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APPROXIMATES SOLUTIONS IN ROBUST MINIMAX PROGRAMMING PROBLEMS WITH APPLICATIONS

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Abstract. This paper is devoted to the study of approximate optimality conditions and duality in robust minimax optimzaion problem under a suitable constraint qualification. Using some advanced tools of variational analysis and generalized differentiation, we establish necessary conditions for approximate solutions of a robust minimax optimization problem. Sufficient conditions for such solutions to the considered problem are also provided by generalized convex functions. We state a dual problem to the primal one and explore weak, strong and converse duality relations between them. Finally, by using the obtained results, we derive necessary and sufficient conditions for weak approximate Pareto solutions to the robust multi-objective optimization problem.

1. Introduction

Optimization problems, in which both a minimization and maximization process are performed, are known in the area of mathematical programming as minimax problems. In the area of game theory, economics, best approximation theory and great variety of situations involving optimal decision making under uncertainty, problems of this type was treated maily. Minimax programming problems have been the subject of immense interest in the past few years. Recently, many researchers have studied optimality conditions and duality theorems for minimax programming problems. For details, see e.g., [1–9]. But in classical optimization models, the data are usually assumed to be known precisely. However, there are numerous situations where the data are uncertain. Most frequently, it is not easy to find an optimal solution which is satisfied all criteria at once. Hence, another important solution concept, namely efficiency and properly efficiency should be taken into consideration.

In 1973, Soyster [10], who was the first to consider, what now is called Robust Linear Programming. To the best of our knowledge, in two subsequent decades there were only two publications on the subject [11, 12]. The activity in the area was revived circa 1997, independently and essentially simultaneously, in the frameworks of both Integer Programming (Kouvelis and Yu [13]) and Preface xvii Convex Programming (Ben-Tal and Nemirovski [14], El Ghaoui et al. [15, 16]). Since 2000, the robust optimization area is witnessing a burst of research activity in both theory and applications, with numerous researchers involved worldwide.

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An increasingly popular approach to optimization problems with data uncertainty is robust optimization, where it is assumed that possible values of data belong to some well-defined uncertainty set. In robust optimization, the goal is to find a solution that satisfies all constraints for any possible scenario from the uncertainty set and optimizes the worst-case value of the objective function. Theoretical and applied aspects in the area of robust optimization have been studied intensively by many researchers; for details, see e.g., [17–21]. The solutions of robust optimization models are "uniformly good" for realization of data from the uncertainty set.

Our approach in this paper is different from the earlier works. That is, we use some advanced tools of variational analysis and generalized differentiation (e.g., the nonsmooth version of Fermat's rule, and the sum rule for the limiting/Mordukhovich subdifferential) (see, [22]) to establish necessary conditions for approximate solutions of a robust minimax programming problem with inequality constraints. Sufficient conditions for such solutions to the considered problem are also provided by means of the use of generalized convexity (see [23]) defined in terms of the limiting subdifferential for locally Lipschitz functions.

Along with optimality conditions, we propose a dual problem to the primal one and examine weak, strong, and converse-like duality relations under assumptions of generalized convexity.

In addition, we employ the necessary and sufficient optimality conditions obtained for the robust minimax programming problem to derive the corresponding ones for a robust multi-objective optimization problem.

This paper is organized as follows. In Sect. 2, we describe some basic definitions from variational analysis and several auxiliary results. In Sect. 3, we present some results on robust minimax programming problem, to including necessary conditions for approximate solutions, sufficient conditions for the such solutions, and duality relations. In Sect. 4, we show the results on robust multi-objective optimization problem, including necessary and sufficient conditions; duality relations. Finally, we give some conclusions in Sect. 5.

2. Preliminaries

Let us recall some notations and preliminary results which will be used throughout this paper; see e.g., [22, 24]. $\mathbb{R}^n$ denotes the Euclidean space equipped with the standard Euclidean norm $\|\cdot\|$. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$. The inner product (or scalar product) in $\mathbb{R}^n$ is defined by $\langle a, b\rangle := a^T b$ for all $a, b \in \mathbb{R}^n$. The symbol $\mathbb{B}(x, \tau)$ means the open ball centered at $x \in \mathbb{R}^n$ with the radius $\tau > 0$. Let $\Pi \subset \mathbb{R}^n$ be a given set, we denote by $\operatorname{co}\Pi$ the convex hull of Π, and the notation $x \xrightarrow{\Pi} \bar{x}$ stands for $x \to \bar{x}$ with $x \in \Pi$. We also denote by Π° the polar cone of a set $\Pi \subset \mathbb{R}^n$, where

$$\Pi^\circ := \{y \in \mathbb{R}^n \mid \langle y, x\rangle \leq 0 \text{ for all } x \in \Pi\}. \tag{2.1}$$

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multifunction (or set-valued mapping), we consider the multifunction F with values $F(x) \subset \mathbb{R}^m$ in the collection of all the subsets of $\mathbb{R}^m$.

The limiting construction

$$\operatorname*{Limsup}_{x\to\bar{x}} F(x) := \Big\{ y \in \mathbb{R}^m \mid \exists\, x_k \to \bar{x},\, y_k \to y \text{ with } y_k \in F(x_k) \text{ for all } k \in \mathbb{N} := \{1, 2, \ldots\} \Big\}$$

is known as the Painlevé–Kuratowski upper/outer limit of F at $\bar{x}$.

Given $\Pi \subset \mathbb{R}^n$, and $\bar{x} \in \Pi$, define the collection of Fréchet/regular normal cone to Π at $\bar{x}$ by

$$\widehat{N}(\bar{x}; \Pi) = \widehat{N}_\Pi(\bar{x}) := \left\{ v \in \mathbb{R}^n \;\middle|\; \limsup_{x \xrightarrow{\Pi} \bar{x}} \frac{\langle v, x - \bar{x}\rangle}{\|x - \bar{x}\|} \le 0 \right\}.$$

If $\bar{x} \notin \Pi$, we put $\widehat{N}(\bar{x}; \Pi) := \emptyset$.

