Journal of Nonlinear and Convex Analysis
Volume 25, Number 5, 2024, 1081–1090

APPLICATIONS AND RESEARCH OF L-FUZZY MATRIX

LAN WANG* AND RUYUE ZHANG

Abstract. This study aimed to extend the fuzzy matrix and propose the concept of an L-fuzzy matrix by means of a completely distributive lattice. Further, the research method of L-fuzzy sets was extended to L-fuzzy matrices, four kinds of cut matrices of L-fuzzy matrices were introduced, and the concepts of L_α-nested matrix and L_β-nested matrix were proposed using maximum and minimum sets. Then, the decomposition and representation theorems of the L-fuzzy matrix were adopted using the aforementioned concepts. Subsequently, the L-fuzzy matrix was related to a Boolean matrix with the help of decomposition and representation theorems. Hence, the L-fuzzy matrix could be studied using a Boolean matrix. Moreover, the definition of L-fuzzy matrix synthesis was introduced based on fuzzy matrix synthesis, and the equivalent representation of L-fuzzy matrix synthesis was introduced using four kinds of cut matrices. The fuzzy matrix can be used to study fuzzy relation equations. Therefore, the L-fuzzy relation equation was further examined with the help of the related properties of L-fuzzy matrix synthesis, and a method for solving the L-fuzzy relation equation was proposed.

1. Introduction

Fuzzy matrices and fuzzy relations are an important part of fuzzy sets. They not only have important value in fuzzy sets but also play an important role in information processing, fuzzy control system modeling, pattern recognition, fuzzy comprehensive evaluation, and other application fields [1, 4]. When the universal set of the research object is a finite set, the fuzzy matrix can be used to express fuzzy relations, and then fuzzy relations can be studied with the help of the fuzzy matrix. Therefore, the study of fuzzy matrices is of great significance. Xu et al. [21] introduced the concept of the cut matrix of the fuzzy matrix and the multiplication operation of the fuzzy matrix and introduced the decomposition theorem of the fuzzy matrix through the operation and cut matrix. Later, Zheng et al. [25] introduced the concept of a nested matrix and established the representation theorem of the fuzzy matrix. The fuzzy matrix was decomposed into a family of Boolean matrices using the cut matrix in the aforementioned studies [21, 25], and then studied using the Boolean matrix. However, as a lattice structure, $[0, 1]$ was dense with total

2020 *Mathematics Subject Classification.* 08A15, 00A69.

Key words and phrases. L-fuzzy matrix, decomposition theorem, representation theorem, L-fuzzy relation equation.

This study is supported by the Scientific Research Project of Mudanjiang Normal University(MNUGP202303), National Natural Science Foundation of China(12271036;11871097), Heilongjiang Education Department(SJGY20210894), Heilongjiang Provincial University Central support Local University Reform and Development Fund (Excellent Young Talents Project)(No.2020YQ07).

*Corresponding author.

order. The use of fuzzy matrix has limitations when solving related problems with discrete structures. To address the given issues, this study extended the concept of a fuzzy matrix to a complete distributive lattice with a discrete structure and examined the related properties of the fuzzy matrix on the complete distributive lattice, thus providing the theoretical basis for the research and application of the fuzzy matrix. Shi [12] introduced four kinds of cut sets of L-fuzzy sets and introduced the concepts of L_β-nested sets and L_α-nested sets. Based on this, the decomposition and representation theorems of L-fuzzy sets were proposed. Hannusch [7] introduced four logical operations between any two vectors of the same length and concluded that if $\sum = [0, 1]$ or $\{0, 1\}$, it is a BZ distributive lattice. Based on the aforementioned findings, the research methods of L-fuzzy sets introduced by Shi [12] and the logical operation introduced by Hannusch [12] were extended to the L-fuzzy matrix. Also, four kinds of cut matrices of the L-fuzzy matrix and the concepts of the L_β-nested matrix and the L_α-nested matrix were proposed. Then, the decomposition and representation theorems of the L-fuzzy matrix were introduced with the help of the cut matrix of the L-fuzzy matrix.

