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SOME RESULTS ON ROUGH $\mathcal{I}_2$-DEFERRED STATISTICALLY CONVERGENT DOUBLE SEQUENCES IN NEUTROSOPHIC NORMED SPACE

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Dedicated to Prof. Ravi P Agarwal on his 76th Birthday

Abstract. For sequences in the neutrosophic normed space ($\mathcal{NNS}$), we introduce the notion of rough $\mathcal{I}_2$-deferred statistical convergence in this work. Additionally, we explore rough $\mathcal{I}_2$-deferred statistical cluster points for sequences in $\mathcal{NNS}$ and analyze the interconnection between the set of these cluster points and the set of rough $\mathcal{I}_2$-deferred statistical limit points associated with the mentioned convergence.

1. Introduction and background

In the realm of mathematics, the concept of sequence convergence has evolved through various extensions, thanks to the introduction of diverse summability techniques. Statistical convergence, initially proposed by Steinhaus [37] and Fast [9], extends the traditional convergence of sequences involving real and complex numbers. Another innovative convergence approach, known as deferred statistical convergence of sequences, was explored by Küçükaslan and Yilmaztürk [26], incorporating deferred density into the statistical convergence definition. Building upon this, Et et al. [8] put forward the concept μ-deferred statistical convergence for real-valued functions, thereby significantly expanding the concept. Refer to [6, 11, 19, 30, 33–35] for a comprehensive understanding of the fundamental characteristics and details associated with these novel ideas.

The introduction of the concepts of roughness degree and rough convergence for sequences in a finite-dimensional normed linear space was pioneered by Phu [31]. Subsequently, Phu extended these notions to an infinite-dimensional normed linear space [32]. Going beyond the investigation of rough convergence, Phu explored analytical properties such as convexity and the proximity of the set of rough limits. Aytar [5] expanded the concept of rough statistical convergence, which includes natural density, and also explored the relationship between the set of rough statistical limit points for a sequence and the set of statistical cluster points. Building on the idea of rough convergence, various authors have further explored rough convergence and statistical rough convergence for sequences in different contexts. This exploration has even extended to the study of rough convergence and rough statistical convergence for double sequences in [27, 28].

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Zadeh [38] introduced the Theory of Fuzzy Sets ($\mathcal{FS}$), a seminal contribution that has had a significant impact on various scientific fields. However, $\mathcal{FS}$ faces challenges in effectively handling uncertain membership degrees. To address this limitation, Atanassov [4] extended the theory to Intuitionistic Fuzzy Sets ($\mathcal{IFS}$). Kramosil and Michalek [20] explored Fuzzy Metric Spaces ($\mathcal{FMS}$) by incorporating concepts from fuzzy and probabilistic metric spaces. Kaleva and Seikkala [12] investigated $\mathcal{FMS}$, considering the distance between two points as a non-negative fuzzy number. George and Veeramani [10] outlined the requirements for $\mathcal{FMS}$. The practical applications of $\mathcal{FMS}$ in fixed-point theory, medical imaging, and decision-making have garnered significant attention.

Smarandache [36] conducted a thorough investigation into the concept of 'Neutrosophic set' ($\mathcal{NS}$) as a generalization of $\mathcal{FS}$ and $\mathcal{IFS}$. The objective was to address uncertainty in practical problem-solving. $\mathcal{NS}$ incorporates membership functions for falsehood (F), indeterminacy (I), and truth (T). The unique aspect of neutrosophy, representing impartial knowledge of thought, sets $\mathcal{NS}$ apart from fuzzy, neutral, logic, and intuitive fuzzy sets.

In $\mathcal{NS}$, uncertainty is characterized independently of the values of truth (T) and falsehood (F), making $\mathcal{NS}$ more comprehensive than $\mathcal{IFS}$ since there are no constraints among the degrees of T, F, and indeterminacy (I). The term neutrosophy signifies impartial knowledge, and the concept of neutrality emphasizes a fundamental distinction from fuzzy, neutral, logic, and intuitive fuzzy sets.

Menger [29] introduced Triangular Norms (t-norms) ($\mathcal{TN}$) as a generalization of probability distributions, incorporating the triangle inequality in terms of metric spaces. Triangular Conorms (t-conorms) ($\mathcal{TC}$), identified as dual operations to $\mathcal{TN}$, play a pivotal role in fuzzy operations, including intersections and unions. $\mathcal{TN}$ and $\mathcal{TC}$ serve as vital components for managing fuzzy operations within the framework of metric spaces.

In $\mathcal{NS}$, uncertainty is distinct from the values of T and F, making $\mathcal{NS}$ more encompassing than $\mathcal{IFS}$ as there are no constraints among the degrees of T, F, and I. The term neutrosophy signifies impartial knowledge, and the notion of neutrality highlights the fundamental distinction from fuzzy, neutral, logic and intuitive fuzzy sets.

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The concept of a neutrosophic metric space, characterized by continuous t-norms and continuous t-conorms, was initially introduced by Kirişci and Şimşek [17]. Expanding on their work, Kirişci and Şimşek [18] further investigated neutrosophic normed spaces ($\mathcal{NNS}$) and explored statistical convergence within the $\mathcal{NNS}$ framework.

Antal et al. [2] introduced the notion of rough statistical convergence for sequences. Rahaman and Mursaleen [3] presented rough deferred statistical convergence for difference sequences in $\mathcal{L}$-fuzzy normed space. In another study [13], the authors proposed a modification to the definition of neutrosophic normed space, originally presented in [17]. Debnath et al. [7] presented the concept of deferred statistical convergence in the $\mathcal{NNS}$. This study introduces the concept of rough deferred statistical convergence of sequences within this adapted space. A significant number of academic publications related to their respective sequence spaces are documented in the literature [14–16, 21–25].

In specific cases, determining the precise values of terms in a convergent sequence (ϕ_{ij}) becomes challenging, especially for large values of u. To overcome this difficulty, an alternative sequence (ω_{ij}) is used for approximation, thereby introducing approximation errors. The concept of rough convergence has been introduced as a solution in these situations.

Our research aims to extend the concept of convergence to sequences within $\mathcal{NNS}$ and explore various algebraic and topological properties. This unique convergence allows the limit to manifest as a set rather than a single point, prompting a comprehensive investigation into the topological (closedness) and geometric properties of the limit set. Additionally, we provided examples, for a given roughness degree $r > 0$, demonstrating that the set of all rough $\mathcal{I}_2$-deferred statistical convergent sequences does not form a linear space. A rough $\mathcal{I}_2$-deferred statistical cluster point in $\mathcal{NNS}$ was also introduced, and a relationship between the cluster point set and the limit set under rough $\mathcal{I}_2$-deferred statistical convergence was developed.

By incorporating the notion of neutrosophy, which accounts for indeterminacy alongside truth and falsehood, this study extends existing convergence theories, providing a nuanced framework for analyzing sequences with uncertain behavior. Additionally, the investigation into the relationship between these cluster points and limit points offers new insights into sequence behavior in $\mathcal{NNS}$, enhancing both theoretical understanding and potential applications in complex data analysis.

2. Auxiliary definitions and notations

A few necessary definitions are provided in this section.

Assuming $\mathcal{F}$ is a linear space over the field $\mathcal{V}$ and $\Diamond$ and $*$ are $\mathcal{TN}$ and $\mathcal{TC}$, respectively. Let Θ, Ω and Ψ be single valued fuzzy sets on $\mathcal{F} \times (0, \infty)$. We designate the 6-tuple $(\mathcal{F}, \Theta, \Omega, \Psi, \Diamond, *)$ as a $\mathcal{NNS}$ if, for all $\omega, \gamma \in \mathcal{F}$ and $\tau, \kappa > 0$, the following conditions are satisfied:

(A1) $\Theta(\omega, \tau) + \Omega(\omega, \tau) + \Psi(\omega, \tau) \leq 3$,

(A2) $\Theta(\omega, \tau) = 1$, $\Omega(\omega, \tau) = 0$ and $\Psi(\omega, \tau) = 0$ iff $\omega = 0$,

(A3) $\Theta(\beta\omega, \tau) = \Theta\left(\omega, \frac{\tau}{|\beta|}\right)$, $\Omega(\beta\omega, \tau) = \Omega\left(\omega, \frac{\tau}{|\beta|}\right)$ and $\Psi(\beta\omega, \tau) = \Psi\left(\omega, \frac{\tau}{|\beta|}\right)$ for any $0 \neq \beta \in \mathcal{F}$,

(A4) $\Theta(\omega + \gamma, \tau + \kappa) \geq \Theta(\omega, \tau) \Diamond \Theta(\gamma, \kappa)$, $\Omega(\omega + \gamma, \tau + \kappa) \leq \Omega(\omega, \tau) * \Omega(\gamma, \kappa)$ and $\Psi(\omega + \gamma, \tau + \kappa) \leq \Psi(\omega, \tau) * \Psi(\gamma, \kappa)$,

(A5) $\Theta(\omega, .)$, $\Omega(\omega, .)$ and $\Psi(\omega, .)$ are continuous on $(0, \infty)$,

(A6) $\lim_{\tau\to\infty} \Theta(\omega, \tau) = 1$, $\lim_{\tau\to\infty} \Omega(\omega, \tau) = 0$ and $\lim_{\mu\to\infty} \Psi(\omega, \tau) = 0$,

(A7) $\lim_{\tau\to 0} \Theta(\omega, \tau) = 0$, $\lim_{\mu\to 0} \Omega(\omega, \tau) = 1$ and $\lim_{\mu\to 0} \Psi(\omega, \tau) = 1$.

In this scenario, we denote the 3-tuple (Θ, Ω, Ψ) as a neutrosophic norm (shortly, $\mathcal{NN}$) on $\mathcal{F}$.

Example 2.1. Let $(\mathcal{F}, \|\|.)$ be a normed space. Consider $\gamma_1 \Diamond \gamma_2 = \gamma_1 \cdot \gamma_2$ and $\gamma_1 * \gamma_2 = \min\{\gamma_1 + \gamma_2, 1\}$, $\forall \gamma_1, \gamma_2 \in [0,1]$. Additionally, define Θ, Ω, and Ψ as follows:

$$\Theta(\omega,\tau) = \frac{\tau}{\tau + \|\omega\|},\ \psi(\omega,\tau) = \frac{\|\omega\|}{\tau + \|\omega\|} \text{ and } \Psi(\omega,\tau) = \frac{2\|\omega\|}{\tau + 2\|\omega\|}$$

for all $\omega \in \mathcal{F}$ and $\tau > 0$. Then $(\mathcal{F}, \Theta, \Omega, \Psi, \Diamond, *)$ is a $\mathcal{NNS}$.

Consider a $\mathcal{NNS}$ $(\mathcal{F}, \Theta, \Omega, \Psi, \Diamond, *)$ and let $\omega \in \mathcal{F}$. For a given $r > 0$ and $\tau \in (0,1)$, the set

$$\mathcal{B}_{\omega}^{(\Theta,\Omega,\Psi)}(r,\tau) = \{\upsilon \in \mathcal{F} : \Theta(\omega - \upsilon, r) > 1 - \tau,\ \Omega(\omega - \upsilon, r) < \tau \text{ and } \Psi(\omega - \upsilon, r) < \tau\}$$

defines an open ball with centered at ω and radius r w.r.t $\tau \in (0,1)$. Define

$$\Im_{(\Theta,\Omega,\Psi)}(\mathcal{F}) = \left\{\mathcal{A} \subset \mathcal{F} : \text{ for all } \omega \in \mathcal{A},\ \exists r > 0 \text{ and } \tau \in (0,1) : \mathcal{B}_{\omega}^{(\Theta,\Omega,\Psi)}(r,\tau) \subset \mathcal{A}\right\}.$$

Then $\Im_{(\Theta,\Omega,\Psi)}(\mathcal{F})$ defines a topology on $\mathcal{F}$, which is induced by NN (Θ, Ω, Ψ). Since

$$\left\{\upsilon \in \mathcal{F} : \Theta\left(\omega - \upsilon, \frac{1}{s}\right) > 1 - \frac{1}{s},\ \Omega\left(\omega - \upsilon, \frac{1}{s}\right) < \frac{1}{s} \text{ and } \Psi\left(\omega - \upsilon, \frac{1}{s}\right) < \frac{1}{s}\right\}$$

is a local base at $\omega \in \mathcal{F}$, the topology $\Im_{(\Theta,\Omega,\Psi)}(\mathcal{F})$ on $\mathcal{F}$ is first countable.

Definition 2.2. Let $(\mathcal{F}, \Theta, \Omega, \Psi, \Diamond, *)$ be a $\mathcal{NNS}$. A sequence (ϕ_i) in $\mathcal{F}$ converges to ϕ_0 w.r.t. $\mathcal{NN}$ (Θ, Ω, Ψ), if

$$\Theta(\phi_i - \phi_0, \tau) \to 1,\ \Omega(\phi_i - \phi_0, \tau) \to 0 \text{ and } \Psi(\phi_i - \phi_0, \tau) \to 0 \text{ as } i \to \infty$$

supplies for each $\tau > 0$. We write the limit by $(\Theta, \Omega, \Psi) - \lim \phi_i = \phi_0$.