The Mordukhovich/limiting normal cone $N(\bar{x}; \Pi)$ to Π at $\bar{x} \in \Pi \subset \mathbb{R}^n$ is obtained from regular normal cones by taking the sequential Painlevé–Kurotowski upper limits as

$$N(\bar{x}; \Pi) := \operatorname*{Limsup}_{x \xrightarrow{\Pi} \bar{x}} \widehat{N}(x; \Pi).$$

If $\bar{x} \notin \Pi$, we put $N(\bar{x}; \Pi) := \emptyset$.

For an extended real-valued function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty] = \mathbb{R} \cup \{\infty\}$, its domain and epigraph are defined by

$$\operatorname{dom} \varphi := \big\{ x \in \mathbb{R}^n \mid \varphi(x) < \infty \big\} \text{ and } \operatorname{epi} \varphi := \big\{ (x, \mu) \in \mathbb{R}^n \times \mathbb{R} \mid \mu \ge \varphi(x) \big\},$$

respectively. We say φ is a lower semicontinuous (l.s.c. in short) function if $\liminf_{y\to x} \varphi(y) \ge \varphi(x)$ for all $x \in \mathbb{R}^n$, in addition, φ is a upper semicontinuous (u.s.c. in short) function if $\limsup_{y\to x} \varphi(y) \le \varphi(x)$ for all $x \in \mathbb{R}^n$.

Let $\varphi \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be finite at $\bar{x} \in \operatorname{dom} \varphi$. Then the collection of basic subgradients, or the (basic/Mordukhovich/limiting) subdifferential, of φ at $\bar{x}$ is defined by

$$\partial \varphi(\bar{x}) := \Big\{ v \in \mathbb{R}^n \mid (v, -1) \in N\big((\bar{x}, \varphi(\bar{x})); \operatorname{epi} \varphi\big) \Big\}.$$

Consider the indicator function $\delta(\cdot; \Pi)$ defined by $\delta(x; \Pi) := 0$ for $x \in \Pi$ and by $\delta(x; \Pi) := \infty$ otherwise, we have a relation between the limiting normal cone and the limiting subdifferential of the indicator function as follows (see [22, Proposition 1.19]):

$$N(\bar{x}; \Pi) = \partial \delta(\bar{x}; \Pi) \text{ for all } \bar{x} \in \Pi.$$

We say a function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ is locally Lipschitz at $\bar{x} \in \mathbb{R}^n$ with rank $L > 0$, i.e., there exists $\tau > 0$ such that

$$|\varphi(x_1) - \varphi(x_2)| \le L\|x_1 - x_2\|, \ \forall x_1, x_2 \in \mathbb{B}(\bar{x}, \tau),$$

and it also holds that [22, Theorem 1.22]

$$\|v\| \le L, \quad \forall v \in \partial \varphi(\bar{x}).$$

The generalized Fermat's rule is formulated as follows [22, Proposition 1.30]: Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be finite at $\bar{x}$. If $\bar{x}$ is a local minimizer of φ, then

$$0 \in \partial\varphi(\bar{x}). \tag{2.2}$$

The following lemmas which are related to the Mordukhovich/limiting subdifferential calculus are very useful.

Lemma 2.1 ([22, Corollary 2.21 and Theorem 4.10(ii)]). (i) *Let $\varphi_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, 2, \ldots, m$, $m \geq 2$ be l.s.c. around $\bar{x} \in \mathbb{R}^n$, and let all but one of these functions be Lipschitz continuous around $\bar{x}$. Then*

$$\partial(\varphi_1 + \varphi_2 + \ldots + \varphi_m)(\bar{x}) \subset \partial\varphi_1(\bar{x}) + \partial\varphi_2(\bar{x}) + \ldots + \partial\varphi_m(\bar{x}).$$

(ii) *Let $\varphi_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, 2, \ldots, m$, $m \geq 2$ be l.s.c. around $\bar{x}$ for $i \in I_{\max}(\bar{x})$ and be u.s.c. at $\bar{x}$ for $i \notin I_{\max}(\bar{x})$, suppose that each φ_i, $i = 1, \ldots, m$, is Lipschitz continuous around $\bar{x}$. Then we have the inclusion*

$$\partial(\max \varphi_i)(\bar{x}) \subset \bigcup \left\{ \partial\Big(\sum_{i \in I_{\max}(\bar{x})} \lambda_i \varphi_i \Big)(\bar{x}) \;\middle|\; (\lambda_1, \ldots, \lambda_m) \in \Lambda(\bar{x}) \right\},$$

where the equality holds and the maximum functions are lower regular at $\bar{x}$ if each φ_i is lower regular at this point and sets $I_{\max}(\bar{x})$ and $\Lambda(\bar{x})$ are defined as follows:

$$I_{\max}(\bar{x}) := \big\{ i \in \{1, \ldots, m\} \mid \varphi_i(\bar{x}) = (\max \varphi_i)(\bar{x}) \big\},$$
$$\Lambda(\bar{x}) := \left\{ (\lambda_1, \ldots, \lambda_m) \mid \lambda_i \geq 0, \ \sum_{i=1}^{m} \lambda_i = 1, \ \lambda_i\big(\varphi_i(\bar{x}) - (\max \varphi_i)(\bar{x})\big) = 0 \right\}.$$

Lemma 2.2 (Mean Value Inequality [22, Corollary 4.14(ii)]). *If φ is Lipschitz continuous on an open set containing $[a, b] \subset \mathbb{R}^n$, then*

$$\langle x^*, b - a \rangle \geq \varphi(b) - \varphi(a) \text{ for some } x^* \in \partial\varphi(c) \text{ with } c \in [a, b)$$

where $[a, b] := \text{co}\{a, b\}$, and $[a, b) := \text{co}\{a, b\} \backslash \{b\}$.

For a function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ being locally Lipschitz continuous at $\bar{x}$, the generalized directional derivative (in the sense of Clarke) of φ at $\bar{x}$ in the direction $v \in \mathbb{R}^n$ is defined as follows:

$$\varphi^\circ(\bar{x}; v) := \limsup_{x \to \bar{x}, \lambda \downarrow 0} \frac{\varphi(x + \lambda v) - \varphi(x)}{\lambda}.$$

In this case, the convexified/Clarke subdifferential of φ at $\bar{x}$ is the set

$$\partial^C \varphi(\bar{x}) := \big\{ x^* \in \mathbb{R}^n \mid \langle x^*, v \rangle \leq \varphi^\circ(\bar{x}; v), \ \forall v \in \mathbb{R}^n \big\},$$

which is nonempty, and the Clarke directional derivative is the support function of the Clarke subdifferential, that is,

$$\varphi^\circ(\bar{x}, v) = \max_{x^* \in \partial^C \varphi(\bar{x})} \langle x^*, v \rangle,$$

for each $v \in \mathbb{R}^n$.