The relation equation is a branch of mathematics whose research object is relations. It has a wide range of applications in studying digital circuits and treating Boolean variables in relational structures. E. Sanchez [10] first proposed the concept of fuzzy relation equations in 1976 for describing complex systems such as medical diagnosis, and established a necessary and sufficient condition for the nonempty solution set of fuzzy relation equations on a complete Brouwer lattice. It was proved that if a solution exists, a maximum solution must exist. Furthermore, an expression for the maximum solution was provided. Subsequently, many domestic scholars carried out extensive research on fuzzy relation equations. Wang [16] proposed a method for solving fuzzy relation equations defined on $[0, 1]$ when the universe is finite. This method accurately indicated all the minimal elements of the relation equations whose solution set was nonempty. Next, Wang [17] provided conditions under which the composition relation equation on the complete Brouwer lattice had a minimal solution less than or equal to it. He also introduced a formula for calculating the number of minimal elements. Fuzzy relation equations can be studied by means of fuzzy relations, and fuzzy relations can be studied by means of the fuzzy matrix. Therefore, in this study, the concept of the fuzzy matrix was extended to complete distributive lattices, and then the L-fuzzy matrix was further used to examine L-fuzzy relation equations, providing a theoretical basis for subsequent applications. Based on the L-fuzzy set composition operation provided by Hannusch [7], the composition operation of the L-fuzzy matrix was further introduced. The equivalent description of the L-fuzzy matrix composition was given using four kinds of cut matrices of the L-fuzzy matrix, that is, the composition of the L-fuzzy matrix could be decomposed into a family of Boolean matrices. On this basis, a method for solving L-fuzzy relation equations using the equivalent description of L-fuzzy matrix synthesis was proposed, that is, the L-fuzzy relation equation was transformed into a family of relation equations composed of Boolean matrices. The L-fuzzy relation equation was solved by solving the relation equations composed of Boolean matrices. At the same time, the feasibility of the method was proved by two examples on $[0, 1]$ and the complete distributive lattice. This method could solve the L-fuzzy relation equations simply and quickly and also provide a new idea for subsequent research of fuzzy relation equations.

2. Preliminaries

Throughout this study, L is a completely distributive lattice, $\mu_{m\times n}$ for all L-fuzzy matrices, $\omega_{m\times n}$ for all Boolean matrices, $M(\mu_{m\times n})$for all nested matrices, $M_{L_\beta}(\mu_{m\times n})$ for all L_β-nested matrices, and $M_{L_\alpha}(\mu_{m\times n})$ for all L_α-nested matrices.

Let L is completely distributive lattice, α is an $\bigwedge$-$\bigcup$ map, and β is a union-preserving map. ([15]).

In 1995, Shi Fugui designed four cut sets of L-fuzzy sets in [12].

Theorem 2.1 ([12]). *Let M is L-fuzzy set, then we have:*

(1) $M = \bigvee_{a\in L} (a \wedge M_{[a]}) = \bigvee_{a\in L} (a \wedge M_{(a)})$.

(2) $M = \bigwedge_{a\in L} (a \vee M^{[a]}) = \bigwedge_{a\in L} (a \vee M^{(a)})$.

3. Decomposition theorem for L-fuzzy matrices

Definition 3.1. If for any $i = 1, 2, 3, \dots, m; j = 1, 2, 3, \dots, n$, $u_{ij} \in L$, we called matrix $U = (u_{ij})_{mn}$ is L-fuzzy matrix.

Definition 3.2. Let $U = (u_{ij})_{mn}$, $V = (v_{ij})_{mn}$ be L-fuzzy matrix, if for any $i = 1, 2, 3, \dots, m; j = 1, 2, 3, \dots, n$, $u_{ij} = v_{ij}$, then $U = V$.

Definition 3.3. Let $U = (u_{ij})_{mn}$, $V = (v_{ij})_{mn}$ be L-fuzzy matrix, if for any $i = 1, 2, 3, \dots, m; j = 1, 2, 3, \dots, n$, $u_{ij} \leqslant v_{ij}$, then $U \leqslant V$.