We refer to the collections of all natural numbers and real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively, throughout this research. Assume that $A \subseteq \mathbb{N}$. The natural or asymptotic density of the set W, represented by $\delta(W)$, may be expressed as follows:

$$\delta(W) = \lim_{u\to\infty} \frac{1}{u} |\{t \leq u : t \in W\}|,$$

given the existence of the limit. Here the cardinality of the set $\{\dots\}$ is shown by $|\{\dots\}|$. If, for any $\varepsilon > 0$, we have

$$\delta(\{u \in \mathbb{N} : |\phi_i - \phi_0| \geq \varepsilon\}) = 0,$$

then a sequence (ϕ_i) of numbers is said to be statistically convergent to ϕ_0 (see [9], [37]).

Definition 2.3. A sequence (ϕ_i) in $\mathcal{F}$ is statistically convergent to $\phi_0 \in \mathcal{F}$ w.r.t $\mathcal{NN}$ (Θ, Ω, Ψ), if for all $\gamma \in (0,1)$ and $\tau > 0$,

$$\lim_{t\to\infty} \frac{1}{t} \big|\{i \leq t : \Theta(\phi_i - \phi_0, \tau) \leq 1 - \gamma \text{ or } \Omega(\phi_i - \phi_0, \tau) \geq \gamma \text{ or } \Psi(\phi_i - \phi_0, \tau) \geq \gamma\}\big| = 0.$$

We represent the limit as $(\Theta, \Omega, \Psi)_{st} - \lim \phi_i = \phi_0$.

Definition 2.4. We define a sequence (ϕ_i) in $\mathcal{F}$ as rough convergent to $\phi_0 \in \mathcal{F}$ w.r.t. $\mathcal{NN}$ (Θ, Ω, Ψ) for some $r \geq 0$ if, for any $\gamma \in (0,1)$ and $\tau > 0$, there exist $i_0 \in \mathbb{N}$ such that

$$\Theta(\phi_i - \phi_0, r+\tau) > 1-\gamma,\ \Omega(\phi_i - \phi_0, r+\tau) < \gamma \text{ and } \Psi(\phi_i - \phi_0, r+\tau) < \gamma,$$

for all $i \geq i_0$. The convergence of the sequence (ϕ_i) is characterized by the limit expressed as $(\Theta, \Omega, \Psi)^r - \lim \phi_i = \phi_0$.

Agnew [1] defined postponed Cesàro mean as follows in 1932, expanding on the idea of Cesàro mean of real (or complex) sequences:

Let $(a_w), (b_w)$ be sequences of non-negative integers satisfying the conditions

$$\begin{gathered} a_w < b_w \\ \lim_{w\to\infty} b_w = \infty. \end{gathered} \tag{2.1}$$

The postponed Cesàro mean of a real (or complex) valued sequence (ϕ_i) is defined by

$$(D_{a,b}(\phi_i))_w := \frac{1}{b_w - a_w} \sum_{i=a_w+1}^{b_w} \phi_i,\ w = 1, 2, \ldots.$$

If the limit is present,

$$D_{a,b}(U) := \lim_{w\to\infty} \frac{1}{b_w - a_w} \left|\{i \in \mathbb{N} : a_w < i \leq b_w,\ i \in U\}\right|,$$

defines the deferred density of U for $U \subseteq \mathbb{N}$. If, for any $\varepsilon > 0$, we have

$$\lim_{w\to\infty} \frac{1}{b_w - a_w} \left|\{i \in \mathbb{N} : 1 + a_w \leq i \leq b_w,\ \ |\phi_i - \phi_0| \geq \varepsilon\}\right| = 0,$$

then a sequence (ϕ_i) of numbers is said to be deferred statistically convergent to ϕ_0 (see [26]).

The aforementioned definition aligns with the statistical convergence of (ϕ_i) as shown in [9] for $a_w = 0$ and $b_w = w$.

Let $\phi = (\phi_{ij})$ be a double sequence and $\psi(u) = q(u) - p(u)$, $\vartheta(v) = s(v) - r(v)$; and assume $\{p(u)\}, \{q(u)\}, \{r(v)\}$ and $\{s(v)\}$ be sequences of nonnegative integers satisfying the conditions

$$\begin{gathered} p(u) < q(u),\ r(v) < s(v) \text{ and} \\ \lim_{u\to\infty} q(u) = \infty,\ \lim_{v\to\infty} s(v) = \infty. \end{gathered} \tag{2.2}$$

Deferred Cesàro mean $D_{\psi,\vartheta}$ of the double sequence $\phi = (\phi_{ij})$ is identified by

$$(D_{\psi,\vartheta}\phi)_{(u,v)} = \frac{1}{\psi(u)\vartheta(v)} \sum_{\substack{i=p(u)+1 \\ j=r(v)+1}}^{q(u),s(v)} \phi_{ij}.$$

3. Main results

Within the context of $\mathcal{NNS}$ $(\mathcal{F},\Theta,\Omega,\Psi,\diamondsuit,*)$, we introduce the concept of rough $\mathcal{I}_2$-deferred statistical convergence for sequences in this section. $\{p(u)\},\{q(u)\}$, $\{r(v)\}$ and $\{s(v)\}$ represent the sequences of non-negative integers that fulfill (2.2) throughout this investigation. Any more limitations on $\{p(u)\},\{q(u)\},\{r(v)\}$ and $\{s(v)\}$ (if any) will be provided in the instances and theorems that correspond to them.

Definition 3.1. We say that a double sequence (ϕ_{ij}) in $\mathcal{F}$ is rough $\mathcal{I}_2$-deferred statistically convergent to $\phi_0 \in \mathcal{F}$ w.r.t. $\mathcal{NN}$ (Θ,Ω,Ψ) for some $r \geq 0$, provided that for each $\gamma,\rho \in (0,1)$ and $\tau > 0$

(3.1)
$$\Big\{(u,v)\in\mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)}\left|\{(i,j): p(u)+1\leq i\leq q(u),\ r(v)+1\leq j\leq s(v),\right.$$
$$\Theta(\phi_{ij}-\phi_0,r+\tau)\leq 1-\gamma,\ \Omega(\phi_{ij}-\phi_0,r+\tau)\geq\gamma$$
$$\left.\text{or } \Psi(\phi_{ij}-\phi_0,r+\tau)\geq\gamma\}\right|\geq\rho\Big\}\in\mathcal{I}_2.$$

In this instance, ϕ_0 is said to be the $\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)$-limit of the the double sequence (ϕ_{ij}) and it is demonstrated by $\phi_{ij} \overset{\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)}{\to} \phi_0$ or $\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\lim\phi_{ij}=\phi_0$.

In the following comment, we discuss how the $\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)$-convergence encompasses certain regular convergence methods within $\mathcal{NNS}$.

Remark 3.2. Let $(\mathcal{F},\Theta,\Omega,\Psi,\diamondsuit,*)$ be an $\mathcal{NNS}$ and $(\phi_{ij})\in\mathcal{F}$. Then
(i) We refer to the $\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)$-convergence of (ϕ_{ij}) as the $\mathcal{I}_2$-deferred statistical convergence w.r.t. $\mathcal{NN}$, given that the condition expressed in (3.1) is satisfied for $r=0$.
(ii) For $p(u)=0$, $q(u)=u$ and $r(v)=0$, $s(v)=v$ in (3.1), we refer to the $\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)$-convergence of (ϕ_{ij}) as the rough $\mathcal{I}_2$-statistical convergence w.r.t. $\mathcal{NN}$.
(iii) Assume $p(u)=k_{u-1}$, $q(u)=k_u$ and $r(v)=l_{v-1}$, $s(v)=l_v$ in (3.1), where $\theta_2=(k_u,l_v)$ is a double lacunary sequence. Then, we refer to the $\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)$-convergence of (ϕ_{ij}) as the rough $\mathcal{I}_2$-lacunary statistical convergence w.r.t. $\mathcal{NN}$.

Let $(\mathcal{F},\Theta,\Omega,\Psi,\diamondsuit,*)$ be an $\mathcal{NNS}$ and $(\phi_{ij})\in\mathcal{F}$. In this context, both $(\Theta,\Omega,\Psi)^r-\lim\phi_{ij}$ and $\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\lim\phi_{ij}$ may not be unique. Hence, we introduce

$$(\Theta,\Omega,\Psi)-\text{LIM}^r(\phi_{ij})=\{\phi_0\in\mathcal{F}:(\Theta,\Omega,\Psi)^r-\lim\phi_{ij}=\phi_0\},$$

and

$$\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\text{LIM}^r(\phi_{ij})=\Big\{\phi_0\in\mathcal{F}:\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\lim\phi_{ij}=\phi_0\Big\}$$

to represent the sets of all $(\Theta,\Omega,\Psi)^r-\lim\phi_{ij}$ and the set of all $\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\lim\phi_{ij}$ of the double sequence (ϕ_{ij}), respectively. We define the double sequence (ϕ_{ij}) as rough convergent w.r.t. $\mathcal{NN}$ if $(\Theta,\Omega,\Psi)-\text{LIM}^r(\phi_{ij})\neq\emptyset$ and as rough

$\mathcal{I}_2$-deferred statistically convergent w.r.t. $\mathcal{NN}$ if $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\text{LIM}^r(\phi_{ij})\neq\emptyset$ for some $r\geq 0$. Certainly, if $0\leq r_1\leq r_2$, then

$$(\Theta,\Omega,\Psi)-\text{LIM}^{r_1}(\phi_{ij})\subset(\Theta,\Omega,\Psi)-\text{LIM}^{r_2}(\phi_{ij}),$$

and

$$\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\text{LIM}^{r_1}(\phi_{ij})\subset\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\text{LIM}^{r_2}(\phi_{ij})$$

for a double sequence (ϕ_{ij}) in $\mathcal{F}$.

Example 3.3. Take $\mathcal{NNS}\left(\mathbb{R}^2,\Theta,\Omega,\Psi,\lozenge,*\right)$, where $\left(\mathbb{R}^2,\|.\|\right)$ is the usual normed space. Consider $\gamma_1\lozenge\gamma_2=\gamma_1\lozenge\gamma_2$ and $\gamma_1*\gamma_2=\min\{\gamma_1+\gamma_2,1\}$, $\forall\gamma_1,\gamma_2\in[0,1]$. Additionally, let $\mathcal{N}_{(\Theta,\Omega,\Psi)}$ denote the neutrosophic fuzzy set on $\mathbb{R}\times(0,\infty)$ characterized by

$$\mathcal{N}_{(\Theta,\Omega,\Psi)}=\left(\frac{\tau}{\tau+\|\omega\|},\frac{\|\omega\|}{\tau+\|\omega\|},\frac{\|\omega\|}{\tau}\right)$$

for all $w\in\mathcal{F}$ and $\tau>0$. The double sequence (ϕ_{ij}) in $\mathbb{R}$ is established as follows:

$$\phi_{ij}=\begin{cases}1, & \text{if } i,j=2t-1\\ -1, & \text{if not}\end{cases},\ t\in\mathbb{N}.$$

Let $\phi_0\in(\Theta,\Omega,\Psi)-\text{LIM}^r(\phi_{ij})$ for some $r>0$. So, we write $\Theta(\phi_{ij}-\phi_0,r+\tau)>1-\gamma$, $\Omega(\phi_{ij}-\phi_0,r+\tau)<\gamma$ and $\Psi(\phi_{ij}-\phi_0,r+\tau)<\gamma$. Hence, we obtain

$$(\tau+r)\frac{\gamma}{(1-\gamma)}>|\phi_{ij}-\phi_0|,\ \forall\gamma\in(0,1)\text{ and }\tau>0.$$

Let κ be extremely small, expressed as $\kappa=\frac{\tau\gamma}{1-\gamma}$ and similarly, let $r'=\frac{r\gamma}{1-\gamma}$. Then

$$r'+\kappa>|\phi_{ij}-\phi_0|\Rightarrow\phi_0\in\left[\phi_{ij}-r',\phi_{ij}+r'\right].$$

For $i,j=2t-1$, we get $\phi_0\in[1-r',1+r']$. When $i,j\neq 2t-1$, then $\phi_0\in[-1-r',-1+r']$. Now

$$\left[1-r',1+r'\right]\cap\left[-1-r',-1+r'\right]=\begin{cases}\emptyset, & \text{if } r'<1\\ \left[1-r',r'-1\right], & \text{if } r'\geq 1.\end{cases}$$

Hence

$$(\Theta,\Omega,\Psi)-\text{LIM}^r(\phi_{ij})=\begin{cases}[1-r,r-1], & \text{if } r\geq 1\\ \emptyset, & \text{if not.}\end{cases}$$

Define the double sequence (ϖ_{ij}) in $\mathbb{R}^2$ as

$$\varpi_{ij}=\begin{cases}ij, & \text{if } i=2^t,j=2^h\\ -1, & \text{if not}\end{cases},\ t,h\in\mathbb{N}.$$

