tau_n, y) = \mathcal{J}(s,t,y)$ exists for each $s,t \in \mathsf{T}$.

Let $\tilde{\Phi}(s) = \int_{-\infty}^{s} \tilde{\mathcal{J}}(s,\tau,\tilde{y}(\tau))\Delta\tau$.
Then

$$\begin{aligned}
&\|\Phi(s+s_n) - \tilde{\Phi}(s)\| \\
&= \left\| \int_{-\infty}^{s+s_n} \mathcal{J}(s+s_n,\tau,y(\tau))\Delta\tau - \int_{-\infty}^{s} \tilde{\mathcal{J}}(s,\tau,\tilde{y}(\tau))\Delta\tau \right\| \\
&= \left\| \int_{-\infty}^{s} \mathcal{J}(s+s_n,\tau+s_n,y(\tau+s_n))\Delta\tau - \int_{-\infty}^{s} \tilde{\mathcal{J}}(s,\tau,\tilde{y}(\tau))\Delta\tau \right\| \\
&\leq \left\| \int_{-\infty}^{s} \mathcal{J}(s+s_n,\tau+s_n,y(\tau+s_n))\Delta\tau - \int_{-\infty}^{s} \mathcal{J}(s+s_n,\tau+s_n,\tilde{y}(\tau))\Delta\tau \right\| \\
&\quad + \left\| \int_{-\infty}^{s} \mathcal{J}(s+s_n,\tau+s_n,\tilde{y}(\tau))\Delta\tau - \int_{-\infty}^{s} \tilde{\mathcal{J}}(s,\tau,\tilde{y}(\tau))\Delta\tau \right\| \\
&\leq \int_{-\infty}^{s} \|\mathcal{J}(s+s_n,\tau+s_n,y(\tau+s_n)) - \mathcal{J}(s+s_n,\tau+s_n,\tilde{y}(\tau))\|\,\Delta\tau \\
&\quad + \int_{-\infty}^{s} \left\| J(s+s_n,\tau+s_n,\tilde{y}(\tau)) - \tilde{\mathcal{J}}(s,\tau,\tilde{y}(\tau)) \right\| \Delta\tau \\
&\leq \int_{-\infty}^{s} L_{\mathcal{J}}(s+s_n)\,\|y(s+s_n) - \tilde{y}(s)\|\,\Delta\tau
\end{aligned}$$

$$+\int_{-\infty}^{s}\left\|\mathcal{J}(s+s_n,\tau+s_n,\tilde{y}(\tau))-\tilde{\mathcal{J}}(s,\tau,\tilde{y}(\tau))\right\|\Delta\tau.$$

Now taking $n\to\infty$, taking into account the fact that $L_{\mathcal{J}}\in BC_{rd}(\mathsf{T},\mathbb{Y})$ as given by $(H_2^{'})$ together with 1) and 3), we see from the above inequality that,

$$\lim_{n\to\infty}\Phi(s+s_n)=\tilde{\Phi}(s)\ \forall s\in\mathsf{T}.$$

Similarly by using $(H_2^{'})$ together with 2) and 4) we can show that

$$\lim_{n\to\infty}\tilde{\Phi}(s-s_n)=\Phi(s)\ \forall s\in\mathsf{T}.$$

Thus we establish $\Phi\in AA(\mathsf{T},\mathbb{Y})$. □

Proposition 3.14. *Let $\mathcal{H}\in bAAA(\mathsf{T}^+\times\mathsf{T}^+\times\mathbb{Y},\mathbb{Y})$ satisfying $(H_2^{'})$. Then for any $y\in AAA(\mathsf{T}^+,\mathbb{Y})$ the function*

$$\Phi(s):=\int_{-\infty}^{s}\mathcal{H}(s,\tau,y(\tau))\Delta\tau,$$

is asymptotically almost automorphic.