Definition 3.4. Let $U = (u_{ij})_{mn}$ be an L-fuzzy matrix and $\forall w \in L$, we define the four kinds of truncated matrices of L-fuzzy matrix :

(1) $U_{[w]} = (u_{ij_{[w]}})_{mn}$, where $u_{ij_{[w]}} = \begin{cases} 1, & u_{ij} \geqslant w \ ; \\ 0, & u_{ij} \ngeqslant w \ . \end{cases}$

(2) $U_{(w)} = (u_{ij_{(w)}})_{mn}$, where $u_{ij_{(w)}} = \begin{cases} 1, & w \in \beta(u_{ij}) \ ; \\ 0, & w \notin \beta(u_{ij}) \ . \end{cases}$

(3) $U^{[w]} = (u_{ij}^{[w]})_{mn}$, where $u_{ij}^{[w]} = \begin{cases} 1, & w \notin \alpha(u_{ij}) \ ; \\ 0, & w \in \alpha(u_{ij}) \ . \end{cases}$

(4) $U^{(w)} = (u_{ij}^{(w)})_{mn}$, where $u_{ij}^{(w)} = \begin{cases} 1, & u_{ij} \nleqslant w \ ; \\ 0, & u_{ij} \leqslant w \ . \end{cases}$

Definition 3.5. Let $U = (u_{ij})_{mn}$ be an L-fuzzy matrix and $\forall w \in L$, we define the following operations :

(1) $w \wedge U_{[w]} = (w \wedge u_{ij_{[w]}})_{mn}$, where $w \wedge u_{ij_{[w]}} = \begin{cases} w, & u_{ij} \geqslant w \ ; \\ 0, & u_{ij} \ngeqslant w \ . \end{cases}$

(2) $w \wedge U_{(w)} = (w \wedge u_{ij_{(w)}})_{mn}$, where $w \wedge u_{ij_{(w)}} = \begin{cases} w, & w \in \beta(u_{ij}) \ ; \\ 0, & w \notin \beta(u_{ij}) \ . \end{cases}$

(3) $w \vee U^{[w]} = (w \vee u_{ij}^{[w]})_{mn}$, where $w \vee u_{ij}^{[w]} = \begin{cases} 1, & w \notin \alpha(u_{ij}) \ ; \\ w, & w \in \alpha(u_{ij}) \ . \end{cases}$

(4) $w \vee U^{(w)} = (u \vee u_{ij}^{(w)})_{mn}$, where $w \vee u_{ij}^{(w)} = \begin{cases} 1, & u_{ij} \nleqslant w \ ; \\ w, & u_{ij} \leqslant w \ . \end{cases}$

Theorem 3.6. *For each $U = (u_{ij})_{mn} \in \mu_{m\times n}$, we have*:

(1) $U = \bigvee_{w\in L} (w \wedge U_{[w]}) = \bigvee_{w\in L} (w \wedge U_{(w)})$.

(2) $U = \bigwedge_{w\in L} (w \vee U^{[w]}) = \bigwedge_{w\in L} (w \vee U^{(w)})$.

Proof. We only give the proof of $U = \bigvee_{w\in L} (w \wedge U_{[w]})$.

Let $\bigvee_{w\in L} (w \wedge U_{[w]}) = V = (v_{ij})_{mn}$, so just prove $v_{ij} = u_{ij}$.

$$\begin{aligned} v_{ij} &= \bigvee_{w\in L} (w \wedge u_{ij_{[w]}}) \\ &= \left(\bigvee_{u_{ij}\geqslant w} (u \wedge u_{ij_{[w]}}) \right) \vee \left(\bigvee_{u_{ij}\ngeqslant w} (w \wedge u_{ij_{[w]}}) \right) \\ &= \left(\bigvee_{u_{ij}\geqslant w} w \right) \vee 0 \\ &= u_{ij}. \end{aligned} \tag{3.1}$$

Because of the arbitrariness of i, j, there is $U = \bigvee_{w\in L} (w \wedge U_{[w]})$. □

Theorem 3.7. *Let $M : L \to \omega_{m\times n}$ be the map, $U = (u_{ij})_{mn}$ be an L-fuzzy matrix, then*

(1) *If $\forall w \in L$, $U_{(w)} \leqslant M(w) \leqslant U_{[w]}$, then $U = \bigvee_{w\in L} (w \wedge M(w))$.*

(2) *If $\forall w \in L$, $U^{(w)} \leqslant M(w) \leqslant U^{[w]}$, then $U = \bigwedge_{w\in L} (w \vee M(w))$.*