It follows from [22] that the relationship between the above subdifferentials of φ at $\bar{x} \in \mathbb{R}^n$ is as follows:

$$\partial\varphi(\bar{x}) \subset \partial^C \varphi(\bar{x}).$$

3. Robust minimax problem

Let us consider the following robust minimax programming problem,

$$\text{(RP)} \qquad \min_{x \in F_r} \max_{k \in K} f_k(x),$$

where F_r is the feasible set of problem (RP), defined by

$$F_r = \{x \in \mathbb{R}^n \mid g_j(x, v_j) \leq 0, \ \forall v_j \in \mathcal{V}_j, \ j \in J\}, \tag{3.1}$$

and the functions $f_k : \mathbb{R}^n \to \mathbb{R}$, $k \in K := \{1, \ldots, l\}$, are locally Lipschitz functions, $g_j : \mathbb{R}^n \times \mathcal{V}_j \to \mathbb{R}$, $j \in J := \{1, \ldots, m\}$ are given functions, $v_j \in \mathcal{V}_j$, $j \in J$ are uncertain parameters, $\mathcal{V}_j \subset \mathbb{R}^q$, $j \in J$ are uncertainty sets.

Definition 3.1. Let $\epsilon \geq 0$ and $\phi(x) := \max_{k \in K} f_k(x)$, $x \in \mathbb{R}^n$. A point $\bar{x} \in F_r$ is called a *local quasi ϵ-solution* of problem (RP) if and only if there is a neighborhood U of $\bar{x}$ such that

$$\phi(\bar{x}) \leq \phi(x) + \epsilon \parallel x - \bar{x} \parallel \quad \forall x \in U \cap F_r.$$

Let us make some assumptions for function g_j, $j \in J$, given in (3.1). We refer the reader to [23] for more details.

(A1) For a fixed $\bar{x} \in \mathbb{R}^n$, there exists $\delta_j^{\bar{x}} > 0$ such that the function $v_j \in \mathcal{V}_j \mapsto g_j(x, v_j) \in \mathbb{R}$ is upper semicontinuous for each $x \in \mathbb{B}(\bar{x}, \delta_j^{\bar{x}})$, and the functions $g_j(\cdot, v_j), v_j \in \mathcal{V}_j$, are Lipschitz of given rank $L_j > 0$ on $\mathbb{B}(\bar{x}, \delta_j^{\bar{x}})$, i.e.,

$$\big| \, g_j(x_1, v_j) - g_j(x_2, v_j) \, \big| \leq L_j \|x_1 - x_2\|, \ \forall x_1, x_2 \in \mathbb{B}(\bar{x}, \delta_j^{\bar{x}}), \ \forall v_j \in \mathcal{V}_j.$$

(A2) The multifunction $(x, v_j) \in \mathbb{B}(\bar{x}, \delta_j^{\bar{x}}) \times \mathcal{V}_j \rightrightarrows \partial_x g_j(x, v_j) \subset \mathbb{R}^n$ is closed at $(\bar{x}, \bar{v}_j)$ for each $\bar{v}_j \in \mathcal{V}_j(\bar{x})$, where the symbol ∂_x stands for the limiting subdifferential operation with respect to x, and the notation $\mathcal{V}_j(\bar{x})$ signifies active indices in $\mathcal{V}_j$ at $\bar{x}$, i.e.,

$$\mathcal{V}_j(\bar{x}) := \big\{ v_j \in \mathcal{V}_j \mid g_j(\bar{x}, v_j) = G_j(\bar{x}) \big\} \tag{3.2}$$

with $G_j(\bar{x}) := \sup_{v_j \in \mathcal{V}_j} g_j(\bar{x}, v_j)$.

The above assumptions have been widely used in nonsmooth analysis and robust multi-objective optimization when dealing with computation of nonsmooth subgradients of a supremum or max function over a compact set.

To obtain the necessary optimality condition of Karush–Kuhn–Tucker type for a local quasi ϵ-solution of problem (RP), we need the constraint qualification that has been introduced, we would recall it as follows.

Definition 3.2 ([23, Definition 3.2])**.** Let $\bar{x} \in F_r$. We say that *constraint qualification* (CQ) is satisfied at $\bar{x}$ if

$$0 \notin \text{co}\big\{ \cup \, \partial_x g_j(\bar{x}, v_j) \mid v_j \in \mathcal{V}_j(\bar{x}), \, j \in J \big\}.$$

Now, we establish Karush–Kuhn–Tucker type of necessary conditions for a local quasi ϵ-solution of problem (RP).

Theorem 3.3. *Let the* (CQ) *be satisfied at* $\bar{x} \in F_r$. *If* $\bar{x}$ *is a local quasi* ϵ*-solution of* (RP), *then there exist* $\alpha_k \geq 0$, $k \in K$ *with* $\sum_{k\in K}\alpha_k = 1$, *and* $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$ *such that*

$$
\begin{aligned}
&0 \in \sum_{k\in K}\alpha_k \partial f_k(\bar{x}) + \sum_{j\in J}\lambda_j \mathrm{co}\left\{ \cup\, \partial_x g_j(\bar{x}, v_j) \mid v_j \in \mathcal{V}_j(\bar{x})\right\} + \epsilon\mathbb{B},\\
&\alpha_k\left(f_k(\bar{x}) - \max_{k\in K} f_k(\bar{x})\right) = 0,\ k \in K,\\
&\lambda_j \sup_{v_j\in\mathcal{V}_j} g_j(\bar{x}, v_j) = 0,\ j \in J.
\end{aligned}
\tag{3.3}
$$

Proof. For $j \in J$, we consider $\delta_j^{\bar{x}}$, L_j, $\mathcal{V}_j(\bar{x})$, and $G_j(\bar{x})$ as in assumptions (A1) and (A2). Along with the proof of [23, Theorem 3.3] again, we can show that

$$\mathrm{co}\left\{ \cup\, \partial_x g_j(\bar{x}, v_j) \mid v_j \in \mathcal{V}_j(\bar{x})\right\}$$

is compact; and

$$\partial G_j(\bar{x}) \subset \mathrm{co}\left\{ \cup\, \partial_x g_j(\bar{x}, v_j) \mid v_j \in \mathcal{V}_j(\bar{x})\right\} \tag{3.4}$$

with the help of Lemma 2.2.