Proof. Since $y\in AAA(\mathsf{T}^+,\mathbb{Y})$. Let us suppose that $y(\tau)=z(\tau)+w(\tau)$, where $z\in AA(\mathsf{T},\mathbb{Y})$ and $w\in C_{rd_0}(\mathsf{T}^+,\mathbb{Y})$. Again for $\mathcal{H}\in bAAA(\mathsf{T}^+\times\mathsf{T}^+\times\mathbb{Y},\mathbb{Y})$, we have

$$\mathcal{H}(s,\tau,y(\tau))=\mathcal{J}(s,\tau,y(\tau))+\mathcal{K}(s,\tau,y(\tau)) \tag{3.24}$$

for some(unique) $\mathcal{J}\in bAA(\mathsf{T}\times\mathsf{T}\times\mathbb{Y},\mathbb{Y})$ and $\mathcal{K}\in C_{rd_0}(\mathsf{T}^+\times\mathsf{T}^+\times\mathbb{Y},\mathbb{Y})$. Now, we have

$$\mathcal{H}(s,\tau,y(\tau))=\mathcal{H}(s,\tau,z(\tau))+[\mathcal{H}(s,\tau,y(\tau))-\mathcal{H}(s,\tau,z(\tau))]. \tag{3.25}$$

It is evident by using similar arguments as in **Proposition 3.11** that $\Phi_1(s,\tau):=\mathcal{H}(s,\tau,z(\tau))\in bAA(\mathsf{T}\times\mathsf{T},\mathbb{Y})$ for z being almost automorphic. Also, by using the condition $(H_2^{'})$ and the fact that $w(s)=x(s)-z(s)$, we can also confirm that,

$$\begin{aligned}\lim_{(s,\tau)\to(\infty,\infty)}\|\mathcal{H}(s,\tau,y(\tau))-\mathcal{H}(s,\tau,z(\tau))\| &= \lim_{\tau\to\infty}|L_{\mathcal{H}}(\tau)|\|y(\tau)-z(\tau)\|\\ &= \lim_{\tau\to\infty}|L_{\mathcal{H}}(\tau)|\|w(\tau)\|=0.\end{aligned}$$

Thus, we have from above discussion, $\mathcal{H}(s,\tau,y(\tau))\in bAAA(\mathsf{T}^+\times\mathsf{T}^+,\mathbb{Y})$.
Now, since $\mathcal{H}(s,\tau\, y(\tau))\in bAAA(\mathsf{T}^+\times\mathsf{T}^+,\mathcal{Y})$, we have unique decomposition of $\mathcal{H}(s,\tau\, y(\tau))$, as given in $(\mathrm{H}_2^{'})$, for $y\in AAA(\mathsf{T}^+,\mathcal{Y})$. Subsequently we get

$$\begin{aligned}\Phi(s):&=\int_{-\infty}^{s}\mathcal{H}(s,\tau,y(\tau))\Delta\tau\\ &=\int_{-\infty}^{s}\left(\mathcal{J}(s,\tau,z(\tau))+[\mathcal{J}(s,\tau,y(\tau))-\mathcal{J}(s,\tau,z(\tau))]\right)\Delta\tau\\ &=\int_{-\infty}^{s}\mathcal{J}(s,\tau,z(\tau))\Delta\tau+\int_{-\infty}^{s}\mathcal{J}(s,\tau,y(\tau))-\mathcal{J}(s,\tau,z(\tau))\Delta\tau.\end{aligned} \tag{3.26}$$

By Proposition 3.13 we have

$$\int_{-\infty}^{s}\mathcal{J}(s,\tau,z(\tau))\Delta\tau\in AA(\mathsf{T},\mathbb{Y}),\ \text{for } z\in AA(\mathsf{T},\mathbb{Y}). \tag{3.27}$$

Also

$$\begin{aligned}
&\lim_{(s,\tau)\to(\infty,\infty)} \left\| \int_{-\infty}^{s} \mathcal{J}(s,\tau,y(\tau)) - \mathcal{J}(s,\tau,z(\tau))\Delta\tau \right\|_{\mathbb{Y}} \\
&\leq \int_{-\infty}^{s} \lim_{(s,\tau)\to(\infty,\infty)} \|\mathcal{J}(s,\tau,y(\tau)) - \mathcal{J}(s,\tau,z(\tau))\|_{\mathbb{Y}}\, \Delta\tau \\
&\leq \int_{-\infty}^{s} \lim_{\tau\to\infty} L_{\mathcal{J}}(\tau)\|y(\tau) - z(\tau)\|_{\mathbb{Y}}\Delta\tau = 0
\end{aligned}$$

as $L_{\mathcal{J}} \in BC_{rd}(\mathsf{T}, \mathcal{Y})$ and $\lim_{\tau\to\infty} \|y(\tau) - z(\tau)\|_{\mathbb{Y}} = 0$.