(3) *If $\forall w \in L$, $U_{(w)} \leqslant M(w) \leqslant U_{[w]}$, then $U_{[w]} = \bigwedge_{s\in\beta(w)} M(s)$.*

(4) *If $\forall w \in L$, $U_{(w)} \leqslant M(w) \leqslant U_{[w]}$, then $U_{(w)} = \bigvee_{w\in\beta(s)} M(s)$.*

(5) *If $\forall w \in L$, $U^{(w)} \leqslant M(w) \leqslant U^{[w]}$, then $U^{[w]} = \bigwedge_{w\in\alpha(s)} M(s)$.*

(6) *If $\forall w \in L$, $U^{(w)} \leqslant M(w) \leqslant U^{[w]}$, then $U^{(w)} = \bigvee_{s\in\alpha(w)} M(s)$.*

Proof. We only give the proof of (4). The others are analogous.

(4) For all $w \in L$ and $w \in \beta(s)$, by $U_{(s)} \leqslant M(s) \leqslant U_{(w)}$, we can obtain $U_{(w)} \geqslant \bigvee_{w\in\beta(s)} M(s)$. On the other hand, for all $w \in \beta(s)$, we have $M(s) \geqslant U_{(s)}$. Further we obtain $\bigvee_{w\in\beta(s)} M(s) \geqslant \bigvee_{w\in\beta(s)} U_{(s)} = U_{(w)}$. Therefore $U_{(w)} = \bigvee_{w\in\beta(s)} M(s)$. □

4. Representation theorem for L-fuzzy matrices

Definition 4.1. Let $M : L \to \omega_{m\times n}$ be a map.

(1) If $w \in \beta(s) \Rightarrow M(s) \leqslant M(w)$, then M is called $L_\beta-$nested matrices.
(2) If $w \in \alpha(s) \Rightarrow M(w) \leqslant M(s)$, then M is called $L_\alpha-$nested matrices.

Theorem 4.2. *Let $M : L \to \omega_{m\times n}$ be a map, $U = (u_{ij})_{mn}$ be an L-fuzzy matrix,*

(1) *If $\forall w \in L, U_{(w)} \leqslant M(w) \leqslant U_{[w]}$, then M is called $L_\beta-$nested matrices.*
(2) *If $\forall w \in L, U^{(w)} \leqslant M(w) \leqslant U^{[w]}$, then M is called $L_\alpha-$nested matrices.*

Theorem 4.3. *Let $M : L \to \omega_{m\times n}$ be a map,*

(1) *If $w \in \beta(s) \Rightarrow M(s) \leqslant M(w)$, then $M(s)_{[w]} = M(s)$; $M(s)_{(w)} = M(s)$.*
(2) *If $w \in \alpha(s) \Rightarrow M(w) \leqslant M(s)$, then $M(s)^{[w]} = M(s)$; $M(s)^{(w)} = M(s)$.*

Definition 4.4. Let $M_\gamma \in M(\mu_{m\times n}), \gamma \in \Gamma$, for all $w \in L$, specify the operation $\vee, \wedge$ in $M(\mu_{m\times n})$:

(1) $\bigvee\limits_{\gamma\in\mathit{\Gamma}} M_\gamma : \left(\bigvee\limits_{\gamma\in\mathit{\Gamma}} M_\gamma\right)(w) \triangleq \bigvee\limits_{\gamma\in\mathit{\Gamma}} M_\gamma(w).$

(2) $\bigwedge\limits_{\gamma\in\mathit{\Gamma}} M_\gamma : \left(\bigwedge\limits_{\gamma\in\mathit{\Gamma}} M_\gamma\right)(w) \triangleq \bigwedge\limits_{\gamma\in\mathit{\Gamma}} M_\gamma(w).$

Theorem 4.5. *Let $f : M_{L_\beta}(\mu_{m\times n}) \to \mathcal{F}(\mu_{m\times n})$. $\forall M \in M_{L_\beta}(\mu_{m\times n})$, we have $f(M) \triangleq \bigvee\limits_{w\in L} (w \wedge M(w))$ and f meets the following conditions*:

(1) $(f(M))_{(w)} \leqslant M(w) \leqslant (f(M))_{[w]}$.
(2) $(f(M))_{[w]} = \bigwedge\limits_{s\in\beta(w)} M(s)$.
(3) $(f(M))_{(w)} = \bigvee\limits_{w\in\beta(s)} M(s)$.
(4) *f is a homomorphic surjection from $M_{L_\beta}(\mu_{m\times n})$ to $\mathcal{F}(\mu_{m\times n})$.*

Proof. First prove that f is surjective.