Take $p(u)=0$, $q(u)=u$ and $r(v)=0$, $s(v)=v^2+1$, $\forall u,v\in\mathbb{N}$. Then

$$\begin{aligned}&\Big\{(u,v)\in\mathbb{N}^2:\frac{1}{\psi(u\ \vartheta(v)}\big|\{(i,j):p(u)+1\leq i\leq q(u),\\ &\quad r(v)+1\leq j\leq s(v)\Theta(\varpi_{ij}-\phi_0,r+\tau)\leq 1-\gamma,\\ &\quad\Omega(\varpi_{ij}-\phi_0,r+\tau)\geq\gamma\text{ or }\Psi(\varpi_{ij}-\phi_0,r+\tau)\geq\gamma\}\big|\geq\rho\Big\}\\ &=\Big\{(u,v)\in\mathbb{N}^2:\frac{1}{\psi(u)\vartheta(v)}\big|\{(i,j):p(u)+1\leq i\leq q(u),\ r(v)+1\leq j\leq s(v)\end{aligned}$$

$$(\tau + r)\frac{\gamma}{(1-\gamma)} > |\varpi_{ij} - \phi_0|\}| \geq \rho\Big\}$$

$$= \Big\{(u,v) \in \mathbb{N}^2 : \frac{1}{\psi(u)\vartheta(v)}\big|\big\{(i,j) : p(u)+1 \leq i \leq q(u),\ r(v)+1 \leq j \leq s(v)$$

$$r' + \kappa > |\varpi_{ij} - \phi_0|\big\}\big| \geq \rho\Big\}.$$

Since $r > 0$, we have $r' \geq 0$. So, for each $r' \geq 0$, we obtain

$$r' + \kappa > |\varpi_{ij} - \phi_0| \text{ implies } r' + \kappa > |1 + \phi_0| \tag{3.2}$$

whenever $p(u) + 1 \leq i \leq q(u)$, $r(v) + 1 \leq j \leq s(v)$ and $i \neq 2^t, j \neq 2^h$. Since

$$\Big\{(u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)}\left|\{(i,j) : p(u)+1 \leq i \leq q(u), r(v)+1 \leq j \leq s(v)\right.$$
$$\left. i \neq 2^t, j \neq 2^h\}\right| < \rho\Big\} \in \mathcal{F}(\mathcal{I}_2),$$

from (3.2), we write

$$\Big\{(u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)}\left|\{(i,j) : p(u)+1 \leq i \leq q(u), r(v)+1 \leq j \leq s(v)\right.$$
$$\left.\phi_0 \in [-1-r', r'-1]\}\right| < \rho\} \in \mathcal{F}(\mathcal{I}_2), \forall r' \geq 0.$$

As a result

$$\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \text{LIM}^r(\varpi_{ij}) = \begin{cases} [-1-r, r-1], & \text{if } r \geq 0 \\ \emptyset, & \text{if not.} \end{cases}$$

Regarding the $\mathcal{NN}$ (Θ,Ω,Ψ), both the double sequences (ϕ_{ij}) and (ϖ_{ij}) do not demonstrate convergence in the ordinary sense. Moreover, the limit $(\Theta,\Omega,\Psi)^r - \lim \varpi_{ij}$ is not valid for $r \geq 0$.

In contrast to the ordinary convergence observed in an $\mathcal{NNS}$, the rough convergence of a double sequence (ϕ_{ij}) w.r.t. $\mathcal{NN}$ does not necessarily imply the rough convergence of a subsequence of (ϕ_{ij}) within the same context. For instance, consider the double sequence $(\phi_{ij}) = (ij)$ in the $\mathcal{NNS}$ specified in Example 3.3. It is evident that $(\Theta,\Omega,\Psi) - \text{LIM}^r(\phi_{ij}) = [1-r, 1+r]$ for $r > 0$. However, when examining the subsequence $(\phi_{i^2j^2}) = (i^2j^2)$ of (ϕ_{ij}), the $(\Theta,\Omega,\Psi) - \text{LIM}^r(\phi_{i^2j^2})$ does not exist for any $r > 0$. This rationale similarly extends to the $\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)$-convergence of a double sequence (ϕ_{ij}) in $\mathcal{NNS}$.

Example 3.4. Take $\mathcal{NNS}$ $(\mathbb{R}^2, \Theta, \Omega, \Psi, \Diamond, *)$, where $(\mathbb{R}^2, \|.\|)$ is the usual normed space. Consider $\gamma_1 \Diamond \gamma_2 = \gamma_1 \cdot \gamma_2$ and $\gamma_1 * \gamma_2 = \min\{\gamma_1 + \gamma_2, 1\}$, $\forall \gamma_1, \gamma_2 \in [0,1]$ and Θ, Ω, Ψ is defined in Example (3.3). Establish the double sequence (ϕ_{ij}) in $\mathbb{R}^2$ as follows:

$$(\phi_{ij}) = \begin{cases} ij, & \text{if } i,j = t^2, \\ 0, & \text{if not} \end{cases}, \ t \in \mathbb{N}.$$

Then $(\phi_{i_k j_l}) = ((1,1),(4,4),(9,9),(16,16),...)$. Take $p(u) = 0$, $q(u) = u$ and $r(v) = 0$, $s(v) = v$, $\forall u, v \in \mathbb{N}$. Then, we have

$$\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \text{LIM}^r(\phi_{ij}) = [-r, r], \ \forall r \geq 0.$$

and

$$\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \text{LIM}^r(\phi_{i_k j_l}) = \emptyset.$$

Now we may provide our auxiliary theorem, which is crucial to the understanding of the subsequent findings. Since the findings are evident, the Lemma's proof is omitted.

Lemma 3.5. *Assume that $(\mathcal{F}, \Theta, \Omega, \Psi, \Diamond, *)$ is a $\mathcal{NNS}$ and that (ϕ_{ij}) is a double sequence in $\mathcal{F}$. For any $\gamma, \rho \in (0,1)$ and $\tau > 0$, the following statements are interchangeable:*

(i) $\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \lim \phi_{ij} = \phi_0$.

(ii)

$$\Big\{(u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p\left(u\right) + 1 \leq i \leq q\left(u\right),\, r\left(v\right) + 1 \leq j \leq s\left(v\right)\right. \\ \Theta\left(\phi_{ij} - \phi_0, r + \tau\right) \leq 1 - \gamma,\, \Omega\left(\phi_{ij} - \phi_0, r + \tau\right) \geq \gamma \\ \left. \textit{or } \Psi\left(\phi_{ij} - \phi_0, r + \tau\right) \geq \gamma\}\right| \geq \rho\} \in \mathcal{I}_2.$$

Theorem 3.6. *Assume $(\mathcal{F}, \Theta, \Omega, \Psi, \Diamond, *)$ be a $\mathcal{NNS}$. Then, for each double sequence (ϕ_{ij}) in $\mathcal{F}$ and $r > 0$, the inclusion*

$$(\Theta, \Omega, \Psi) - \mathrm{LIM}^r\left(\phi_{ij}\right) \subset \mathcal{I}_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r\left(\phi_{ij}\right).$$

supplies.

Proof. Let $\phi_0 \in (\Theta, \Omega, \Psi) - \mathrm{LIM}^r\left(\phi_{ij}\right)$. For all $\gamma \in (0,1)$ and $\tau > 0$, $\exists u_0 \in \mathbb{N}$ such that

$$\Theta\left(\phi_{ij} - \phi_0, r + \tau\right) > 1 - \gamma,\ \Omega\left(\phi_{ij} - \phi_0, r + \tau\right) < \gamma \text{ and } \Psi\left(\phi_{ij} - \phi_0, r + \tau\right) < \gamma,$$

for all $i, j \geq u_0$. Thus

$$\Big\{(i,j) : \Theta(\phi_{ij} - \phi_0, r + \tau) \leq 1 - \gamma,\, \Omega(\phi_{ij} - \phi_0, r + \tau) \geq \gamma \text{ or} \\ \Psi(\phi_{ij} - \phi_0, r + \tau) \geq \gamma\Big\} \subset (\{1, 2, ..., u_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, ..., u_0\}).$$

Since

$$\Big\{(u,v) \in \mathbb{N} \times \mathbb{N} : \tfrac{1}{\psi(u)\vartheta(v)}\big|\{(i,j) : p(u) + 1 \leq i \leq q(u),\, r(v) + 1 \leq j \leq s(v), \\ (i,j) \in (\{1, 2, ..., u_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, ..., u_0\})\}\big| \geq \rho\Big\} \in \mathcal{I}_2.$$

We obtain

$$\Big\{(u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)}\big|\{(i,j) : p(u) + 1 \leq i \leq q(u),\, r(v) + 1 \leq j \leq s(v), \\ \Theta(\phi_{ij} - \phi_0, r + \tau) \leq 1 - \gamma,\, \Omega(\phi_{ij} - \phi_0, r + \tau) \geq \gamma \\ \text{or } \Psi(\phi_{ij} - \phi_0, r + \tau) \geq \gamma\}\big| \geq \rho\Big\} \in \mathcal{I}_2.$$

Therefore, $\phi_0 \in \mathcal{I}_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r\left(\phi_{ij}\right)$. As a result, we obtain

$$(\Theta, \Omega, \Psi) - \mathrm{LIM}^r\left(\phi_{ij}\right) \subset \mathcal{I}_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \mathrm{LIM}^r\left(\phi_{ij}\right).$$

□

The inclusion connection indicated above is in fact rigorous, as Example 3.3 shows.

Theorem 3.7. *Let* $(\mathcal{F}, \Theta, \Omega, \Psi, \Diamond, *)$ *be a* $\mathcal{NNS}$ *and* (ϕ_{ij}) *be a double sequence in* $\mathcal{F}$*. It follows that for all* $r > 0$ *and* $\gamma \in (0,1)$*, there is no pair of elements* $\phi_1, \phi_2 \in \mathcal{I}_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \text{LIM}^r (\phi_{ij})$ *such that* $\Theta(\phi_1 - \phi_2, qr) \leq 1 - \gamma$ *or* $\Omega(\phi_1 - \phi_2, qr) \geq \gamma$ *or* $\Psi(\phi_1 - \phi_2, qr) \geq \gamma$ *for* $q > 2$*.*

Proof. For any $\gamma \in (0,1)$, there exists $\gamma_1 \in (0,1)$ such that $(1-\gamma_1) \Diamond (1-\gamma_1) > 1-\gamma$ and $\gamma_1 * \gamma_1 < \gamma$. We establish this result through a proof by contradiction. Consequently, there exist elements $\phi_1, \phi_2 \in \mathcal{I}_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \text{LIM}^r$ such that

$$\Theta(\phi_1 - \phi_2, qr) \leq 1 - \gamma \text{ or } \Omega(\phi_1 - \phi_2, qr) \geq \gamma \text{ or } \Psi(\phi_1 - \phi_2, qr) \geq \gamma, \tag{3.3}$$

for $q > 2$. For each $\tau > 0$ and construct the following sets

$$\begin{aligned} K &= \Big\{(u,v) \in \mathbb{N}^2 : \Theta(\phi_{ij} - \phi_1, r + \tfrac{\tau}{2}) \leq 1 - \gamma_1, \\ &\quad \Omega(\phi_{ij} - \phi_1, r + \tfrac{\tau}{2}) \geq \gamma_1 \text{ or } \Psi(\phi_{ij} - \phi_1, r + \tfrac{\tau}{2}) \geq \gamma_1\Big\}, \end{aligned}$$

and

$$\begin{aligned} L = &\Big\{(u,v) \in \mathbb{N}^2 : \Theta(\phi_{ij} - \phi_2, r + \tfrac{\tau}{2}) \leq 1 - \gamma_1, \\ &\quad \Omega(\phi_{ij} - \phi_2, r + \tfrac{\tau}{2}) \geq \gamma_1 \text{ or } \Psi(\phi_{ij} - \phi_2, r + \tfrac{\tau}{2}) \geq \gamma_1\Big\}, \end{aligned}$$

Hence, based on Lemma 3.5, we get

$$\Big\{(u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)}\big|\{(i,j) : p(u)+1 \leq i \leq q(u),\, r(v)+1 \leq j \leq s(v) \; (i,j) \in K\}\big| \geq \rho\Big\} \in \mathcal{I}_2,$$

and

$$\Big\{(u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)}\big|\{(i,j) : p(u)+1 \leq i \leq q(u),\, r(v)+1 \leq j \leq s(v) \; (i,j) \in L\}\big| \geq \rho\Big\} \in \mathcal{I}_2.$$

for $\rho > 0$. Now

$$\begin{aligned} &\Big\{(u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)}\big|\{(i,j) : p(u)+1 \leq i \leq q(u),\, r(v)+1 \leq j \leq s(v) \\ &(i,j) \in K \cup L\}\big| \geq \rho\Big\} \in \mathcal{I}_2, \\ &\subseteq \Big\{(u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)}\big|\{(i,j) : p(u)+1 \leq i \leq q(u),\, r(v)+1 \leq j \leq s(v) \\ &(i,j) \in K\}\big| \geq \rho\Big\} \\ &\cup\Big\{(u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)}\big|\{(i,j) : p(u)+1 \leq i \leq q(u),\, r(v)+1 \leq j \leq s(v) \\ &(i,j) \in L\}\big| \geq \rho\Big\} \in \mathcal{I}_2. \end{aligned}$$

Hence

$$M = \Big\{(u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)}|\{(i,j) : p(u)+1 \leq i \leq q(u),\, r(v)+1 \leq j \leq s(v) \; (i,j) \notin K \cup L\}| < \rho\Big\} \notin \mathcal{I}_2.$$