Let $\bar{x}$ be a local quasi ϵ-solution of problem (RP). Then $\bar{x}$ is a local minimizer of the following problem

$$\min_{x\in U\cap F_r} h(x), \tag{3.5}$$

where $h(x) := \phi(x) + \epsilon \parallel x - \bar{x} \parallel$, and $\phi(x) := \max_{k\in K} f_k(x)$. Define a real-valued function ψ by

$$\psi(x) := \max_{j\in J}\left\{h(x) - h(\bar{x}), G_j(x)\right\},\ x \in \mathbb{R}^n.$$

We claim that

$$\psi(\bar{x}) = 0 \leq \psi(x),\ \forall x \in U. \tag{3.6}$$

Indeed, it is easy to see that the equality in (3.6) holds due to $\bar{x} \in F_r$. Let us justify the inequality therein. If $x \in U \cap F_r$, then $\psi(x) \geq 0$. Otherwise, $\psi(x) < 0$ leads to that

$$h(x) - h(\bar{x}) < 0,$$

which is a contradiction to (3.5). If $x \in U \backslash F_r$, then there is $j_0 \in J$ such that $G_{j_0}(x) > 0$, which entails that $\psi(x) > 0$.

Thus, (3.6) is valid, and this infers that $\bar{x}$ is a local minimizer for ψ. Invoking now the nonsmooth version of Fermat's rule (2.2), we have

$$0 \in \partial\psi(\bar{x}).$$

Applying further the formula for the limiting subdifferential of maximum functions and the limiting subdifferential sum rule for local Lipschitz functions (Lemma 2.1), we get

$$0 \in \left\{\mu_0 \partial h(\bar{x}) + \sum_{j\in J}\mu_j \partial G_j(\bar{x}) \mid \mu_0 \geq 0,\ \mu_j \geq 0,\ \mu_j G_j(\bar{x}) = 0,\ j \in J, \mu_0 + \sum_{j\in j}\mu_j = 1\right\}.$$

From (3.4) together with (3.2), we derive

$$
\begin{aligned}
0 \in \Big\{ & \mu_0 \partial h(\bar{x}) + \sum_{j\in J} \mu_j \mathrm{co}\,\big\{ \cup \partial_x g_j(\bar{x}, v_j) \mid v_j \in \mathcal{V}_j(\bar{x})\big\} \\
& \Big| \; \mu_0 \geq 0,\ \mu_j \geq 0,\ \mu_j \sup_{v_j\in\mathcal{V}_j} g_j(\bar{x}, v_j) = 0,\ j \in J,\ \mu_0 + \sum_{j\in J}\mu_j = 1 \Big\}.
\end{aligned}
\tag{3.7}
$$

Note further that $\partial \parallel \cdot - \bar{x} \parallel (\bar{x}) = \mathbb{B}$, we get

$$
\begin{aligned}
\partial h(\bar{x}) =& \partial(\max_{k\in K} f_k + \epsilon \parallel \cdot - \bar{x} \parallel)(\bar{x}) \subset \partial(\max_{k\in K} f_k)(\bar{x}) + \epsilon\mathbb{B} \\
\subset & \Big\{ \sum_{k\in K(\bar{x})} \alpha_k \partial f_k(\bar{x}) + \epsilon\mathbb{B} \;\Big|\; \alpha_k \geq 0,\ \alpha_k\Big(f_k(\bar{x}) - (\max f_k)(\bar{x})\Big) = 0, \\
& \qquad k \in K(\bar{x}),\ \sum_{k\in K(\bar{x})} \alpha_k = 1 \Big\},
\end{aligned}
\tag{3.8}
$$

where $K(\bar{x}) := \{k \in K \mid f_k(\bar{x}) = \phi(\bar{x})\} \neq \emptyset$. Now, letting $\alpha_k := 0$ for $k \in K \backslash K(\bar{x})$. If the (CQ) be satisfied at $\bar{x}$, then $\mu_0 \neq 0$, set $\lambda_j = \frac{\mu_j}{\mu_0}$, $j \in J$, and (3.7) with (3.8) establishes (3.3), which completes the proof. ☐

Definition 3.4. We say that (f, g) is *generalized convex* at $\bar{x} \in \mathbb{R}^n$ if for any $x \in \mathbb{R}^n$, $\xi_k \in \partial f_k(\bar{x})$, $k \in K$, and $\eta_j \in \partial_x g_j(\bar{x}, v_j)$, $v_j \in \mathcal{V}_j(\bar{x})$, $j \in J$, there exists $h \in \mathbb{R}^n$ such that

$$
\begin{aligned}
f_k(x) - f_k(\bar{x}) &\geq \langle \xi_k, h\rangle, \quad k \in K, \\
g_j(x, v_j) - g_j(\bar{x}, v_j) &\geq \langle \eta_j, h\rangle, \quad v_j \in \mathcal{V}_j(\bar{x}),\ j \in J,
\end{aligned}
$$

and

$$
\langle \vartheta, h\rangle \leq \parallel x - \bar{x} \parallel, \quad \vartheta \in \mathbb{B},
$$

where $\mathcal{V}_j(\bar{x})$, $j \in J$, are defined as in (3.2).

With the help of the generalized convexity, we can establish the following theorem.

Theorem 3.5. *Let $\bar{x} \in F_r$ satisfy the condition* (3.3). *If (f, g) is generalized convex at $\bar{x}$, then $\bar{x}$ is a global quasi ϵ-solution of problem* (RP).