$\forall M \in M_{L_\beta}(\mu_{m\times n}), f(M) = \bigvee\limits_{w\in L} (w \wedge M(u)) \in \mathcal{F}(\mu_{m\times n})$ can be uniquely certain, so f is a mapping from $M_{L_\beta}(\mu_{m\times n})$ to $\mathcal{F}(\mu_{m\times n})$.

$\forall U \in \mathcal{F}(\mu_{m\times n})$, let $M(w) = U_{(w)}$, we have $M \in M_{L_\beta}(\mu_{m\times n})$ and f(M)=U. Therefore f is surjective from $M_{L_\beta}(\mu_{m\times n})$ to $\mathcal{F}(\mu_{m\times n})$.

(1) Let $M(w) = (c_{ij}), M(s) = (d_{ij}),\ u_{ij} \in L, f(M) = (u_{ij})$, we have $(f(M))_{[w]} = (u_{ij_{[w]}})$.

$$
\begin{aligned}
(f(M))_{[w]} &= \left(\bigvee_{s \in L} (s \wedge M(s)) \right)_{[w]} \\
&= \left(\bigvee_{s \geqslant w} (s_{[w]} \wedge M(s)_{[w]}) \right) \vee \left(\bigvee_{s \ngeqslant w} (s_{[w]} \wedge M(s)_{[w]}) \right) \\
&= \bigvee_{s \geqslant w} M(s)_{[w]} \\
&\geqslant M(w).
\end{aligned}
\tag{4.1}
$$

The proof is similar to $(f(M))_{[w]} \geqslant M(w)$.

So we can obtain $(f(M))_{(w)} \leqslant M(w) \leqslant (f(M))_{[w]}$.

(2) It follows from 3.7.

(4) In order to prove the f is homomorphic surjection from $M_{L_\beta}(\mu_{m\times n})$ to $\mathcal{F}(\mu_{m\times n})$, just prove that f holds the operation for all $w \in L$.

$$
\begin{aligned}
\left(f\left(\bigwedge_{\gamma \in \Gamma} M_\gamma \right) \right)_{[w]} &= \bigwedge_{s \in \beta(w)} \left(\bigwedge_{\gamma \in \Gamma} M_\gamma \right)(s) \\
&= \bigwedge_{\gamma \in \Gamma} \left(\bigwedge_{s \in \beta(w)} M_\gamma(s) \right) \\
&= \left(\bigwedge_{\gamma \in \Gamma} f(M_\gamma) \right)_{[w]} .
\end{aligned}
\tag{4.2}
$$

So $f\left(\bigwedge_{\gamma \in \Gamma} M_\gamma \right) = \bigwedge_{\gamma \in \Gamma} f(M_\gamma)$. It can also be shown $f\left(\bigvee_{\gamma \in \Gamma} M_\gamma \right) = \bigvee_{\gamma \in \Gamma} f(M_\gamma)$. □

In summary we can be obtained f is homomorphic surjection from $M_{L_\beta}(\mu_{m\times n})$ to $\mathcal{F}(\mu_{m\times n})$.

Theorem 4.6. *Let* $g : M_{L_\alpha}(\mu_{m\times n}) \to \mathcal{F}(\mu_{m\times n})$. $\forall M \in M_{L_\alpha}(\mu_{m\times n})$, *there is* $g(M) \triangleq \bigwedge_{a \in L} (a \vee M(a))$ *and* g *meets the following conditions*:

(1) $(g(M))^{(a)} \leqslant M(a) \leqslant (g(M))^{[a]}$.

(2) $(g(M))^{[a]} = \bigwedge_{b \in \beta(a)} M(b)$.

(3) $(g(M))^{(a)} = \bigvee_{a \in \beta(b)} M(b)$.