Since $q > 2$, take $qr = 2r + \tau$ for some $\tau > 0$. Let $(i,j) \in M = K^c \cap L^c$. Take $\Theta(\phi_1 - \phi_2, qr) \leq 1 - \gamma$ for $q > 2$. Then, we write

$$\begin{aligned} 1 - \gamma \geq \Theta(\phi_1 - \phi_2; 2r + \tau) &\geq \Theta\left(\phi_{ij} - \phi_1, r + \tfrac{\tau}{2}\right) \Diamond \Theta\left(\phi_{ij} - \phi_2, r + \tfrac{\tau}{2}\right) \\ &> (1-\gamma_1) \Diamond (1-\gamma_1) > 1 - \gamma, \end{aligned}$$

which is absurd. If $\Omega(\phi_1 - \phi_2, qr) \geq \gamma$ for $q > 2$, then we have

$$\begin{aligned}\gamma \leq \Omega\left(\phi_1 - \phi_2; 2r + \tau\right) &\leq \Omega\left(\phi_{ij} - \phi_1, r + \tfrac{\tau}{2}\right) * \Omega\left(\phi_{ij} - \phi_2, \tfrac{\tau}{2}\right)\\ &< \gamma_1 * \gamma_1 < \gamma,\end{aligned}$$

which is absurd. If $\Psi(\phi_1 - \phi_2, qr) \geq \gamma$ for $q > 2$, then

$$\begin{aligned}\gamma \leq \Psi\left(\phi_1 - \phi_2, r + \tau\right) &\leq \Psi\left(\phi_{ij} - \phi_1, r + \tfrac{\tau}{2}\right) * \Psi\left(\phi_{ij} - \phi_2, \tfrac{\tau}{2}\right)\\ &< \gamma_1 * \gamma_1 < \gamma.\end{aligned}$$

which is absurd. Hence,

$$\Theta(\phi_1 - \phi_2; 2r + \tau) > 1 - \gamma \text{ and } \Omega(\phi_1 - \phi_2; 2r + \tau) < \gamma,\ \Psi(\phi_1 - \phi_2; 2r + \tau) < \gamma. \tag{3.4}$$

Then, from (3.4) we get

$$\Theta(\phi_1 - \phi_2; qr) > 1 - \gamma \text{ and } \Psi(\phi_1 - \phi_2; qr) < \gamma,\ \Omega(\phi_1 - \phi_2; qr) < \gamma \text{ for } q > 2$$

which is a contradiction to (3.3). Thus, no $\phi_1, \phi_2 \in \mathcal{I}_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \text{LIM}^r$ exist such that $\Theta(\phi_1 - \phi_2; qr) \leq 1 - \gamma$ or $\Psi(\phi_1 - \phi_2; qr) \geq \gamma$, $\Omega(\phi_1 - \phi_2; qr) \geq \gamma$ for $q > 2$. □

Proposition 3.8. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \Diamond, *)$ be a $\mathcal{NNS}$ and (ϕ_{ij}), (ω_{ij}) be double sequences in $\mathcal{F}$. If $\mathcal{I}^{r_1}_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \lim \phi_{ij} = \phi_0$ and $\mathcal{I}^{r_2}_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \lim \omega_{ij} = \omega_0$ for some $r_1, r_2 \geq 0$, then*

$$\mathcal{I}^{(r_1+r_2)}_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \lim\left(\phi_{ij} + \omega_{ij}\right) = \phi_0 + \omega_0.$$

Proof. Given $\gamma \in (0,1)$, there exists $\gamma_1 \in (0,1)$ such that $(1 - \gamma_1) \Diamond (1 - \gamma_1) > 1 - \gamma$ and $\gamma_1 * \gamma_1 < \gamma$. Suppose $\mathcal{I}^{r_1}_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \lim \phi_{ij} = \phi_0$ and $\mathcal{I}^{r_2}_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \lim \omega_{ij} = \omega_0$ for a certain $r_1, r_2 \geq 0$. For any $\tau > 0$, take into

$$\begin{aligned}P = \Big\{ &(i,j) : \Theta\left(\phi_{ij} - \phi_0, r_1 + \tfrac{\tau}{2}\right) > 1 - \gamma_1,\\ &\Omega\left(\phi_{ij} - \phi_0, r_1 + \tfrac{\tau}{2}\right) < \gamma_1 \text{ and } \Psi\left(\phi_{ij} - \phi_0, r_1 + \tfrac{\tau}{2}\right) < \gamma_1\Big\},\end{aligned}$$

and

$$\begin{aligned}Q = \Big\{ &(i,j) : \Theta\left(\omega_{ij} - \omega_0, r_2 + \tfrac{\tau}{2}\right) > 1 - \gamma_1,\\ &\Omega\left(\omega_{ij} - \omega_0, r_2 + \tfrac{\tau}{2}\right) < \gamma_1 \text{ and } \Psi\left(\omega_{ij} - \omega_0, r_2 + \tfrac{\tau}{2}\right) < \gamma_1\Big\}.\end{aligned}$$

Then, we deduce

$$\begin{aligned}\Big\{ &(u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p\left(u\right) + 1 \leq i \leq q\left(u\right),\right.\\ &\left. r\left(v\right) + 1 \leq j \leq s\left(v\right),\ (i,j) \in P\}\right| < \rho\Big\} \in \mathcal{F}\left(\mathcal{I}_2\right),\end{aligned}$$

and

$$\begin{aligned}\Big\{ &(u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p\left(u\right) + 1 \leq i \leq q\left(u\right),\right.\\ &\left. r\left(v\right) + 1 \leq j \leq s\left(v\right),\ (i,j) \in Q\}\right| < \rho\Big\} \in \mathcal{F}\left(\mathcal{I}_2\right).\end{aligned}$$

Construct

$$R = \left\{(i,j) : (i,j) \in P \cap Q\right\}.$$

Then, we get

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left| \{(i,j) : p(u) + 1 \le i \le q(u), \right. \\ \left. r(v) + 1 \le j \le s(v),\ (i,j) \in R\} \right| < \rho \Big\} \in \mathcal{F}(\mathcal{I}_2).$$

This yields that $R \neq \emptyset$. Let $(i,j) \in R$. Then

$$\begin{aligned} &\Theta\left((\phi_{ij} + \omega_{ij}) - (\phi_0 + \omega_0), r_1 + r_2 + \tau\right) \\ &\ge \Theta\left(\phi_{ij} - \phi_0, r_1 + \frac{\tau}{2}\right) \lozenge \Theta\left(\omega_{ij} - \omega_0, r_2 + \frac{\tau}{2}\right) \\ &> (1 - \gamma_1) \lozenge (1 - \gamma_1) \\ &> 1 - \gamma, \end{aligned}$$

$$\begin{aligned} &\Omega\left((\phi_{ij} + \omega_{ij}) - (\phi_0 + \omega_0), r_1 + r_2 + \tau\right) \\ &\le \Omega\left(\phi_{ij} - \phi_0, r_1 + \frac{\tau}{2}\right) * \Omega\left(\omega_{ij} - \omega_0, r_2 + \frac{\tau}{2}\right) \\ &< \gamma_1 * \gamma_1 \\ &< \gamma, \end{aligned}$$

and

$$\begin{aligned} &\Psi\left((\phi_{ij} + \omega_{ij}) - (\phi_0 + \omega_0), r_1 + r_2 + \tau\right) \\ &\le \Psi\left(\phi_{ij} - \phi_0, r_1 + \frac{\tau}{2}\right) * \Psi\left(\omega_{ij} - \omega_0, r_2 + \frac{\tau}{2}\right) \\ &< \gamma_1 * \gamma_1 \\ &< \gamma. \end{aligned}$$

Hence

$$\begin{aligned} P \cap Q \subseteq \Big\{ (i,j) : &\Theta\left((\phi_{ij} + \omega_{ij}) - (\phi_0 + \omega_0), r_1 + r_2 + \tau\right) > 1 - \gamma, \\ &\Omega\left((\phi_{ij} + \omega_{ij}) - (\phi_0 + \omega_0), r_1 + r_2 + \tau\right) < \gamma \\ &\text{and } \Psi\left((\phi_{ij} + \omega_{ij}) - (\phi_0 + \omega_0), r_1 + r_2 + \tau\right) < \gamma \Big\}. \end{aligned}$$

This gives that

$$\begin{aligned} &\Big\{(u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \big| \big\{(i,j) : p(u) + 1 \le i \le q(u), \\ &\qquad r(v) + 1 \le j \le s(v),\ (i,j) \in P \cap Q\big\} \big| < \rho \Big\} \\ &\subseteq \Big\{(u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \big| \big\{(i,j) : p(u) + 1 \le i \le q(u), \\ &\quad r(v) + 1 \le j \le s(v),\ \Theta((\phi_{ij} + \omega_{ij}) - (\phi_0 + \omega_0), r_1 + r_2 + \tau) > 1 - \gamma, \\ &\qquad \Omega((\phi_{ij} + \omega_{ij}) - (\phi_0 + \omega_0), r_1 + r_2 + \tau) < \gamma, \\ &\qquad\quad \text{and } \Psi((\phi_{ij} + \omega_{ij}) - (\phi_0 + \omega_0), r_1 + r_2 + \tau) < \gamma \big| < \rho \Big\}. \end{aligned}$$

Thus, we obtain

$$\Big\{(u,v)\in\mathbb{N}^2:\tfrac{1}{\psi(u)\vartheta(v)}\big|\big\{(i,j):p(u)+1\le i\le q(u),\ r(v)+1\le j\le s(v),\ \Theta((\phi_{ij}+\omega_{ij})-(\phi_0+\omega_0),r_1+r_2+\tau)>1-\gamma,\ \Omega((\phi_{ij}+\omega_{ij})-(\phi_0+\omega_0),r_1+r_2+\tau)<\gamma,\ \text{and } \Psi((\phi_{ij}+\omega_{ij})-(\phi_0+\omega_0),r_1+r_2+\tau)<\gamma\big|<\rho\Big\}\in\mathcal{F}(\mathcal{I}_2).$$

As a result, $\mathcal{I}^{(r_1+r_2)}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\lim(\phi_{ij}+\omega_{ij})=\phi_0+\omega_0$. □

Remark 3.9. Proposition 3.8 is not valid for $0<r<r_1+r_2$ when at least one of r_1 and r_2 is non-zero, namely, for $r_1\neq 0$ or $r_2\neq 0$ when $\mathcal{I}^{r_1}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\lim\phi_{ij}=\phi_0$ and $\mathcal{I}^{r_2}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\lim\omega_{ij}=\omega_0$, then $\mathcal{I}^{(r_1+r_2)}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\lim(\phi_{ij}+\omega_{ij})$ need not to be equal to $\phi_0+\omega_0$ for $0<r<r_1+r_2$.

Example 3.10. Consider the $\mathcal{NNS}$ $(\mathbb{R},\Theta,\Omega,\Psi,\Diamond,*)$ as outlined in Example 3.3. Construct

$$\phi_{ij}=\begin{cases}(-1)^{i+j}, & \text{if } i,j\neq\rho^2\\ ij, & \text{if not}\end{cases},\quad \rho\in\mathbb{N}$$

and

$$\omega_{ij}=\begin{cases}(-2)^{i+j}, & \text{if } i,j\neq\rho^2,\\ 0 & \text{if not}\end{cases},\quad \rho\in\mathbb{N}.$$

Let $p(u)=0$, $q(u)=u$ and $r(v)=0$, $s(v)=3v$, $\forall u,v\in\mathbb{N}$. Adopting the approach illustrated in Example 3.3, we obtain

$$\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}^{r_1}\phi_{ij}=\begin{cases}[1-r_1,r_1-1], & \text{if } r_1\ge 1\\ \emptyset, & \text{if not,}\end{cases}$$

and

$$\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}^{r_2}(\omega_{ij})=\begin{cases}[2-r_2,r_2-2], & \text{if } r_2\ge 2\\ \emptyset, & \text{if not.}\end{cases}$$

Now

$$(\phi_{ij}+\omega_{ij})=\begin{cases}(-3)^{i+j} & \text{if } i,j\neq\rho^2,\\ ij, & \text{if not}\end{cases},\quad \rho\in\mathbb{N}.$$

Then

$$\mathcal{I}^{r}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}(\phi_{ij}+\omega_{ij})=\begin{cases}[3-r,r-3], & \text{if } r\ge 3\\ \emptyset, & \text{if not.}\end{cases}$$

$\mathcal{I}^{r_1}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\lim\phi_{ij}$ and $\mathcal{I}^{r_2}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\lim\omega_{ij}$ are equal to 0 if $r_1=1$ and $r_2=2$. We obtain $\mathcal{I}^{(r_1+r_2)}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}^{r}(\phi_{ij}+\omega_{ij})=\emptyset$ for $0<r<r_1+r_2=3$.