By the help (3.27) and (3.28), we conclude from (3.26) that, $\Phi \in AAA(\mathsf{T}^+, \mathcal{Y})$. Hence the result. □

Proposition 3.15. *Under the hypothesis* $(H_1^{'})-$ $(H_3^{'})$*, the function* F_3 *defined as*

$$\Psi(s) := \int_{-\infty}^{s} T(s - \sigma(t))\mathcal{F}\left(t, y(t), \int_{-\infty}^{t} \mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t$$

is also aysmptotically almost automorphic.

Since $\mathcal{F} \in AAA(\mathsf{T}^+ \times \mathbb{Y} \times \mathbb{Y}, \mathbb{Y})$, we can easily show that the function $\Psi \in AAA(\mathsf{T}^+, \mathbb{Y})$.
For reference of proof, we refer **Proposition 3.6** in [33] and **Lemma 3.3** and **Lemma 3.4** of [11].
Now we establish our main result of this section.

Theorem 3.16. *Under the given hypothesis* $(H_1^{'})-(H_4^{'})$ *the integral-dynamic equation given by* (3.1)-(3.2) *admits a unique solution which is also asymptotically almost automorphic, provided* $0 < M_{\mathcal{F}} = \sup\left\{\|\mathcal{F}(s,0,z)\|_{\mathbb{Y}};\ s \in \mathcal{J},\ z \in \mathbb{Y}\right\} < \infty$.

Let $\Upsilon : AAA(\mathsf{T}^+, \mathbb{Y}) \to AAA(\mathsf{T}^+, \mathbb{Y})$, given by

$$\Upsilon(y)(s) = \int_{-\infty}^{s} T(s - \sigma(t)\mathcal{F}\left(t, y(t), \int_{-\infty}^{t} \mathcal{H}(t,\tau,y(\tau))\Delta\tau\right)\Delta t,\ \forall y \in AAA((\mathbb{Y}, \mathbb{X})).$$

We show that Υ, defined as above has a unique fixed point. For the same, we follow the following steps:

Step: 1 Υ is well defined. Let $y \in AAA(\mathsf{T}^+, \mathbb{Y})$. Then using the hypothesis $(H_2^{'})$, by **Proposition 3.14**, the function $\Phi(s) := \int_{-\infty}^{s} \mathcal{H}(s,\tau,y(\tau))\Delta\tau \in AAA(\mathsf{T}^+, \mathbb{Y})$. And hence by **Proposition 3.12**, the function $\Gamma(s) := \mathcal{F}\big(s, y(s), \int_{-\infty}^{s} \mathcal{H}(s,\tau,y(\tau))\, \Delta\tau\big) \in AAA(\mathsf{T}^+, \mathbb{Y})$. Then by using **Proposition** 3.15, the function $\Psi(s) := \int_{-\infty}^{s} T(s - \sigma(t))\mathcal{F}\big(t, y(t), \int_{-\infty}^{t} \mathcal{H}(t,\tau,y(\tau))\Delta\tau\big)\Delta t \in AAA(\mathsf{T}^+, \mathbb{Y})$. Hence $\Upsilon(y)(s) \in AAA(\mathsf{T}^+, \mathbb{Y})$. This concludes the first step.

Step: 2 By $(H_4^{'})$, there exists $r > 0$ such that

$$\frac{\varepsilon r}{M} - r(1 + \tilde{\mu}\varepsilon)\left(L_{\mathcal{F}}^{*} + {L_{\mathcal{H}}^{1}}^{*}\right) > (1 + \tilde{\mu}\varepsilon)M_F. \tag{3.28}$$