(4) g *is a homomorphic surjection from* $M_{L_\alpha}(\mu_{m\times n})$ *to* $\mathcal{F}(\mu_{m\times n})$.

Proof. The proof is similar to Theorem 4.5. □

5. L-fuzzy relation equation

Definition 5.1. Let $U=(u_{ij})_{ms}, V=(v_{ij})_{sn}$ be the L-fuzzy matrix, then $U \circ V = \bigvee_{k=1}^{s}(u_{ik} \wedge v_{kj})$.

Theorem 5.2. *Let $U=(u_{ij})_{ms}$, $V=(v_{ij})_{sn}$ be L-fuzzy matrix. The following conditions are true for all $w \in L$:*

(1) $(U \circ V)_{(w)} \leqslant U_{(w)} \circ V_{(w)} \leqslant U_{[w]} \circ V_{[w]} \leqslant (U \circ V)_{[w]}$.
(2) $(U \circ V)^{(w)} \leqslant U^{(w)} \circ V^{(w)} \leqslant U^{[w]} \circ V^{[w]} \leqslant (U \circ V)^{[w]}$.
(3) $(U \circ V) = \bigvee_{w \in L}\left(w \wedge U_{[w]} \circ V_{[w]}\right) = \bigvee_{w \in L}\left(w \wedge U_{(w)} \circ V_{(w)}\right)$.
(4) $(U \circ V) = \bigwedge_{w \in L}\left(w \vee U^{[w]} \circ V^{[w]}\right) = \bigwedge_{w \in L}\left(w \vee U^{(w)} \circ V^{(w)}\right)$.

Proof. We only give the proof of (1). The others are analogous.

In this order, let $(U \circ V) = C = (c_{ij})$.

(1) First of all, assuming that $c_{ij_{(w)}} = 1$.

$$
\begin{aligned}
c_{ij_{(w)}} = 1 &\Rightarrow w \in \beta(c_{ij}) \\
&\Rightarrow w \in \beta\left(\bigvee_{k=1}^{s}(u_{ik} \wedge v_{kj})\right) \\
&\Rightarrow \exists k, w \in \beta(u_{ik} \wedge v_{kj}) \\
&\Rightarrow \exists k, w \in \beta(u_{ik}) \textit{ and } w \in \beta(v_{kj}) \\
&\Rightarrow \bigvee_{k=1}^{s}(u_{ik_{(w)}} \wedge v_{kj(w)}) = 1.
\end{aligned} \tag{5.1}
$$

So $(U \circ V)_{(w)} \leqslant U_{(w)} \circ V_{(w)}$; $U_{(w)} \circ V_{(w)} \leqslant U_{[w]} \circ V_{[w]}$ is obviously.

Then assuming that $c_{ij_{[w]}} = 0$.

$$
\begin{aligned}
c_{ij_{[w]}} = 0 &\Rightarrow c_{ij} \ngeqslant w \\
&\Rightarrow \bigvee_{k=1}^{s}(u_{ik} \wedge v_{kj}) \ngeqslant w \\
&\Rightarrow \forall k, u_{ik} \wedge v_{kj} \ngeqslant w \\
&\Rightarrow \forall k, u_{ik} \ngeqslant u, v_{kj} \ngeqslant w \\
&\Rightarrow \bigvee_{k=1}^{s}(u_{ik_{[w]}} \wedge v_{kj[w]}) = 0.
\end{aligned} \tag{5.2}
$$

Thus $U_{[w]} \circ V_{[w]} \leqslant (U \circ V)_{[w]}$. (1) is proven. □

The L-fuzzy relation equations discussed in this study are as follows:

Let $U=(u_{ij})_{mn}, X=(x_j)_{n1}$, and $V=(v_i)_{m1}$, then $U \circ X = V$. Where $U=(u_{ij})_{mn}$, $V=(v_i)_{m1}$ is the L-fuzzy matrix, $X=(x_j)_{n1}$is L-fuzzy matrix be determined.

The above theorem shows that the solution of L-fuzzy relation equation $U \circ X = V$ can be transformed into a family of classical relation equation $U_{[w]} \circ X_{[w]} = V_{[w]}$ by the decomposition theorem of matrix.