Proposition 3.11. *Assume $(\mathcal{F},\Theta,\Omega,\Psi,\Diamond,*)$ is an $\mathcal{NNS}$ and let (ϕ_{ij}) be a double sequence in $\mathcal{F}$. If $\mathcal{I}^{r}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\lim\phi_{ij}=\phi_0$ for some $r\ge 0$, then $\mathcal{I}^{|c|r}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\lim c\phi_{ij}=c\phi_0$ for any $c\in\mathbb{R}$.*

Proof. The result is evident when $0 = c \in \mathbb{R}$. Let $0 \neq c \in \mathbb{R}$. For given $\gamma \in (0,1)$, one has $\gamma_2 \in (0,1)$ such that $1-\gamma_2 > 1-\gamma$. Since $\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \lim \phi_{ij} = \phi_0$, we can consider the set

$$U = \Big\{(i,j) : \Theta\left(\phi_{ij} - \phi_0, r + \tfrac{\tau}{2|c|}\right) > 1-\gamma_2,$$
$$\Omega\left(\phi_{ij} - \phi_0, r + \tfrac{\tau}{2|c|}\right) < \gamma_2 \text{ and } \Psi\left(\phi_{ij} - \phi_0, r + \tfrac{\tau}{2|c|}\right) < \gamma_2\Big\}$$

with

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p(u)+1 \leq i \leq q(u),\right.$$
$$\left. r(v)+1 \leq j \leq s(v),\ (i,j) \in U\}\right| < \rho\Big\} \in \mathcal{F}(\mathcal{I}_2).$$

Consider $(i,j) \in U$. Then

$$\begin{aligned}\Theta(c\phi_{ij} - c\phi_0, |c|r + \tau) &= \Theta\left(\phi_{ij} - \phi_0, r + \frac{\tau}{|c|}\right)\\ &\geq \Theta\left(\phi_{ij} - \phi_0, r + \frac{\tau}{2|c|}\right)\\ &> 1-\gamma_2 > 1-\gamma,\end{aligned}$$

$$\begin{aligned}\Omega(c\phi_{ij} - c\phi_0, |c|r + \tau) &= \Omega\left(\phi_{ij} - \phi_0, r + \frac{\tau}{|c|}\right)\\ &\leq \Omega\left(\phi_{ij} - \phi_0, r + \frac{\tau}{2|c|}\right)\\ &< \gamma_2 < \gamma,\end{aligned}$$

and

$$\begin{aligned}\Psi(c\phi_{ij} - c\phi_0, |c|r + \tau) &= \Psi\left(\phi_{ij} - \phi_0, r + \frac{\tau}{|c|}\right)\\ &\leq \Psi\left(\phi_{ij} - \phi_0, r + \frac{\tau}{2|c|}\right)\\ &< \gamma_2 < \gamma.\end{aligned}$$

Consequently,

$$U \subset \{(i,j) : \Theta(c\phi_{ij} - c\phi_0, |c|r + \tau) > 1-\gamma,$$
$$\Omega(c\phi_{ij} - c\phi_0, |c|r + \tau) < \gamma \text{ and } \Psi(c\phi_{ij} - c\phi_0, |c|r + \tau) < \gamma\}.$$

Therefore,

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p(u)+1 \leq i \leq q(u),\ r(v)+1 \leq j \leq s(v)\right.$$
$$\Theta(c\phi_{ij} - c\phi_0, |c|r + \tau) > 1-\gamma,\ \Omega(c\phi_{ij} - c\phi_0, |c|r + \tau) < \gamma$$
$$\left.\text{and } \Psi(c\phi_{ij} - c\phi_0, |c|r + \tau) < \gamma\}\right| < \rho\Big\} \in \mathcal{F}(\mathcal{I}_2).$$

Consequently, $\mathcal{I}^{|c|r}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \lim c\phi_{ij} = c\phi_0$. □

Remark 3.12. When $0 < t < |c|r$, Proposition 3.11 is invalid for $r > 0$. Specifically, for some $r > 0$ if $\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \lim \phi_{ij} = \phi_0$, then $\mathcal{I}^t_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \lim c\phi_{ij}$ need not to be equal $c\phi_0$ for $0 < t < |c|r$ and $0 \neq c \in \mathbb{R}$.

Example 3.13. Take a look at Example 3.10 and assume $c = 2$. It is obvious that

$$2\phi_{ij} = \begin{cases} 2ij, & \text{if } i,j = 5^k \\ (-1)^k\, 2, & \text{if not} \end{cases}, \quad k \in \mathbb{N},$$

and

$$\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \text{LIM}^t\,(2\phi_{ij}) = \begin{cases} [2-t, t-2], & \text{if } t \geq 2 \\ \emptyset, & \text{if not.} \end{cases}$$

Let $r_1 = 2$. Then

$$\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \text{LIM}^2\,(\phi_{ij}) = [-1,1]$$

and

$$\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \text{LIM}^{2c}\,(c\phi_{ij}) = [-2,2] = c\,[-1,1]\,.$$

Conversely, $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \text{LIM}^t\,(c\phi_{ij}) = [2-t, t-2] \neq c[-1,1]$ is obtained if $2 \leq t < 4$.

Theorem 3.14. *Assume $(\mathcal{F},\Theta,\Omega,\Psi,\lozenge,*)$ is a $\mathcal{NNS}$ and (ϕ_{ij}) is a double sequence in $\mathcal{F}$. If there is a double sequence (ω_{ij}) in $\mathcal{F}$ with $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \lim \omega_{ij} = \phi_0$ such that for each $\gamma,\rho \in (0,1)$ we have $\Theta\,(\phi_{ij} - \omega_{ij}, r) > 1-\gamma$, $\Omega\,(\phi_{ij} - \omega_{ij}, r) < \gamma$ and $\Psi\,(\phi_{ij} - \omega_{ij}, r) < \gamma$ for all $i,j \in \mathbb{N}$, then $\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \lim \phi_{ij} = \phi_0$ for some $r \geq 0$.*

Proof. For given $\gamma \in (0,1)$, choose $\gamma_1 \in (0,1)$ such that $(1-\gamma_1)\lozenge(1-\gamma_1) > 1-\gamma$ and $\gamma_1 * \gamma_1 < \gamma$. Assume $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \lim \omega_{ij} = \phi_0$ and

$$\Theta\,(\phi_{ij} - \omega_{ij}, r) > 1-\gamma,\ \Omega\,(\phi_{ij} - \omega_{ij}, r) < \gamma \text{ and } \Psi\,(\phi_{ij} - \omega_{ij}, r) < \gamma$$

for each $\gamma \in (0,1)$ and for all $i,j \in \mathbb{N}$. For every $\tau > 0$ and the sets

$$U = \{(i,j) : \Theta\,(\omega_{ij} - \phi_0, \tau) \leq 1-\gamma_1,\ \Omega\,(\omega_{ij} - \phi_0, \tau) \geq \gamma_1 \text{ or } \Psi\,(\omega_{ij} - \phi_0, \tau) \geq \gamma_1\}\,,$$

and

$$V = \{(i,j) : \Theta\,(\phi_{ij} - \omega_{ij}, r) \leq 1-\gamma_1,\ \Omega\,(\phi_{ij} - \omega_{ij}, r) \geq \gamma_1 \text{ or } \Psi\,(\phi_{ij} - \omega_{ij}, r) \geq \gamma_1\}\,,$$

we get

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)}\,|\{(i,j) : p\,(u)+1 \leq i \leq q\,(u)\,,\ r\,(v)+1 \leq j \leq s\,(v)\ (i,j) \in U\}| \geq \rho \Big\} \in \mathcal{I}_2,$$

and

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)}\,|\{(i,j) : p\,(u)+1 \leq i \leq q\,(u)\,,\ r\,(v)+1 \leq j \leq s\,(v)\ (i,j) \in V\}| \geq \rho \Big\} \in \mathcal{I}_2.$$

Hence

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)}\,|\{(i,j) : p\,(u)+1 \leq i \leq q\,(u)\,,\ r\,(v)+1 \leq j \leq s\,(v)\ (i,j) \in U^c\}| < \rho \Big\} \in \mathcal{F}\,(\mathcal{I}_2)\,,$$

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)}\,|\{(i,j) : p\,(u)+1 \leq i \leq q\,(u)\,,\ r\,(v)+1 \leq j \leq s\,(v)\ (i,j) \in V^c\}| < \rho \Big\} \in \mathcal{F}\,(\mathcal{I}_2)\,.$$

Evidently, $U^c \cap V^c \neq \emptyset$ and

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p(u)+1 \leq i \leq q(u)\,,\; r(v)+1 \leq j \leq s(v)\right. \\ \left.(i,j) \in U^c \cap V^c\}\right| < \rho \Big\} \in \mathcal{F}(\mathcal{I}_2)\,.$$

Consider $(i,j) \in U^c \cap V^c$. Then

$$\begin{aligned} \Theta(\phi_{ij} - \phi_0, r+\tau) &\geq \Theta(\phi_{ij} - \omega_{ij}, r) \lozenge \Theta(\omega_{ij} - \phi_0, \tau) \\ &> (1-\gamma_1) \lozenge (1-\gamma_1) > 1-\gamma, \\ \Omega(\phi_{ij} - \phi_0, r+\tau) &\leq \Omega(\phi_{ij} - \omega_{ij}, r) * \Omega(\omega_{ij} - \phi_0, \tau) \\ &< \gamma_1 * \gamma_1 < \gamma, \end{aligned}$$

and

$$\Psi(\phi_{ij} - \phi_0, r+\tau) \leq \Psi(\phi_{ij} - \omega_{ij}, r) * \Psi(\omega_{ij} - \phi_0, \tau) < \gamma_1 * \gamma_1 < \gamma.$$

So,

$$U^c \cap V^c \subset \{(i,j) : \Theta(\phi_{ij} - \phi_0, r+\tau) > 1-\gamma, \\ \Omega(\phi_{ij} - \phi_0, r+\tau) < \gamma \text{ and } \Psi(\phi_{ij} - \phi_0, r+\tau) < \gamma\}\,,$$

follows, implying

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p(u)+1 \leq i \leq q(u)\,,\right. \\ r(v)+1 \leq j \leq s(v)\,,\; \Theta(\phi_{ij} - \phi_0, r+\tau) > 1-\gamma, \\ \left.\Omega(\phi_{ij} - \phi_0, r+\tau) < \gamma \text{ and } \Psi(\phi_{ij} - \phi_0, r+\tau) < \gamma\}\right| < \rho \Big\} \in \mathcal{F}(\mathcal{I}_2)\,.$$

Thus $\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \lim \phi_{ij} = \phi_0$. □

Definition 3.15. A double sequence (ϕ_{ij}) in $\mathcal{F}$ is said to be $\mathcal{I}_2$-deferred statistically bounded w.r.t. $\mathcal{NN}$ (Θ,Ω,Ψ), if for all $\gamma, \rho \in (0,1)$, $\exists \beta > 0$ such that

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p(u)+1 \leq i \leq q(u)\,,\; r(v)+1 \leq j \leq s(v)\right. \\ \left.\Theta(\phi_{ij}, \beta) \leq 1-\gamma \text{ or } \Omega(\phi_{ij}, \beta) \geq \gamma,\; \Psi(\phi_{ij}, \beta) \geq \gamma\}\right| \geq \rho \Big\} \in \mathcal{I}_2.$$

Theorem 3.16. *Let (ϕ_{ij}) be a double sequence in $\mathcal{F}$ and $(\mathcal{F}, \Theta, \Omega, \Psi, \lozenge, *)$ be a $\mathcal{NNS}$. In such case, for some $r \geq 0$, (ϕ_{ij}) is $\mathcal{I}_2$-deferred statistically bounded iff $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \mathrm{LIM}^r \phi_{ij} \neq \emptyset$ for some $r \geq 0$.*

Proof. Assume that (ϕ_{ij}) is $\mathcal{I}_2$-deferred statistically bounded. For each $\gamma, \rho \in (0,1)$, there exists $\beta > 0$ such that the set

$$\mathcal{K} = \{(i,j) : \Theta(\phi_{ij}, \beta) \leq 1-\gamma \text{ or } \Omega(\phi_{ij}, \beta) \geq \gamma,\; \Psi(\phi_{ij}, \beta) \geq \gamma\}$$

has

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p(u)+1 \leq i \leq q(u)\,,\right. \\ \left. r(v)+1 \leq j \leq s(v)\,,\; (i,j) \in \mathcal{K}\}\right| \geq \rho \Big\} \in \mathcal{I}_2.$$

Thus, we obtain

$$\mathcal{K}^c = \{(i,j) : \Theta(\phi_{ij}, \beta) > 1-\gamma \text{ and } \Omega(\phi_{ij}, \beta) < \gamma,\; \Psi(\phi_{ij}, \beta) < \gamma\}$$

and

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p\,(u) + 1 \le i \le q\,(u)\,, \right. \\ \left. r\,(v) + 1 \le j \le s\,(v)\,,\ (i,j) \in \mathcal{K}^c\}\right| < \rho \Big\} \in \mathcal{F}\,(\mathcal{I}_2)\,.$$

Consider $(i,j) \in \mathcal{K}^c$. For each $\tau > 0$, we get

$$\begin{aligned} \Theta\,(\phi_{ij}, \beta + \tau) &\ge \Theta\,(\phi_{ij}, \beta)\,\Diamond\Theta(0,\tau) \\ &> (1-\gamma)\Diamond 1 \\ &= 1-\gamma, \end{aligned}$$

$$\begin{aligned} \Omega\,(\phi_{ij}, \beta + \tau) &\le \Omega\,(\phi_{ij}, \beta) * \Omega(0,\tau) \\ &< \gamma * 0, \\ &= \gamma, \end{aligned}$$

and

$$\begin{aligned} \Psi\,(\phi_{ij}, \beta + \tau) &\le \Psi\,(\phi_{ij}, \beta) * \Psi(0,\tau) \\ &< \gamma * 0, \\ &= \gamma. \end{aligned}$$

So, we obtain

$$\mathcal{K}^c \subset \{(i,j) : \Theta\,(\phi_{ij}, \beta + \tau) > 1-\gamma, \\ \Omega\,(\phi_{ij}, \beta + \tau) < \gamma \text{ and } \Psi\,(\phi_{ij}, \beta + \tau) < \gamma\}$$

and

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p\,(u) + 1 \le i \le q\,(u)\,, \right. \\ r\,(v) + 1 \le j \le s\,(v)\,, \Theta\,(\phi_{ij}, \beta + \tau) > 1-\gamma, \\ \left. \Omega\,(\phi_{ij}, \beta + \tau) < \gamma \text{ and } \Psi\,(\phi_{ij}, \beta + \tau) < \gamma\}\right| < \rho \Big\} \in \mathcal{F}\,(\mathcal{I}_2)\,.$$

Consequently, we have $0 \in \mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \mathrm{LIM}^{\beta}\,\phi_{ij}$. So, $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \mathrm{LIM}^{\beta}\,\phi_{ij} \neq \emptyset$.