Let us consider the set, $\mathcal{B}_r = \{y \in AAA(\mathsf{T}^+, \mathbb{Y}) : \|y(s)\| \leq r\} \subset AAA(\mathsf{T}^+, \mathbb{Y})$. For $y \in \mathcal{B}_r$, let us assume that $z(t) = \int_{-\infty}^{t} \mathcal{H}(t,\tau,y(\tau))\Delta\tau$ and $z_0(t) = \int_{-\infty}^{t} \mathcal{H}(t,\tau,0)\Delta\tau$

such that,

$$\begin{aligned}
\|z(t)-z_0(t)\| &= \|\int_{-\infty}^{t}\mathcal{H}(t,\tau,y(\tau))\Delta\tau-\int_{-\infty}^{t}\mathcal{H}(t,\tau,0)\Delta\tau\| \\
&\leq \int_{-\infty}^{t}\|\mathcal{H}(t,\tau,y(\tau))-\mathcal{H}(t,\tau,0)\|\Delta\tau \\
&\leq \int_{-\infty}^{t}L_{\mathcal{H}}(\tau)\|y(\tau)\|\Delta\tau.
\end{aligned} \tag{3.29}$$

Then we have,

$$\begin{aligned}
\|\Upsilon(y)(t)\| &= \left\|\int_{-\infty}^{s}T(s-\sigma(t))\mathcal{F}\left(t,y(t),z(t)\right))\Delta t\right\| \\
&= \left\|\int_{-\infty}^{s}T(s-\sigma(t))\left[\mathcal{F}(t,y(t),z(t)-\mathcal{F}(t,0,z_0(t)\right]\Delta t\right. \\
&\qquad\qquad \left.+\int_{-\infty}^{s}T(s-\sigma(t))\mathcal{F}(t,0,z_0(t)\Delta t\right\| \\
&= \int_{-\infty}^{s}\|T(s-\sigma(t))\left[\mathcal{F}(t,y(t),z(t)-\mathcal{F}(t,0,z_0(t)\right]\|_{\mathbb{Y}}\Delta t \\
&\qquad\qquad +\int_{-\infty}^{s}\|T(s-\sigma(t))\mathcal{F}(t,0,z_0(t)\|_{\mathbb{Y}}\Delta t \\
&\leq \int_{-\infty}^{s}Me_{\varepsilon}(s-\sigma(t))\left\{L_{\mathcal{F}}(t)(\|y(t)\|+L^1_{\mathcal{H}}(t)\|y\|)\right\}\Delta t \\
&\quad +MM_{\mathcal{F}}\int_{-\infty}^{s}e_{\varepsilon}(s-\sigma(t)) \\
&\leq M(L^*_{\mathcal{F}}+{L^1_{\mathcal{H}}}^*)r\left(\frac{1+\tilde{\mu}\varepsilon}{\varepsilon}\right)+MM_{\mathcal{F}}\left(\frac{1+\tilde{\mu}\varepsilon}{\varepsilon}\right) \\
&= M\left(\frac{1+\tilde{\mu}\varepsilon}{\varepsilon}\right)\left((L^*_{\mathcal{F}}+{L^1_{\mathcal{H}}}^*)r+M_{\mathcal{F}}\right) \\
&< r,
\end{aligned} \tag{3.30}$$

which proves that, $\Upsilon:=\mathcal{B}_r\to\mathcal{B}_r$.
Also from (3.28), we have

$$\begin{aligned}
&\frac{\varepsilon r}{M}-r(1+\tilde{\mu}\varepsilon)\left(L^*_{\mathcal{F}}+{L^1_{\mathcal{H}}}^*\right)>0 \\
&\Rightarrow \gamma=M(L^*_{\mathcal{F}}+{L^1_{\mathcal{H}}}^*)\left(\frac{1+\tilde{\mu}\varepsilon}{\varepsilon}\right)<1
\end{aligned} \tag{3.31}$$