In order to verify the correctness of the above method, the example in [17] is selected for verification.

Example 5.3. Let $L = [0, 1]$, $U \circ X = V$ is fuzzy relation equation, where $U= \begin{pmatrix} 0.3 & 0.2 & 0 \\ 0.5 & 0 & 0.6 \\ 0.2 & 0.1 & 0.1 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $V = \begin{pmatrix} 0.2 \\ 0.4 \\ 0.2 \end{pmatrix}$.

Proof. When $w = 0$, we know that $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \circ X^{[w]} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. It is easy to verify that $\begin{cases} x_1^{[w]} \vee x_2^{[w]} \vee x_3^{[w]} = 1; \\ x_1^{[w]} \vee x_2^{[w]} \vee x_3^{[w]} = 1; \\ x_1^{[w]} \vee x_2^{[w]} \vee x_3^{[w]} = 1. \end{cases}$

When $w \in (0, 0.1]$, we know that $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \circ X^{[w]} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. We can obtain $\begin{cases} x_1^{[w]} \vee x_2^{[w]} = 1; \\ x_1^{[w]} \vee x_3^{[w]} = 1; \\ x_1^{[w]} \vee x_2^{[w]} \vee x_3^{[w]} = 1. \end{cases}$

When $w \in (0.1, 0.2]$, $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \circ X^{[w]} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. We can obtain $\begin{cases} x_1^{[w]} \vee x_2^{[w]} = 1; \\ x_1^{[w]} \vee x_3^{[w]} = 1; \\ x_1^{[w]} = 1. \end{cases}$

When $w \in (0.2, 0.3]$, $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \circ X^{[w]} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. This shows that $\begin{cases} x_1^{[w]} = 0; \\ x_1^{[w]} \vee x_3^{[w]} = 1. \end{cases}$

When $w \in (0.3, 0.4]$, $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \circ X^{[w]} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. It is easy to verify that $x_1^{[w]} \vee x_3^{[w]} = 1 \Rightarrow x_3^{[w]} = 1$.

When $w \in (0.4, 0.5]$, $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \circ X^{[w]} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. It can be seen that $x_1^{[w]} \vee x_3^{[w]} = 0 \Rightarrow x_1^{[w]} = 0, x_3^{[w]} = 0$.

When $w > 0.5$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \circ X^{[w]} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

According to the decomposition theorem of L-fuzzy matrix, it can be known that the maximum solution of the equation of L-fuzzy relation is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 1 \\ 0.4 \end{pmatrix}$. □

Example 5.4. Let completely distributive lattice $L = \{a, b, c, d\}$, the order defined in L is as follows: $a < b, a < c, a < d, c < d, b < d, b \nleq c, c \nleq b$. $U \circ X = V$ is L-fuzzy relation equation, where $U = \begin{pmatrix} a & d \\ b & c \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, V = \begin{pmatrix} d \\ c \end{pmatrix}$.

Proof. When $w = a$, we know that $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \circ X_{[w]} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We can obtain $\begin{cases} x_{1_{[w]}} \vee x_{2_{[w]}} = 1; \\ x_{1_{[w]}} \vee x_{2_{[w]}} = 1. \end{cases}$

When $w = b$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ X_{[w]} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We can obtain $\begin{cases} x_{2_{[w]}} = 1; \\ x_{1_{[w]}} = 0. \end{cases}$

When $w = c$, $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \circ X_{[w]} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This shows that $\begin{cases} x_{2_{[w]}} = 1; \\ x_{1_{[w]}} = 0. \end{cases}$ or $\begin{cases} x_{2_{[w]}} = 1; \\ x_{1_{[w]}} = 1. \end{cases}$

When $w = d$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \circ X_{[w]} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We know that $x_{2_{[w]}} = 1$.

In conclusion, the maximum solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$. □

References

[1] U. Ahmed, J. C. W. Lin and G. Srivastava, *Deep fuzzy contrast-set deviation point representation and trajectory detection*, IEEE Transactions on Fuzzy Systems **31** (2022), 571–581.

[2] A. Campagner, D. Ciucci and V. Dorigatti, *Uncertainty representation in dynamical systems using rough set theory*, Theoretical Computer Science **908** (2022), 28–42.