On the contrary, assume $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \mathrm{LIM}^{r}\,\phi_{ij} \neq \emptyset$ for some $r \ge 0$. So, there exist $\phi_0 \in \mathcal{F}$ such that $\phi_0 \in \mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \mathrm{LIM}^{r}\,\phi_{ij}$. Therefore, for each $\gamma \in (0,1)$ and $\tau > 0$, we get

$$\mathcal{L} = \{(i,j) : \Theta\,(\phi_{ij} - \phi_0, r + \tau) > 1-\gamma \\ \Omega\,(\phi_{ij} - \phi_0, r + \tau) < \gamma \text{ and } \Psi\,(\phi_{ij} - \phi_0, r + \tau) < \gamma\}$$

with

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p\,(u) + 1 \le i \le q\,(u)\,, \right. \\ \left. r\,(v) + 1 \le j \le s\,(v)\,,\ (i,j) \in \mathcal{L}\}\right| < \rho \Big\} \in \mathcal{F}\,(\mathcal{I}_2)\,.$$

Select an $T > 0$ large enough such that $Q = T - (r+\tau) > 0$, $\Theta(\phi_0, Q) = 1$ and $\Omega(\phi_0, Q) = \Psi(\phi_0, Q) = 0$. Let $(i,j) \in \mathcal{L}$. Then

$$\begin{aligned} \Theta\,(\phi_{ij}, T) &\ge \Theta\,(\phi_{ij} - \phi_0, r + \tau)\,\Diamond\Theta(\phi_0, Q) \\ &> (1-\gamma)\Diamond 1 \\ &= 1-\gamma, \end{aligned}$$

$$\begin{aligned}\Omega\left(\phi_{ij},T\right) &\le \Omega\left(\phi_{ij}-\phi_0,r+\tau\right)*\Omega(\phi_0,Q)\\ &< \gamma * 0\\ &= \gamma.\end{aligned}$$

Likewise, we obtain $\Psi\left(\phi_{ij},T\right)<\gamma$. Thus,

$$\mathcal{L}\subset\left\{(i,j):\Theta\left(\phi_{ij},T\right)>1-\gamma,\ \Omega\left(\phi_{ij},T\right)<\gamma \text{ and } \Psi\left(\phi_{ij},T\right)<\gamma\right\}.$$

and so, we get

$$\Big\{(u,v)\in\mathbb{N}^2:\tfrac{1}{\psi(u)\vartheta(v)}\left|\{(i,j):p(u)+1\le i\le q(u),\ r(v)+1\le j\le s(v),\right.$$
$$\left.\Theta\left(\phi_{ij},T\right)>1-\gamma,\ \Omega\left(\phi_{ij},T\right)<\gamma \text{ and } \Psi\left(\phi_{ij},T\right)<\gamma\}\right|<\rho\Big\}\in\mathcal{F}\left(\mathcal{I}_2\right).$$

As a result, the double sequence (ϕ_{ij}) is $\mathcal{I}_2$-deferred statistically bounded. □

Theorem 3.17. *Let $(\mathcal{F},\Theta,\Omega,\Psi,\Diamond,*)$ be a $\mathcal{NNS}$ and (ϕ_{ij}) be a double sequence in $\mathcal{F}$. Then, $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}^r\left(\phi_{ij}\right)$ is a convex set for each $r\ge 0$.*

Proof. Assume that $\gamma\in(0,1)$ and $\phi_1,\phi_2\in\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}^r\left(\phi_{ij}\right)$. Then, there exists $\gamma_1\in(0,1)$ such that $(1-\gamma_1)\Diamond(1-\gamma_1)>1-\gamma$ and $\gamma_1*\gamma_1<\gamma$. We show that

$$\eta\phi_1+(1-\eta)\phi_2\in\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}^r\left(\phi_{ij}\right)$$

for any $\eta\in[0,1]$. The proof is straightforward when $\eta=0$ and $\eta=1$. Consider $\eta\in(0,1)$. For any $\tau>0$, we define

$$T=\Big\{(i,j):\Theta\left(\phi_{ij}-\phi_1,r+\tfrac{\tau}{2\eta}\right)>1-\gamma_1$$
$$\Omega\left(\phi_{ij}-\phi_1,r+\tfrac{\tau}{2\eta}\right)<\gamma_1 \text{ and } \Psi\left(\phi_{ij}-\phi_1,r+\tfrac{\tau}{2\eta}\right)<\gamma_1\Big\},$$

and

$$V=\Big\{(i,j):\Theta\left(\phi_{ij}-\phi_2,r+\tfrac{\tau}{2(1-\eta)}\right)>1-\gamma_1,$$
$$\Omega\left(\phi_{ij}-\phi_2,r+\tfrac{\tau}{2(1-\eta)}\right)<\gamma_1 \text{ and } \Psi\left(\phi_{ij}-\phi_2,r+\tfrac{\tau}{2(1-\eta)}\right)<\gamma_1\Big\}.$$

Since $\phi_1,\phi_2\in\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}^r\left(\phi_{ij}\right)$, we get

$$\Big\{(u,v)\in\mathbb{N}^2:\tfrac{1}{\psi(u)\vartheta(v)}\left|\{(i,j):p(u)+1\le i\le q(u),\right.$$
$$\left.r(v)+1\le j\le s(v),\ (i,j)\in T\}\right|<\rho\Big\}\in\mathcal{F}\left(\mathcal{I}_2\right),$$

and

$$\Big\{(u,v)\in\mathbb{N}^2:\tfrac{1}{\psi(u)\vartheta(v)}\left|\{(i,j):p(u)+1\le i\le q(u),\right.$$
$$\left.r(v)+1\le j\le s(v),\ (i,j)\in V\}\right|<\rho\Big\}\in\mathcal{F}\left(\mathcal{I}_2\right).$$

So, $T\cap V\neq\emptyset$ and

$$\Big\{(u,v)\in\mathbb{N}^2:\tfrac{1}{\psi(u)\vartheta(v)}\left|\{(i,j):p(u)+1\le i\le q(u),\right.$$
$$\left.r(v)+1\le j\le s(v),\ (i,j)\in T\cap V\}\right|<\rho\Big\}\in\mathcal{F}\left(\mathcal{I}_2\right).$$

Consider $(i,j)\in T\cap V$. Next

$$\begin{aligned}&\Theta\left(\phi_{ij}-[\eta\phi_1+(1-\eta)\phi_2],r+\tau\right)\\ &=\Theta\left((1-\eta)\left(\phi_{ij}-\phi_2\right)+\eta\left(\phi_{ij}-\phi_1\right),(1-\eta)r+\eta r+\tau\right)\end{aligned}$$

$$\geq \Theta\left((1-\eta)\left(\phi_{ij}-\phi_2\right),(1-\eta)r+\frac{\tau}{2}\right)\Diamond\Theta\left(\eta\left(\phi_{ij}-\phi_1\right),\eta r+\frac{\tau}{2}\right)$$
$$=\Theta\left(\phi_{ij}-\phi_2,r+\frac{\tau}{2(1-\eta)}\right)\Diamond\Theta\left(\phi_{ij}-\phi_1,r+\frac{\tau}{2\eta}\right)$$
$$>(1-\gamma_1)\Diamond(1-\gamma_1)>1-\gamma,$$

and

$$\Omega\left(\phi_{ij}-[\eta\phi_1+(1-\eta)\phi_2],r+\tau\right)$$
$$=\Omega\left((1-\eta)\left(\phi_{ij}-\phi_2\right)+\eta\left(\phi_{ij}-\phi_1\right),(1-\eta)r+\eta r+\tau\right)$$
$$\leq \Omega\left((1-\eta)\left(\phi_{ij}-\phi_2\right),(1-\eta)r+\frac{\tau}{2}\right)*\Omega\left(\eta\left(\phi_{ij}-\phi_1\right),\eta r+\frac{\tau}{2}\right)$$
$$=\Omega\left(\phi_{ij}-\phi_2,r+\frac{\tau}{2(1-\eta)}\right)*\Omega\left(\phi_{ij}-\phi_1,r+\frac{\tau}{2\eta}\right)$$
$$<\gamma_1*\gamma_1<\gamma.$$

In a similar vein

$$\Psi\left(\phi_{ij}-[\eta\phi_1+(1-\eta)\phi_2],r+\tau\right)<\gamma.$$

This indicates that the set

$$\Big\{(i,j):\Theta\left(\phi_{ij}-[\eta\phi_1+(1-\eta)\phi_2],r+\tau\right)>1-\gamma,\ \Omega\left(\phi_{ij}-[\eta\phi_1+(1-\eta)\phi_2],r+\tau\right)<\gamma\ \text{and}\ \Psi\left(\phi_{ij}-[\eta\phi_1+(1-\eta)\phi_2],r+\tau\right)<\gamma\Big\}$$

contains $T\cap V$ as a subset. Consequently, we have

$$\Big\{(u,v)\in\mathbb{N}^2:\tfrac{1}{\psi(u)\vartheta(v)}\left|\{(i,j):p(u)+1\leq i\leq q(u),\ r(v)+1\leq j\leq s(v),\ \Theta\left(\phi_{ij}-[\eta\phi_1+(1-\eta)\phi_2],r+\tau\right)>1-\gamma,\ \Omega\left(\phi_{ij}-[\eta\phi_1+(1-\eta)\phi_2],r+\tau\right)<\gamma\ \text{and}\ \Psi\left(\phi_{ij}-[\eta\phi_1+(1-\eta)\phi_2],r+\tau\right)<\gamma\}\right|<\rho\Big\}\in\mathcal{F}\left(\mathcal{I}_2\right).$$

for each $\gamma,\rho\in(0,1)$ and for all $\tau>0$. Thus

$$\eta\phi_1+(1-\eta)\phi_2\in\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}^r\left(\phi_{ij}\right).$$

□

Theorem 3.18. *Let $(\mathcal{F},\Theta,\Omega,\Psi,\Diamond,*)$ be a $\mathcal{NNS}$ and (ϕ_{ij}) be a double sequence in $\mathcal{F}$. After that, for each $r\geq 0$, $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}^r$ is closed.*

Proof. We do not need to establish a proof since $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}^r\left(\phi_{ij}\right)$ is an empty set. Suppose $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}^r\left(\phi_{ij}\right)\neq\emptyset$ for some $r>0$. Let $\phi_0\in\overline{\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}^r\left(\phi_{ij}\right)}$. Then, we have a convergent double sequence (ω_{ij}) in $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}^r\left(\phi_{ij}\right)$ w.r.t. $\mathfrak{NN}$ such that $\omega_{ij}\overset{(\Theta,\Omega,\Psi)}{\to}\phi_0$. Select $\gamma_1\in(0,1)$ for $\gamma\in(0,1)$ such that $(1-\gamma_1)\Diamond(1-\gamma_1)>(1-\gamma)$ and $\gamma_1*\gamma_1<\gamma$. Since $\omega_{ij}\overset{(\Theta,\Omega,\Psi)}{\to}\phi_0$, then, for all $\tau>0$ and $i_0\in\mathbb{N}$ such that

$$\Theta\left(\omega_{ij}-\phi_0,\frac{\tau}{2}\right)>1-\gamma_1,\ \Omega\left(\omega_{ij}-\phi_0,\frac{\tau}{2}\right)<\gamma_1\ \text{and}\ \Psi\left(\omega_{ij}-\phi_0,\frac{\tau}{2}\right)<\gamma_1$$

for all $i, j \geq i_0$. Adjust $t, k > i_0$ so that $w_{tk} \in \mathcal{I}_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \text{LIM}^r(\phi_{ij})$. Thus, using

$$\begin{aligned}(3.5)\qquad &\Big\{(u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p(u)+1 \leq i \leq q(u),\right.\\ &\quad \left. r(v)+1 \leq j \leq s(v),\ (i,j) \in \mathcal{K}\}\right| < \rho\Big\} \in \mathcal{F}(\mathcal{I}_2),\end{aligned}$$

we obtain

$$\begin{aligned}\mathcal{K} = \Big\{ &(i,j) : \Theta\left(\phi_{ij} - w_{tk}, r + \frac{\tau}{2}\right) > 1 - \gamma_1,\\ &\Omega\left(\phi_{ij} - w_{tk}, r + \frac{\tau}{2}\right) < \gamma_1 \text{ and } \Psi\left(\phi_{ij} - w_{tk}, r + \frac{\tau}{2}\right) < \gamma_1\Big\}.\end{aligned}$$