Step: 3 Now for $y_1, y_2 \in \mathcal{B}_r$, let $z_1(t) = \int_{-\infty}^{t} \mathcal{H}(t,\tau,y_1(\tau))\Delta\tau$ and $z_2(t) = \int_{-\infty}^{t} \mathcal{H}(t,\tau,y_2(\tau))\Delta\tau$. Then we get,

$$\begin{aligned}
\|z_2(t) - z_1(t)\|_{\mathbb{Y}} &= \left\| \int_{-\infty}^{t} \mathcal{H}(t,\tau,y_2(\tau))\Delta\tau - \int_{-\infty}^{t} \mathcal{H}(t,\tau,y_1(\tau))\Delta\tau \right\|_{\mathbb{Y}} \\
&\le \int_{-\infty}^{t} \|\mathcal{H}(t,\tau,y_2(\tau)) - \mathcal{H}(t,\tau,y_1(\tau))\|_{\mathbb{Y}}\Delta\tau \\
&\le \int_{-\infty}^{t} L_{\mathcal{H}}(\tau)\|y_2(\tau) - y_1(\tau)\|_{\mathbb{Y}}\Delta\tau
\end{aligned} \tag{3.32}$$

Now

$$\begin{aligned}
&\|\Upsilon(y_2)(t) - \Upsilon(y_1)(t)\|_{\mathbb{Y}} \\
&= \left\| \int_{-\infty}^{s} T(s-\sigma(t))\mathcal{F}(t, y_2(t), z_2(t))\Delta t \right. \\
&\quad \left. - \int_{-\infty}^{s} T(s-\sigma(t))\mathcal{F}(t, y_1(t), z_1(t))\Delta t \right\|_{\mathbb{Y}} \\
&\le \int_{-\infty}^{s} \|T(s-\sigma(t))\| \|\mathcal{F}(t, y_2(t), z_2(t)) - \mathcal{F}(t, y_1(t), z_1(t))\|_{\mathbb{Y}}\Delta t \\
&\le \int_{-\infty}^{s} Me_{\varepsilon}(s-\sigma(t)) \left\{ L_{\mathcal{F}}(t) \left(\|y_2(t) - y_1(t)\|_{\mathbb{Y}} + \|z_2(t) - z_1(t)\|_{\mathbb{Y}} \right) \right\} \Delta t \\
&\le \int_{-\infty}^{s} Me_{\varepsilon}(s-\sigma(t)) \left\{ L_{\mathcal{F}}(t) \big(\|y_2(t) - y_1(t)\|_{\mathbb{Y}} \right. \\
&\qquad\qquad \left. + \int_{-\infty}^{t} L_{\mathcal{H}}(\tau)\|y_2(\tau) - y_1(\tau)\|_{\mathbb{Y}}\Delta\tau \big) \right\} \Delta t \\
&\le \int_{-\infty}^{s} Me_{\varepsilon}(s-\sigma(t)) \left\{ L_{\mathcal{F}}(t) \big(\|y_2(t) - y_1(t)\|_{\mathbb{Y}} + L_{\mathcal{H}}^{1}(t)\|y_2 - y_1\|_{\mathbb{Y}} \big) \right\} \Delta t \\
&\le M(L_{\mathcal{F}}^{*} + {L_{\mathcal{H}}^{1}}^{*})\|y_2 - y_1\| \int_{-\infty}^{t} e_{\varepsilon}(s-\sigma(t)) \\
&\le M(L_{\mathcal{F}}^{*} + {L_{\mathcal{H}}^{1}}^{*}) \left(\frac{1+\tilde{\mu}\varepsilon}{\varepsilon} \right) \|y_2 - y_1\|_{\mathbb{Y}} \\
&\le \gamma\|y_2 - y_1\|_{\mathbb{Y}} \text{ (using equation (3.31)).}
\end{aligned}$$

Thus we have established that the function Υ is a contraction and thus by the Banach Contraction Principle, there exists a unique $y \in AAA(\mathsf{T}^{+}, \mathbb{Y})$ for which $\Upsilon y = y$.

Example 3.17. Consider the time scale,

$$\mathbb{P}_{a,b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a].$$

This time scale is invariant under translation and contains 0. This time scale is one of the most useful time scale which is used to model population dynamics of certain species with certain life span, whose measurements are given in terms of a and b. Let us now consider the dynamic equation

$$y^{\Delta}(s) = Ay(s) + \mathcal{F}\left(s, y(s), \int_{s_0}^{s} \mathcal{H}(s, \tau, y(\tau))\,\Delta\tau\right) \tag{3.33}$$