[3] T. C. K. Chak, *Study on the tecomposition of fuzzy matrix under* $(max - \cdot)$ *operator*, Journal of Northwest University for Nationalities (Natural Science Edition) **44** (2023), 06–09.

[4] P. Chaudhary, Y. V. Varshney, G. Srivastava and S. Bhatia, *Motor imagery classification using sparse nonnegative matrix factorization and convolutional neural networks*, Neural Computing and Applications **34** (2022), 1–11.

[5] X. M. Deng, *The solution of fuzzy relation equation on complete brouwer lattice and the number problem of transfer relation*, Sichuan Normal University, 2010.

[6] C. T. Fan. *Fuzzy Matrix Theory and Application*, Science Press, Beijing, 2006.

[7] C. Hannusch and T. Mihálydeák, *Algebraic structures gained from rough approximation in incomplete information systems*, Annals of Computer Science and Information Systems **32** (2022), 73–77.

[8] L. T. Kóczy, M. E. Cornejo and J. Medina, *Algebraic structure of fuzzy signatures*, Fuzzy Sets and Systems **418** (2021), 25–50.

[9] C. Z. Luo, *Introduction to Fuzzy Sets*, Beijing Normal University Press, Beijing, 1989.

[10] E. Sanchez, *Resolution of composite fuzzy relation equations*, Information and control **30** (1976), 38–48.

[11] F. Shen, G. X. Qiu and J. S. Wang, *Fuzzy equivalence relation and equivalence of sets*, University Mathematics **22** (2006), 108–110.

[12] F. G. Shi, L_β*-set cover with* L_α*-collection set theory and its application*, Fuzzy Systems and Mathematics **09** (1995), 65–72.

[13] F. G. Shi, *L-fuzzy relations and L-fuzzy subgroups*, Journal of Fuzzy Mathematics **08** (2002), 491–499.
[14] M. Van, *Theory of Convex Structures*, Elsevier Press, North-Holland, New York, 1993.
[15] G. J. Wang, *L-Topological Space Theory*, Shanxi Normal University Press, Shanxi, 1988.
[16] X. P. Wang, *A method of solving fuzzy relation equation on lattice* $[0,1]$, Journal of Higher Applied Aathematics (Chinese edition) **15** (2000), 127–133.
[17] X. P. Wang, *Conditions for minimum solutions of fuzzy relation equations on complete Brouwerian lattices*, Advances in Mathematics **31** (2002), 220–228.
[18] X. P. Wang, *Infnite fuzzy relational equations on a complete brouwerian lattice*. Fuzzy Sets and Systems **49** (2003), 657–666.
[19] X. P. Wang, *Research progress of fuzzy relation equations on complete lattice*, Journal of Sichuan Normal University (Natural Science Edition) **32** (2009), 365–376.
[20] Y. L. Wang and Q. Q. Xiong, *The solution set of fuzzy bilinear relation equation synthesized by* $(max - \cdot)$, Journal of Sichuan Normal University Science. **42** (2019), 176–183.
[21] F. Xu, M. Zhang and J. Y. Peng, *The representation theorem of fuzzy matrix. Journal of Neijiang Normal University* **27** (2012), 04–07.
[22] J. J. Xie, *Fuzzy Mathematical Method and Its Application*, Huazhong University of Science and Technology Press, Wuhan, 2005.
[23] L. A. Zadeh, *Fuzzy sets*, Information and Control **03** (1965), 318–353.
[24] X. D. Zhang and F. G. Shi, *LF set representation theorem and its application.* Natural Sciences, Heilongjiang University **19** (2003), 23–26.
[25] Z. Zheng, S. J. Dai, M. Zhang and J. Y. Peng, *Decomposition theorem of fuzzy matrix. Journal of Neijiang Normal University* **27** (2012), 04–06.

Manuscript received August 29, 2023
revised November 29, 2023

L. Wang
School of Mathematical Science, Mudanjiang Normal University, Heilongjiang 157011, P.R. China
E-mail address: `wanglantongtong@126.com`

R. Zhang
School of Mathematical Science, Mudanjiang Normal University, Heilongjiang 157011, P.R. China
E-mail address: `zhangruyue1998@163.com`