If $(i,j) \in \mathcal{K}$, then we get

$$\begin{aligned}\Theta(\phi_{ij} - \phi_0, r + \tau) &\geq \Theta\left(\phi_{ij} - w_{tk}, r + \frac{\tau}{2}\right) \Diamond \Theta\left(w_{tk} - \phi_0, \frac{\tau}{2}\right)\\ &> (1 - \gamma_1) \Diamond (1 - \gamma_1) > 1 - \gamma,\end{aligned}$$

$$\begin{aligned}\Omega(\phi_{ij} - \phi_0, r + \tau) &\leq \Omega\left(\phi_{ij} - w_{tk}, r + \frac{\tau}{2}\right) * \Omega\left(w_{tk} - \phi_0, \frac{\tau}{2}\right)\\ &< \gamma_1 * \gamma_1 < \gamma,\end{aligned}$$

and

$$\begin{aligned}\Psi(\phi_{ij} - \phi_0, r + \tau) &\leq \Psi\left(\phi_{ij} - w_{tk}, r + \frac{\tau}{2}\right) * \Psi\left(w_{tk} - \phi_0, \frac{\tau}{2}\right)\\ &< \gamma_1 * \gamma_1 < \gamma.\end{aligned}$$

As a result

$$\begin{aligned}\mathcal{K} \subseteq \{&(i,j) : \Theta(\phi_{ij} - \phi_0, r + \tau) > 1 - \gamma,\\ &\Omega(\phi_{ij} - \phi_0, r + \tau) < \gamma \text{ and } \Psi(\phi_{ij} - \phi_0, r + \tau) < \gamma\}.\end{aligned}$$

According to the (3.5), we obtain

$$\begin{aligned}&\Big\{(u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p(u)+1 \leq i \leq q(u),\right.\\ &\quad r(v)+1 \leq j \leq s(v),\ \Theta(\phi_{ij} - \phi_0, r + \tau) > 1 - \gamma,\ \Omega(\phi_{ij} - \phi_0, r + \tau) < \gamma\\ &\qquad \left.\text{and } \Psi(\phi_{ij} - \phi_0, r + \tau) < \gamma\}\right| < \rho\Big\} \in \mathcal{F}(\mathcal{I}_2),\end{aligned}$$

or $\phi_0 \in \mathcal{I}_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \text{LIM}^r(\phi_{ij})$. Consequently, the outcome guarantees. □

Theorem 3.19. *In the event that* $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \lim(\phi_{ij}) = \phi_0$, $\tau \in (0,1)$ *occurs such that, for some* $r > 0$, $\overline{\mathcal{B}_{\phi_0}^{(\Theta,\Omega,\Psi)}(r,\tau)} \subset \mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \lim(\phi_{ij})$.

Proof. If $\gamma \in (0,1)$ is known, find $\exists \gamma_1 \in (0,1)$ such that $(1 - \gamma_1) \Diamond (1 - \gamma_1) > 1 - \gamma$ and $\gamma_1 * \gamma_1 < \gamma$. Assume that $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta, \Omega, \Psi) - \lim(\phi_{ij}) = \phi_0$. For every $\tau > 0$ and consider the set

$$\begin{aligned}\mathcal{L} = \Big\{ &(i,j) : \Theta(\phi_{ij} - \phi_0, \tau) > 1 - \gamma_1\\ &\Omega(\phi_{ij} - \phi_0, \tau) < \gamma_1 \text{ and } \Psi(\phi_{ij} - \phi_0, \tau) < \gamma_1\Big\}.\end{aligned}$$

Then, we get

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p(u)+1 \le i \le q(u), \right. \\ \left. r(v)+1 \le j \le s(v),\ (i,j) \in \mathcal{L}\}\right| < \rho \Big\} \in \mathcal{F}(\mathcal{I}_2).$$

Select α such that $\alpha \in \overline{\mathcal{B}_{\phi_0}^{(\Theta,\Omega,\Psi)}(r,\tau_1)}$, $r > 0$.

$$\Theta(\phi_0 - \alpha, r) \ge 1 - \gamma_1,\ \Omega(\phi_0 - \alpha, r) \le \gamma_1 \text{ and } \Psi(\phi_0 - \alpha, r) \le \gamma_1$$

in such case. Likewise, for $(i,j) \in \mathcal{L}$, we get

$$\Theta(\phi_{ij} - \alpha, r+\tau) > 1-\gamma,\ \Omega(\phi_{ij} - \alpha, r+\tau) < \gamma \text{ and } \Psi(\phi_{ij} - \alpha, r+\tau) < \gamma.$$

Consequently,

$$\mathcal{L} \subset \{(i,j) : \Theta(\phi_{ij} - \alpha, r+\tau) > 1-\gamma \\ \Omega(\phi_{ij} - \alpha, r+\tau) < \gamma \text{ and } \Psi(\phi_{ij} - \alpha, r+\tau) < \gamma\}.$$

So, we obtain

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p(u)+1 \le i \le q(u),\ r(v)+1 \le j \le s(v), \right. \\ \Theta(\phi_{ij} - \alpha, r+\tau) > 1-\gamma,\ \Omega(\phi_{ij} - \alpha, r+\tau) < \gamma \\ \left. \text{and } \Psi(\phi_{ij} - \alpha, r+\tau) < \gamma\}\right| < \rho \Big\} \in \mathcal{F}(\mathcal{I}_2).$$

Thus, $\alpha \in \mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \lim(\phi_{ij})$. This gives that

$$\overline{\mathcal{B}_{\phi_0}^{(\Theta,\Omega,\Psi)}(r,\tau)} \subset \mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi) - \lim(\phi_{ij}).$$

□

Now, let's introduce and explore the concept of a rough deferred statistical cluster point in a $\mathfrak{NNS}$ as stated below:

Definition 3.20. Let $(\mathcal{F},\Theta,\Omega,\Psi,\Diamond,*)$ be a $\mathcal{NNS}$ and consider (ϕ_{ij}) as a double sequence in $\mathcal{F}$. For each $r \ge 0$, we define $p \in \mathcal{F}$ as a rough $\mathcal{I}_2$-deferred statistical cluster point of (ϕ_{ij}) w.r.t. $\mathcal{NN}$ (Θ,Ω,Ψ) if

$$d_{\mathcal{I}_2}\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p(u)+1 \le i \le q(u),\ r(v)+1 \le j \le s(v), \right. \\ \Theta(\phi_{ij} - p, r+\tau) > 1-\gamma,\ \Omega(\phi_{ij} - p, r+\tau) < \gamma, \\ \left. \Psi(\phi_{ij} - p, r+\tau) < \gamma\}\right| < \rho \Big\} \neq 0,$$

where

$$d_{\mathcal{I}_2}(A) = \mathcal{I}_2 - \lim_{u,v\to\infty} \frac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p(u)+1 \le i \le q(u), \right. \\ \left. r(v)+1 \le j \le s(v),\ (i,j) \in A\}\right|.$$

holds for every $\tau > 0$ and $\gamma, \rho \in (0,1)$. Equivalently

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \Big|\Big\{ (i,j) : p(u)+1 \le i \le q(u), \\ r(v)+1 \le j \le s(v),\ \Theta(\phi_{ij} - p, r+\tau) > 1-\gamma, \\ \Omega(\phi_{ij} - p, r+\tau) < \gamma \text{ and } \Psi(\phi_{ij} - p, r+\tau) < \gamma\Big\}\Big| < \rho \Big\} \in \mathcal{F}(\mathcal{I}_2).$$

We use $\Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})$ to represent the collection of all $\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)$-cluster points of the double sequence (ϕ_{ij}).

When r equals 0, we refer to the rough $\mathcal{I}_2$-deferred statistical cluster point of a double sequence (ϕ_{ij}) in $\mathcal{F}$ as the deferred $\mathcal{I}_2$-statistical cluster point of (ϕ_{ij}) w.r.t $\mathcal{NN}(\Theta,\Omega,\Psi)$, denoted as $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)$-cluster point. In this scenario, we represent the collection of all $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)$-cluster points of (ϕ_{ij}) by $\Gamma_{DS[\psi,\vartheta]}(\mathcal{I})(\phi_{ij})$.

This is how we now display the set $\Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})$'s topological property:

Theorem 3.21. *Let $(\mathcal{F},\Theta,\Omega,\Psi,\Diamond,*)$ be a $\mathcal{NNS}$ and consider (ϕ_{ij}) as a double sequence in $\mathcal{F}$. It follows that for any $r\geq 0$, $\Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})$ is a closed set.*

Proof. Assume that $\gamma\in(0,1)$. Then, there exists $\gamma_1\in(0,1)$ such that
$$(1-\gamma_1)\Diamond(1-\gamma_1)>1-\gamma \text{ and } \gamma_1*\gamma_1<\gamma.$$
Suppose $\omega\in\overline{\Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})}$. Then, there is a double sequence (ω_{ij}) of members in $\Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})$ such that $(\Theta,\Omega,\Psi)-\lim\omega_{ij}=\omega$. Thus, for each $\tau>0$, $\exists i,j\in\mathbb{N}$ such that
$$\Theta\left(\omega_{ij}-\omega,\frac{\tau}{2}\right)>1-\gamma_1,\ \Omega\left(\omega_{ij}-\omega,\frac{\tau}{2}\right)<\gamma_1 \text{ and } \Psi\left(\omega_{ij}-\omega,\frac{\tau}{2}\right)<\gamma_1$$
for all $i\geq i_0$ and $j\geq j_0$. Assign $t>i_0$, $k>j_0$. Next,
$$\Theta\left(\omega_{tk}-\omega,\frac{\tau}{2}\right)>1-\gamma_1,\ \Omega\left(\omega_{tk}-\omega,\frac{\tau}{2}\right)<\gamma_1 \text{ and } \Psi\left(\omega_{tk}-\omega,\frac{\tau}{2}\right)<\gamma_1.$$
Also, we have
$$\begin{aligned}W=\Big\{(i,j):\Theta\left(\phi_{ij}-\omega_{tk},r+\frac{\tau}{2}\right)&>1-\gamma_1,\\ \Omega\left(\phi_{ij}-\omega_{tk},r+\frac{\tau}{2}\right)<\gamma_1 &\text{ and } \Psi\left(\phi_{ij}-\omega_{tk},r+\frac{\tau}{2}\right)<\gamma_1\Big\}\end{aligned}$$
with
$$\begin{aligned}\Big\{(u,v)\in\mathbb{N}^2:&\tfrac{1}{\psi(u)\vartheta(v)}\left|\{(i,j):p(u)+1\leq i\leq q(u),\right.\\ &\left. r(v)+1\leq j\leq s(v),\ (i,j)\in W\}\right|<\rho\Big\}\in\mathcal{F}(\mathcal{I}_2).\end{aligned}$$
If $(i,j)\in W$, then we get
$$\begin{aligned}\Theta(\phi_{ij}-\omega,r+\tau)&\geq\Theta\left(\phi_{ij}-\omega_{tk},r+\frac{\tau}{2}\right)\Diamond\Theta\left(\omega_{tk}-\omega,\frac{\tau}{2}\right)\\ &>(1-\gamma_1)\Diamond(1-\gamma_1)>1-\gamma,\end{aligned}$$
$$\begin{aligned}\Omega(\phi_{ij}-\omega,r+\tau)&\leq\Omega\left(\phi_{ij}-\omega_{tk},r+\frac{\tau}{2}\right)*\Omega\left(\omega_{tk}-\omega,\frac{\tau}{2}\right)\\ &<\gamma_1*\gamma_1<\gamma,\end{aligned}$$
and
$$\begin{aligned}\Psi(\phi_{ij}-\omega,r+\tau)&\leq\Psi\left(\phi_{ij}-\omega_{tk},r+\frac{\tau}{2}\right)*\Psi\left(\omega_{tk}-\omega,\frac{\tau}{2}\right)\\ &<\gamma_1*\gamma_1<\gamma.\end{aligned}$$
Thus, we get

$W\subset\{(i,j):\Theta(\phi_{ij}-\omega,r+\tau)>1-\gamma$

$$\Omega(\phi_{ij}-\omega,r+\tau)<\gamma \text{ and } \Psi(\phi_{ij}-\omega,r+\tau)<\gamma\}$$
$$\Longrightarrow \Big\{(u,v)\in\mathbb{N}^2:\frac{1}{\psi(u)\vartheta(v)}\left|\{(i,j):p(u)+1\le i\le q(u),\,r(v)+1\le j\le s(v),\right.$$
$$\Theta(\phi_{ij}-\omega,r+\tau)>1-\gamma,\ \Omega(\phi_{ij}-\omega,r+\tau)<\gamma$$
$$\left.\text{and } \Psi(\phi_{ij}-\omega,r+\tau)<\gamma\}\right|<\rho\Big\}\in\mathcal{F}(\mathcal{I}_2).$$

$\omega\in\Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})$ as a result, and $\Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})$ is closed. □

Theorem 3.22. *Let $(\mathcal{F},\Theta,\Omega,\Psi,\Diamond,*)$ be a $\mathcal{NNS}$ and let (ϕ_{ij}) be a double sequence in $\mathcal{F}$. Assume $\kappa\in\Gamma_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})$. If, for each $\gamma\in(0,1)$,*

$$\Theta(\varsigma-\kappa,r)>1-\gamma,\ \Omega(\varsigma-\kappa,r)<\gamma \text{ and } \Psi(\varsigma-\kappa,r)<\gamma.$$

hold for some $r\ge 0$, then $\varsigma\in\Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})$.