Proof. Let $\phi(x) := \max_{k\in K} f_k(x)$. Since $\bar{x} \in F_r$ satisfies condition (3.3), there exist $\alpha_k \geq 0$, $\sum_{k\in K}\alpha_k = 1$, $\xi_k \in \partial f_k(\bar{x})$, $k \in K$, and $\lambda_j \geq 0$, $j \in J$, $\lambda_{ji} \geq 0$, $\eta_{ji} \in \partial_x g_j(\bar{x}, v_{ji})$, $v_{ji} \in \mathcal{V}_j(\bar{x})$, $i \in I_j = \{1, \ldots, i_j\}$, $i_j \in \mathbb{N}$, $\sum_{i\in I_j}\lambda_{ji} = 1$, and $\vartheta \in \mathbb{B}$ such that

$$
0 = \sum_{k\in K} \alpha_k \xi_k + \sum_{j\in J} \lambda_j \big(\sum_{i\in I_j} \lambda_{ji}\eta_{ji}\big) + \epsilon\vartheta, \tag{3.9}
$$

$$
\alpha_k \Big(f_k(\bar{x}) - \max_{k\in K} f_k(\bar{x})\Big) = 0,\ k \in K, \tag{3.10}
$$

$$
\lambda_j \sup_{v_j\in\mathcal{V}_j} g_j(\bar{x}, v_j) = 0,\ j \in J. \tag{3.11}
$$

Assume to the contrary that $\bar{x}$ is not a global quasi ϵ-solution of problem (RP), then there is $\hat{x} \in F_r$ such that

$$\phi(\bar{x}) > \phi(\hat{x}) + \epsilon \parallel \hat{x} - \bar{x} \parallel . \tag{3.12}$$

By the generalized convexity of (f, g) at $\bar{x}$, we deduce from (3.9) that for such $\hat{x}$ there is $h \in \mathbb{R}^n$ such that

$$\begin{aligned} 0 &= \sum_{k\in K}\alpha_k\langle\xi_k, h\rangle + \sum_{j\in J}\lambda_j\left(\sum_{i\in I_j}\lambda_{ji}\langle\eta_{ji}, h\rangle\right) + \epsilon\langle\vartheta, h\rangle \\ &\leq \sum_{k\in K}\alpha_k\Big[f_k(\hat{x}) - f_k(\bar{x})\Big] + \sum_{j\in J}\lambda_j\left(\sum_{i\in I_j}\lambda_{ji}\big[g_j(\hat{x}, v_{ji}) - g_j(\bar{x}, v_{ji})\big]\right) + \epsilon \parallel \hat{x} - \bar{x} \parallel . \end{aligned}$$

Hence

$$\sum_{k\in K}\alpha_k f_k(\bar{x}) + \sum_{j\in J}\lambda_j\left(\sum_{i\in I_j}\lambda_{ji}g_j(\bar{x}, v_{ji})\right) \tag{3.13}$$

$$\leq \sum_{k\in K}\alpha_k f_k(\hat{x}) + \sum_{j\in J}\lambda_j\left(\sum_{i\in I_j}\lambda_{ji}g_j(\hat{x}, v_{ji})\right) + \epsilon \parallel \hat{x} - \bar{x} \parallel . \tag{3.14}$$

Since $v_{ji} \in \mathcal{V}_j(\bar{x})$,

$$g_j(\bar{x}, v_{ji}) = \sup_{v_j\in\mathcal{V}_j} g_j(\bar{x}, v_j), \ \forall j \in J, \forall i \in I_j.$$

Thus, it follows from (3.11) that $\lambda_j g_j(\bar{x}, v_{ji}) = 0$ for $j \in J$ and $i \in I_j$. In addition, due to $\hat{x} \in F_r$, $\lambda_j g_j(\hat{x}, v_{ji}) \leq 0$ for $j \in J$ and $i \in I_j$. So, we get by (3.13), (3.14) that

$$\begin{aligned} \sum_{k\in K}\alpha_k f_k(\bar{x}) &= \sum_{k\in K}\alpha_k f_k(\bar{x}) + \sum_{j\in J}\left(\sum_{i\in I_j}\lambda_{ji}\lambda_j g_j(\bar{x}, v_{ji})\right) \\ &\leq \sum_{k\in K}\alpha_k f_k(\hat{x}) + \sum_{j\in J}\lambda_j\left(\sum_{i\in I_j}\lambda_{ji}g_j(\hat{x}, v_{ji})\right) + \epsilon \parallel \hat{x} - \bar{x} \parallel \\ &\leq \sum_{k\in K}\alpha_k f_k(\hat{x}) + \epsilon \parallel \hat{x} - \bar{x} \parallel . \end{aligned} \tag{3.15}$$

From (3.10), we derive

$$\alpha_k f_k(\bar{x}) = \alpha_k(\max_{k\in K} f_k)(\bar{x}) = \alpha_k\phi(\bar{x}). \tag{3.16}$$

Hence, combine (3.15) and (3.16), we get

$$\begin{aligned} \sum_{k\in K}\alpha_k\phi(\bar{x}) = \sum_{k\in K}\alpha_k f_k(\bar{x}) &\leq \sum_{k\in K}\alpha_k f_k(\hat{x}) + \epsilon \parallel \hat{x} - \bar{x} \parallel \\ &\leq \sum_{k\in K}\alpha_k\phi(\hat{x}) + \epsilon \parallel \hat{x} - \bar{x} \parallel . \end{aligned}$$

This implies that

$$\phi(\bar{x}) \leq \phi(\hat{x}) + \epsilon \parallel \hat{x} - \bar{x} \parallel, \tag{3.17}$$

due to $\sum_{k\in K}\alpha_k = 1$. Obviously, (3.17) contradicts (3.12), and so the proof is complete. □

Now, we formulate a dual problem for the robust minimax programming problems, and explore weak and strong duality relations between them. Let $z \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{R}^l_+$ with $\sum_{k\in K} \alpha_k = 1$,

$$\lambda \in \mathbb{R}^{\mathbb{N}}_+ := \Big\{ \lambda := (\lambda_j, \lambda_{ji}),\, j \in J,\, i \in I_j = \{1, \ldots, i_j\} \;\big|\; i_j \in \mathbb{N}, \\ \lambda_j \geq 0,\, \lambda_{ji} \geq 0,\, \sum_{i\in I_j} \lambda_{ji} = 1 \Big\}.$$