$$y(s_0) = y_0, \tag{3.34}$$

on the time scale $\mathbb{P}_{a,b}$, where A is some generator of a exponentially stable C_0-semigroup, $\{T(s) : s \in \mathsf{T}\}$ such that $\|T(s - s_0)\| \leq Me_{\ominus\varepsilon}(s - s_0)$. We take $s_0 = 0$ and $S = 2m+1$, for some $m \in \mathbb{N}$. Let us take, $\mathcal{F}(s, x, y) = \varepsilon_1 \sin\left(\frac{1}{2+\cos s+\cos\sqrt{2}s}\right)[\sin x + y] + \varepsilon_2 e_{\ominus\varepsilon}(s, s_0)$, where $\varepsilon > 0, \varepsilon_1 \in \left(0, \frac{1}{2M(2m+1)(1+2(2m+1))}\right), \varepsilon_2$ are some constant and $\mathcal{H}(s, t, y) = \sin s \cos t + \sin y + \cos y$. At the first instance we note that $\mathcal{F}$, given above is an asymptotically almost automorphic function, where $\mathcal{P}$ is given by $\mathcal{P}(s, x, y) = \sin\left(\frac{1}{2+\cos s+\cos\sqrt{2}s}\right)[\sin x + y] \in AA(\mathsf{T} \times \mathbb{Y} \times \mathbb{Y}, \mathbb{Y})$. Also $\lim_{s\to\infty} e_{\ominus\varepsilon}(s, s_0) = 0$ (by **Lemma 2.14**).
We can also verify that $\mathcal{F}$ satisfies Lipschitz condition given by $(H_1^{'})$, as

$$\begin{aligned}
&\left\|\mathcal{F}(s, x_1, y_1) - \mathcal{F}(s, x_2, y_2)\right\|_2^2 \\
&= \int_0^{\pi} \varepsilon_1^2 \left| sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)[\sin x_1 + y_1] + e_{\ominus}(t, s_0) \right. \\
&\left. - \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)[\sin x_2 + y_2] + e_{\ominus}(t, s_0)\right|^2 \Delta t \\
&\Rightarrow \left\|\mathcal{F}(s, x_1, y_1) - \mathcal{F}(s, x_2, y_2)\right\|_2^2 \\
&\quad \leq \varepsilon_1^2 \left|\sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)\right|^2 |[\sin x_1 + y_1] - [\sin x_2 + y_2]|^2 \\
&\quad \leq \varepsilon_1^2 \|x_1 - x_2\|_2^2 + \|y_1 - y - 2\|_2^2, \text{ for some } 1 \geq c \in \mathbb{R}.
\end{aligned}$$

i.e., $\left\|\mathcal{F}(s, x_1, y_1) - \mathcal{F}(s, x_2, y_2)\right\|_2 \leq \varepsilon_1 \|x_1 - x_2\|_2 + \|y_1 - y - 2\|_2$
From the above equation we can verify that, $\mathcal{F}$ satisfies the Lipschitz condition given by H_1 as well as $(H_1^{'})$. Furthermore, $\mathcal{H} \in bAA(\mathsf{T} \times \mathsf{T} \times \mathbb{Y}, \mathbb{Y}) \Rightarrow \mathcal{H} \in bAAA(\mathsf{T} \times \mathsf{T} \times \mathbb{Y}, \mathbb{Y})$. Also

$$\begin{aligned}
&\|\mathcal{H}(s, t, y_1) - \mathcal{H}(s, t, y_2)\|_2 \\
&\quad = \|\sin s \cos t + \sin y_1 + \cos y_1 - \sin s \cos t + \sin y_2 + \cos y_2\|_2 \\
&\quad \leq \|\sin y_1 - \sin y_2\|_2 + \|\cos y_1 - \cos y_2\|_2 \\
&\quad \leq 2\|y_1 - y_2\|_2
\end{aligned} \tag{3.35}$$

Therefore H_2 as well as $(H_2^{'})$ is also satisfied. Also H_3 which is also same as $(H_3^{'})$ is evident by assumptions on A. Now $M(S - s))L_{\mathcal{F}}^{*}(1 + L_{\mathcal{H}}(S - s_0)) = M(2m+1)\varepsilon_1(1 + 2(2m+1)) < 1$, which verifies the hypothesis H_4. Hence, theorem 3.2 ensures us a unique solution to the given equation.

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