Proof. For given $\gamma\in(0,1)$, $\exists\gamma_1\in(0,1)$ such that $(1-\gamma_1)\Diamond(1-\gamma_1)>1-\gamma$ and $\gamma_1*\gamma_1<\gamma$. Suppose that $\kappa\in\Gamma_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\phi_{ij})$. Then, for all $\tau>0$, the set

$$T=\{(i,j):\Theta(\phi_{ij}-\kappa,\tau)>1-\gamma_1,$$
$$\Omega(\phi_{ij}-\kappa,\tau)<\gamma_1 \text{ and } \Psi(\phi_{ij}-\kappa,\tau)<\gamma_1\}$$

has

$$\Big\{(u,v)\in\mathbb{N}^2:\tfrac{1}{\psi(u)\vartheta(v)}\left|\{(i,j):p(u)+1\le i\le q(u),\right.$$
$$\left. r(v)+1\le j\le s(v),\ (i,j)\in T\}\right|<\rho\Big\}\in\mathcal{F}(\mathcal{I}_2), \tag{3.6}$$

for each $\gamma,\rho\in(0,1)$ and $\tau>0$. Consider $\varsigma\in\mathcal{F}$ such that

$$\Theta(\varsigma-\kappa,r)>1-\gamma_1,\ \Omega(\varsigma-\kappa,r)<\gamma_1 \text{ and } \Psi(\varsigma-\kappa,r)<\gamma_1$$

for some $r\ge 0$. For any pair $(i,j)\in T$, following a similar approach as mentioned above, we derive

$$\Theta(\phi_{ij}-\varsigma,r+\tau)>1-\gamma,\ \Omega(\phi_{ij}-\varsigma,r+\tau)<\gamma \text{ and } \Psi(\phi_{ij}-\varsigma,r+\tau)<\gamma.$$

Therefore,

$$T\subset\{(i,j):\Theta(\phi_{ij}-\varsigma,r+\tau)>1-\gamma,$$
$$\Omega(\phi_{ij}-\varsigma,r+\tau)<\gamma \text{ and } \Psi(\phi_{ij}-\varsigma,r+\tau)<\gamma\}$$
$$\Longrightarrow\Big\{(u,v)\in\mathbb{N}^2:\frac{1}{\psi(u)\vartheta(v)}\left|\{(i,j):p(u)+1\le i\le q(u),\right.$$
$$\left. r(v)+1\le j\le s(v),\ (i,j)\in T\}\right|<\rho\Big\}$$
$$\subset\Big\{(u,v)\in\mathbb{N}^2:\frac{1}{\psi(u)\vartheta(v)}\left|\{(i,j):p(u)+1\le i\le q(u),\right.$$
$$r(v)+1\le j\le s(v),\ \Theta(\phi_{ij}-\varsigma,r+\tau)>1-\gamma,$$
$$\left.\Omega(\phi_{ij}-\varsigma,r+\tau)<\gamma\text{and } \Psi(\phi_{ij}-\varsigma,r+\tau)<\gamma\}\right|<\rho\Big\}.$$

Since the (3.6) holds, we obtain

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p(u)+1 \le i \le q(u),\right.$$
$$r(v)+1 \le j \le s(v),\ \Theta(\phi_{ij} - \varsigma, r+\tau) > 1-\gamma,\ \Omega(\phi_{ij} - \varsigma, r+\tau) < \gamma$$
$$\left.\text{and } \Psi(\phi_{ij} - \varsigma, r+\tau) < \gamma\}\right| < \rho \Big\} \in \mathcal{F}(\mathcal{I}_2).$$

So, $\varsigma \in \Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})$. □

The aforementioned theorem makes it abundantly evident that there is a corresponding rough $\mathcal{I}_2$-deferred statistical cluster point for each deferred statistical cluster point in a double sequence in a $\mathcal{NNS}$. The following theorem is presented in view of this fact.

Theorem 3.23.

$$\Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij}) = \bigcup_{\phi_0 \in \Gamma_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})} \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{\phi_0}(r,\gamma)}$$

exists for some $r > 0$ and $\gamma \in (0,1)$.

Proof. Suppose $\gamma \in (0,1)$ is given. So, $\exists \gamma_1 \in (0,1)$ such that $(1-\gamma_1) \lozenge (1-\gamma_1) > 1-\gamma$ and $\gamma_1 * \gamma_1 < \gamma$. For some $r > 0$, let

$$\varsigma \in \bigcup_{\phi_0 \in \Gamma_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})} \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{\phi_0}(r,\gamma_1)}.$$

Then, $\exists \phi_0 \in \Gamma_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})$ such that $\varsigma \in \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{\phi_0}(r,\gamma_1)}$, that is,

$$\Theta(\phi_0 - \varsigma, r) \ge 1-\gamma_1,\ \Omega(\phi_0 - \varsigma, r) \le \gamma_1 \text{ and } \Psi(\phi_0 - \varsigma, r) \le \gamma_1.$$

By $\phi_0 \in \Gamma_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})$, for each $\tau > 0$ and the set

$$H = \{(i,j) : \Theta(\phi_{ij} - \phi_0, \tau) > 1-\gamma_1,$$
$$\Omega(\phi_{ij} - \phi_0, \tau) < \gamma_1 \text{ and } \Psi(\phi_{ij} - \phi_0, \tau) < \gamma_1\}$$

we get

$$\Big\{ (u,v) \in \mathbb{N}^2 : \tfrac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p(u)+1 \le i \le q(u),\right.$$
$$\left. r(v)+1 \le j \le s(v),\ (i,j) \in H\}\right| < \rho \Big\} \in \mathcal{F}(\mathcal{I}_2).$$

Consider $(i,j) \in H$. In a similar vein, we get

$$\Theta(\phi_{ij} - \varsigma, r+\tau) > 1-\gamma,\ \Omega(\phi_{ij} - \varsigma, r+\tau) < \gamma \text{ and } \Psi(\phi_{ij} - \varsigma, r+\tau) < \gamma,$$

as mentioned earlier. Thus,

$$H \subset \{(i,j) : \Theta(\phi_{ij} - \varsigma, r+\tau) > 1-\gamma,$$
$$\Omega(\phi_{ij} - \varsigma, r+\tau) < \gamma \text{ and } \Psi(\phi_{ij} - \varsigma, r+\tau) < \gamma\}$$
$$\Rightarrow \Big\{ (u,v) \in \mathbb{N}^2 : \frac{1}{\psi(u)\vartheta(v)} \left|\{(i,j) : p(u)+1 \le i \le q(u),\right.$$
$$r(v)+1 \le j \le s(v),\ \Theta(\phi_{ij} - \varsigma, r+\tau) > 1-\gamma,\ \Omega(\phi_{ij} - \varsigma, r+\tau) < \gamma$$
$$\left.\text{and } \Psi(\phi_{ij} - \varsigma, r+\tau) < \gamma\}\right| < \rho \Big\} \in \mathcal{F}(\mathcal{I}_2),$$

that is, $\varsigma \in \Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})$. So, we have

$$\bigcup_{\phi_0 \in \Gamma_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})} \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{\phi_0}(r,\gamma_1)} \subset \Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij}). \tag{3.7}$$

Conversely, if $\varsigma \in \Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})$, then let on contrary

$$\varsigma \notin \bigcup_{\phi_0 \in \Gamma_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})} \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{\phi_0}(r,\gamma_1)}.$$

So, for all $\phi_0 \in \Gamma_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})$, we obtain $\varsigma \notin \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{\phi_0}(r,\gamma_1)}$, i.e.,

$$\Theta(s_0-\varsigma,r) < 1-\gamma_1,\ \Omega(s_0-\varsigma,r) > \gamma_1 \text{ and } \Psi(s_0-\varsigma,r) > \gamma_1.$$

Therefore, by Theorem 3.22, we have $\varsigma \notin \Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})$, which goes against what we assumed. Thus

$$\Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij}) \subset \bigcup_{\phi_0 \in \Gamma_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})} \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{\phi_0}(r,\gamma_1)}. \tag{3.8}$$

Combining (3.7) and (3.8), yields the following outcome. □

Theorem 3.24. *Let $(\mathcal{F},\Theta,\Omega,\Psi,\Diamond,*)$ be a $\mathcal{NNS}$. Given a double sequence (ϕ_{ij}) in $\mathcal{F}$, let $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\lim \phi_{ij} = \phi_0$. Then $\Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij}) \subset \mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}^r(\phi_{ij})$ for some $r>0$.*

Proof. Assume $\mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\lim \phi_{ij} = \phi_0$. Thus $\phi_0 \in \Gamma_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij})$. By Theorem 3.23, for some $r>0$ and $\gamma \in (0,1)$,

$$\Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij}) = \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{\phi_0}(r,\gamma)}. \tag{3.9}$$

Also, by Theorem 3.19,

$$\overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{\phi_0}(r,\gamma_1)} \subset \mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}^r(\phi_{ij}). \tag{3.10}$$

Hence by (3.9) and (3.10), we have

$$\Gamma^r_{DS[\psi,\vartheta](\Theta,\Omega,\Psi)}(\mathcal{I})(\phi_{ij}) \subset \mathcal{I}_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\mathrm{LIM}^r(\phi_{ij}).$$

□

Definition 3.25. A double sequence (ϕ_{ij}) in $\mathcal{F}$ is said to be strongly $\mathcal{I}_2$-deferred Cesàro convergent or $\mathcal{I}^r_{D[\psi,\vartheta]}(\Theta,\Omega,\Psi)$ summable to ϕ_0 w.r.t. the $\mathcal{NN}$ (Θ,Ω,Ψ) for some $r \geq 0$, provided that for each $\gamma \in (0,1)$ and $\tau > 0$, there exist $i_0, j_0 \in \mathbb{N}$ such that

$$\left\{(u,v)\in\mathbb{N}^2:\frac{1}{\psi(u)\vartheta(v)}\sum_{\substack{i=p(u)+1\\ j=r(v)+1}}^{q(u),s(v)}\Theta\left(\phi_{ij}-\phi_0,r+\tau\right)\leq 1-\gamma\right.$$

$$\text{or }\frac{1}{\psi(u)\vartheta(v)}\sum_{\substack{i=p(u)+1\\ j=r(v)+1}}^{q(u),s(v)}\Omega\left(\phi_{ij}-\phi_0,r+\tau\right)\geq\gamma,$$

$$\left.\frac{1}{\psi(u)\vartheta(v)}\sum_{\substack{i=p(u)+1\\ j=r(v)+1}}^{q(u),s(v)}\Psi\left(\phi_{ij}-\phi_0,r+\tau\right)\geq\sigma\right\}\in\mathcal{I}_2.$$

Using symbols, we represent the expression as $\phi_{ij}\to\phi_0\big(\mathcal{I}^r_{D[\psi,\vartheta]}(\Theta,\Omega,\Psi)\big)$.

From Definition 3.25 and as a consequence of the previous theorems, we can give the following result.

Theorem 3.26. *Let $(\mathcal{F},\Theta,\Omega,\Psi,\diamondsuit,*)$ be a $\mathcal{NNS}$, and let (ϕ_{ij}) be a double sequence in $\mathcal{F}$.*

(i) *If $(\Theta,\Omega,\Psi)^r-\lim\phi_{ij}=\phi_0$ holds, then $\phi_{ij}\to\phi_0\big(\mathcal{I}^r_{D[\psi,\vartheta]}(\Theta,\Omega,\Psi)\big)$.*

(ii) *$\phi_{ij}\to\phi_0\big(\mathcal{I}^r_{D[\psi,\vartheta]}(\Theta,\Omega,\Psi)\big)$ implies $\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\lim\phi_{ij}=\phi_0$.*

(iii) *$\mathcal{I}^r_{DS[\psi,\vartheta]}(\Theta,\Omega,\Psi)-\lim\phi_{ij}=\phi_0$ implies $\phi_{ij}\to\phi_0\big(\mathcal{I}^r_{D[\psi,\vartheta]}(\Theta,\Omega,\Psi)\big)$ for (ϕ_{ij}) to be bounded.*

4. Conclusion

In the context of a convergent double sequence (ϕ_{ij}), where estimating terms becomes challenging for sufficiently large i,j, an auxiliary sequence (ω_{ij}) is employed to approximate values, introducing errors. To address this challenge, rough convergence has emerged as a solution. Numerous mathematicians actively explore the relationship between statistical convergence and various convergence concepts within neutrosophic normed space. However, the more general idea in this theory remains insufficiently investigated, particularly with consideration given to the Pringsheim limit. This study, extending neutrosophic theory, significantly contributes to the existing literature. It introduces two valuable additions in the realm of neutrosophic theory for double sequences in $\mathcal{NNS}$: (i) a form of rough $\mathcal{I}_2$-deferred statistical convergence; (ii) rough $\mathcal{I}_2$-deferred statistical limit and cluster points. These concepts and findings can serve as theoretical tools for examining optimal approaches within the framework of turnpike theory in a fuzzy environment.

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