In connection with the robust minimax programming problem (RP), we consider a dual problem in the following form:

$$\text{(RD)} \qquad \max_{(z,\alpha,\lambda)\in F_D} \Big\{ \tilde{\phi}(z, \alpha, \lambda) := \phi(z) + \sum_{j\in J} \lambda_j \Big(\sum_{i\in I_j} \lambda_{ji} g_j(z, v_{ji}) \Big) \Big\}.$$

Here, we denote $\phi(z) := \max_{k\in K} f_k(z)$ and

$$F_D := \Big\{ (z, \alpha, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l_+ \backslash \{0\} \times \mathbb{R}^{\mathbb{N}}_+ \mid 0 \in \sum_{k\in K} \alpha_k \partial f_k(z) + \\ \sum_{j\in J} \lambda_j \big(\sum_{i\in I_j} \lambda_{ji} \eta_{ji} \big) + \epsilon\mathbb{B},\, \eta_{ji} \in \{ \cup \partial_x g_j(z, v_{ji}) \mid v_{ji} \in \mathcal{V}_j(z) \}, \\ \alpha_k \Big(f_k(z) - \phi(z) \Big) = 0,\ k \in K,\ \sum_{k\in K} \alpha_k = 1 \Big\},$$

where $\mathcal{V}_j(z)$ is defined as in (3.2) by replacing $\bar{x}$ by z.

Definition 3.6. A point $(\bar{z}, \bar{\alpha}, \bar{\lambda}) \in F_D$ is called a *global quasi ϵ-solution* of problem (RD) if and only if

$$\tilde{\phi}(\bar{z}, \bar{\alpha}, \bar{\lambda}) + \epsilon \parallel (\bar{z}, \bar{\alpha}, \bar{\lambda}) - (z, \alpha, \lambda) \parallel \geq \tilde{\phi}(z, \alpha, \lambda),\ \forall (z, \alpha, \lambda) \in F_D.$$

The following theorem gives weak robust duality relations between (RP) and (RD).

Theorem 3.7. *Let $x \in F_r$, and let $(z, \alpha, \lambda) \in F_D$. If (f, g) is generalized convex at z, then*

$$\phi(x) + \epsilon \parallel x - z \parallel \geq \tilde{\phi}(z, \alpha, \lambda).$$

Proof. Since $(z, \alpha, \lambda) \in F_D$, there exist $\alpha_k \geq 0$, $k \in K$ with $\sum_{k\in K} \alpha_k = 1$, $\xi_k \in \partial f_k(z)$, $k \in K$, and $\lambda_j \geq 0$, $j \in J$, $\lambda_{ji} \geq 0$, $\eta_{ji} \in \partial_x g_j(z, v_{ji})$, $v_{ji} \in \mathcal{V}_j(z)$, $i \in I_j = \{1, \ldots, i_j\}$, $i_j \in \mathbb{N}$, $\sum_{i\in I_j} \lambda_{ji} = 1$, and $\vartheta \in \mathbb{B}$ such that

$$0 = \sum_{k\in K} \alpha_k \xi_k + \sum_{j\in J} \lambda_j \big(\sum_{i\in I_j} \lambda_{ji} \eta_{ji} \big) + \epsilon\vartheta, \tag{3.18}$$

$$\alpha_k \Big(f_k(z) - \phi(z) \Big) = 0,\ k \in K. \tag{3.19}$$

Assume to the contrary that

$$\phi(x) + \epsilon \parallel x - z \parallel < \tilde{\phi}(z, \alpha, \lambda),$$

which is equivalent to

$$\phi(x)+\epsilon \parallel x-z \parallel < \phi(z)+\sum_{j\in J}\lambda_j\Big(\sum_{i\in I_j}\lambda_{ji}g_j(z,v_{ji})\Big). \tag{3.20}$$

By the generalized convex property of (f,g) at z, we deduce from (3.18) that for such x there is $h\in\mathbb{R}^n$ such that

$$\begin{aligned}0&=\sum_{k\in K}\alpha_k\langle\xi_k,h\rangle+\sum_{j\in J}\lambda_j\Big(\sum_{i\in I_j}\lambda_{ji}\langle\eta_{ji},h\rangle\Big)+\epsilon\langle\vartheta,h\rangle,\\ &\le\sum_{k\in K}\alpha_k\Big[f_k(x)-f_k(z)\Big]+\sum_{j\in J}\lambda_j\Big(\sum_{i\in I_j}\lambda_{ji}[g_j(x,v_{ji})-g_j(z,v_{ji})]\Big)+\epsilon \parallel x-z \parallel .\end{aligned} \tag{3.21}$$

It stems from $x\in F_r$ that

$$\sum_{j\in J}\lambda_j\Big(\sum_{i\in I_j}\lambda_{ji}g_j(x,v_{ji})\Big)\le 0.$$

Thus, (3.21) gives us

$$0\le\sum_{k\in K}\alpha_k\Big[f_k(x)-f_k(z)\Big]-\sum_{j\in J}\lambda_j\Big(\sum_{i\in I_j}\lambda_{ji}g_j(z,v_{ji})\Big)+\epsilon \parallel x-z \parallel . \tag{3.22}$$

Combining now (3.19) with (3.22), we have

$$0\le\sum_{k\in K}\alpha_kf_k(x)-\sum_{k\in K}\alpha_k\phi(z)-\sum_{j\in J}\lambda_j\Big(\sum_{i\in I_j}\lambda_{ji}g_j(z,v_{ji})\Big)+\epsilon \parallel x-z \parallel .$$

This gives us

$$\begin{aligned}\phi(z)+\sum_{j\in J}\lambda_j\Big(\sum_{i\in I_j}\lambda_{ji}g_j(z,v_{ji})\Big)&=\sum_{k\in K}\alpha_k\phi(z)+\sum_{j\in J}\lambda_j\Big(\sum_{i\in I_j}\lambda_{ji}g_j(z,v_{ji})\Big)\\ &\le\sum_{k\in K}\alpha_kf_k(x)+\epsilon \parallel x-z \parallel\\ &\le\sum_{k\in K}\alpha_k\max_{k\in K}f_k(x)+\epsilon \parallel x-z \parallel\\ &=\phi(x)+\epsilon \parallel x-z \parallel,\end{aligned} \tag{3.23}$$

where the equality in (3.23) holds due to $\sum_{k\in K}\alpha_k=1$, which contradicts to (3.20). The proof of the theorem is complete. □

The forthcoming theorem describes strong robust duality relations between (RP) and (RD).

Theorem 3.8. *Let $\bar{x}\in F_r$ be a local quasi ϵ-solution of problem* (RP) *such that the* (CQ) *is satisfied at this point. Then there exists $(\bar{\alpha},\bar{\lambda})\in(\mathbb{R}^l_+\backslash\{0\})\times\mathbb{R}^{\mathbb{N}}_+$ such that $(\bar{x},\bar{\alpha},\bar{\lambda})\in F_D$ and*

$$\phi(\bar{x})=\tilde{\phi}(\bar{x},\bar{\alpha},\bar{\lambda}).$$

Furthermore, if (f, g) *is generalized convex at any* $z \in \mathbb{R}^n$*, then* $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ *is a global quasi* ϵ*-solution of problem* (RD).

Proof. Applying Theorem 3.3, we find $\alpha_k \geq 0$, $k \in K$ with $\sum_{k \in K} \alpha_k = 1$, $\xi_k \in \partial f_k(\bar{x})$, $k \in K$, and $\lambda_j \geq 0$, $j \in J$, $\lambda_{ji} \geq 0$, $\eta_{ji} \in \partial_x g_j(\bar{x}, v_{ji})$, $v_{ji} \in \mathcal{V}_j(\bar{x})$, $i \in I_j = \{1, \ldots, i_j\}$, $i_j \in \mathbb{N}$, $\sum_{i \in I_j} \lambda_{ji} = 1$, and $\vartheta \in \mathbb{B}$ such that

$$0 = \sum_{k \in K} \alpha_k \xi_k + \sum_{j \in J} \lambda_j \big(\sum_{i \in I_j} \lambda_{ji} \eta_{ji} \big) + \epsilon \vartheta, \tag{3.24}$$

$$\alpha_k \left(f_k(\bar{x}) - \max_{k \in K} f_k(\bar{x}) \right) = 0, \ k \in K,$$

$$\lambda_j \sup_{v_j \in \mathcal{V}_j} g_j(\bar{x}, v_j) = 0, \ j \in J. \tag{3.25}$$

Letting $\bar{\alpha} := (\alpha_1, \ldots, \alpha_l)$, and $\bar{\lambda} := (\lambda_j, \lambda_{ji})$, we have $(\bar{\alpha}, \bar{\lambda}) \in (\mathbb{R}^l_+ \backslash \{0\}) \times \mathbb{R}^{\mathbb{N}}_+$, and so $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_D$ due to (3.24). Since $v_{ji} \in \mathcal{V}_j(\bar{x})$,

$$g_j(\bar{x}, v_{ji}) = \sup_{v_j \in \mathcal{V}_j} g_j(\bar{x}, v_j)$$

for $j \in J$, and $i \in I_j = 1, \ldots, i_j$. Thus, it stems from (3.25) that $\lambda_j g_j(\bar{x}, v_{ji}) = 0$ for $j \in J$, and $k \in I_j$. This entails that

$$\sum_{j \in J} \lambda_j \left(\sum_{i \in I_j} \lambda_{ji} g_j(\bar{x}, v_{ji}) \right) = \sum_{j \in J} \left(\sum_{i \in I_j} \lambda_{ji} \lambda_j g_j(\bar{x}, v_{ji}) \right) = 0,$$

and therefore,

$$\phi(\bar{x}) = \phi(\bar{x}) + \sum_{j \in J} \lambda_j \left(\sum_{i \in I_j} \lambda_{ji} g_j(\bar{x}, v_{ji}) \right) = \tilde{\phi}(\bar{x}, \bar{\alpha}, \bar{\lambda}).$$

Since (f, g) is generalized convex at any $z \in \mathbb{R}^n$, by invoking Theorem 3.7, we obtain

$$\begin{aligned} \tilde{\phi}(\bar{x}, \bar{\alpha}, \bar{\lambda}) + \epsilon \parallel (\bar{x}, \bar{\alpha}, \bar{\lambda}) - (z, \alpha, \lambda) \parallel &= \phi(\bar{x}) + \epsilon \parallel (\bar{x}, \bar{\alpha}, \bar{\lambda}) - (z, \alpha, \lambda) \parallel \\ &\geq \tilde{\phi}(z, \alpha, \lambda) \end{aligned}$$

for any $(z, \alpha, \lambda) \in F_D$. This means that $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is a global quasi ϵ-solution of problem (RD). $\square$

The forthcoming theorem declares converse-like robust duality relations between (RP) and (RD).

Theorem 3.9. *Let* $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_D$ *be such that* $\phi(\bar{x}) = \tilde{\phi}(\bar{x}, \bar{\alpha}, \bar{\lambda})$*. If* $\bar{x} \in F_r$ *and* (f, g) *is generalized convex at* $\bar{x}$*, then* $\bar{x}$ *is a global quasi* ϵ*-solution of problem* (RP).

Proof. Since $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_D$, there exist $\bar{\alpha} := (\alpha_1, \ldots, \alpha_l) \in \mathbb{R}^l_+ \backslash \{0\}$, $\xi_k \in \partial f_k(z)$, $k \in K$, $\bar{\lambda} := (\lambda_j, \lambda_{ji})$, $\lambda_j \geq 0$, $j \in J$, $\lambda_{ji} \geq 0$, $\eta_{ji} \in \partial_x g_j(\bar{x}, v_{ji})$, $v_{ji} \in \mathcal{V}_j(\bar{x})$,

$i \in I_j = \{1, \dots, i_j\}$, $i_j \in \mathbb{N}$, $\sum_{i \in I_j} \lambda_{ji} = 1$, and $\vartheta \in \mathbb{B}$ such that

$$0 = \sum_{k \in K} \alpha_k \xi_k + \sum_{j \in J} \lambda_j \big(\sum_{i \in I_j} \lambda_{ji} \eta_{ji} \big) + \epsilon \vartheta, \tag{3.26}$$

$$\alpha_k \bigg(f_k(\bar{x}) - \phi(\bar{x}) \bigg) = 0, \ k \in K. \tag{3.27}$$

As $\phi(\bar{x}) = \tilde{\phi}(\bar{x}, \bar{\alpha}, \bar{\lambda})$, we have

$$\phi(\bar{x}) = \phi(\bar{x}) + \sum_{j \in J} \lambda_j \bigg(\sum_{i \in I_j} \lambda_{ji} g_j(\bar{x}, v_{ji}) \bigg) = \tilde{\phi}(\bar{x}, \bar{\alpha}, \bar{\lambda}),$$

i.e.,

$$\sum_{j \in J} \lambda_j \bigg(\sum_{i \in I_j} \lambda_{ji} g_j(\bar{x}, v_{ji}) \bigg) = 0. \tag{3.28}$$

Since $v_{ji} \in \mathcal{V}_j(\bar{x})$,

$$g_j(\bar{x}, v_{ji}) = \sup_{v_j \in \mathcal{V}_j} g_j(\bar{x}, v_j) \tag{3.29}$$

for $j \in J$, and $i \in I_j = 1, \dots, i_j$. Combining now (3.28), (3.29) with $\sum_{i \in I_j} \lambda_{ji} = 1$, we derive

$$\begin{aligned} \sum_{j \in J} \lambda_j \bigg(\sum_{i \in I_j} \lambda_{ji} g_j(\bar{x}, v_{ji}) \bigg) &= \sum_{j \in J} \bigg(\sum_{i \in I_j} \lambda_{ji} \lambda_j g_j(\bar{x}, v_{ji}) \bigg) \\ &= \sum_{j \in J} \bigg(\lambda_j \sup_{v_j \in \mathcal{V}_j} g_j(\bar{x}, v_j) \bigg) = 0. \end{aligned} \tag{3.30}$$

Let $\bar{x} \in F_r$. Then $g_j(\bar{x}, v_j) \le 0$, for all $v_j \in \mathcal{V}_j$, $j = 1, \dots, m$. Thus, deducing from (3.30), we obtain

$$\lambda_j \sup_{v_j \in \mathcal{V}_j} g_j(\bar{x}, v_j) = 0, \ j \in J.$$

This together with (3.26) and (3.27) confirms that $\bar{x}$ satisfies condition (3.3). To finish the proof, it remains to apply Theorem 3.5. □

4. Robust multi-objective problem

This section is devoted to optimality conditions of robust minimax programming problem to robust multi-objective optimization problems. More precisely, we employ necessary and sufficient conditions obtained for (RP) in the previous sections to derive the corresponding ones for (RMOP).

A multi-objective optimization problem with locally Lipschitzian data in the face of data uncertainty in the constraints is of the form

$$\text{(UMOP)} \qquad \text{Min}_{\mathbb{R}^l_+} \{ f(x) \mid g_j(x, v_j) \le 0, \ j \in J \},$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, $f(x) := (f_1(x), \dots, f_l(x))$, $f_k : \mathbb{R}^n \to \mathbb{R}$, $k \in K := \{1, \dots, l\}$, are locally Lipschitz functions, $g_j : \mathbb{R}^n \times \mathcal{V}_j \to \mathbb{R}$,

$j \in J := \{1, \dots, m\}$ are given functions, $v_j \in \mathcal{V}_j$, $j \in J$ are uncertain parameters, $\mathcal{V}_j \subset \mathbb{R}^q$, $j \in J$ are uncertainty sets.

We will treat problem (UMOP) by robust optimization approach, its robust counterpart is as follows,

$$\text{(RMOP)} \qquad \mathrm{Min}_{\mathbb{R}^l_+} \{f(x) \mid x \in F_r\},$$

where F_r is the feasible set of problem (RMOP), defined by

$$F_r = \{x \in \mathbb{R}^n \mid g_j(x, v_j) \leq 0, \ \forall v_j \in \mathcal{V}_j, \ j \in J\}.$$

Definition 4.1. Let $\varepsilon := (\epsilon_1, \dots, \epsilon_l) \in \mathbb{R}^l_+$, we say $\bar{x} \in F_r$ is a *weakly quasi ε-Pareto solution* of problem (RMOP) if and only if

$$f(x) + \varepsilon \parallel x - \bar{x} \parallel - f(\bar{x}) \notin -\mathrm{int}\, \mathbb{R}^l_+, \ \forall x \in F_r.$$

The following result is a Karush–Kuhn–Tucker (KKT) necessary condition for weakly quasi ε-Pareto solutions of problem (RMOP).

Theorem 4.2. *Let $\varepsilon := (\epsilon_1, \dots, \epsilon_l) \in \mathbb{R}^l_+$, and the* (CQ) *be satisfied at $\bar{x} \in \mathbb{R}^n$. If $\bar{x}$ is a weakly quasi ε-Pareto solution of problem* (RMOP), *then there exist $\alpha_k \geq 0$, $k \in K$ with $\sum_{k \in K} \alpha_k = 1$, $\lambda_j \geq 0$, $j \in J$, such that*

$$\text{(4.1)} \qquad \begin{aligned} &0 \in \sum_{k \in K} \alpha_k \partial f_k(\bar{x}) + \sum_{j \in J} \lambda_j \mathrm{co} \big\{ \cup \partial_x g_j(\bar{x}, v_j) \mid v_j \in \mathcal{V}_j(\bar{x}) \big\} + \epsilon_{\bar{k}} \mathbb{B}, \\ &\lambda_j \sup_{v_j \in \mathcal{V}_j} g_j(\bar{x}, v_j) = 0, \ j \in J, \end{aligned}$$

where $\epsilon_{\bar{k}} = \max_{k \in K} \{\epsilon_k\}$.

Proof. Let $\bar{x}$ be a weakly quasi ε-Pareto solution of problem (RMOP) and let

$$\widehat{f}_k(x) := f_k(x) - f_k(\bar{x}), \quad k \in K, \ x \in \mathbb{R}^n.$$

We will show that $\bar{x}$ is a quasi $\epsilon_{\bar{k}}$-solution of the minimax programming problem

$$\text{(4.2)} \qquad \min_{x \in F_r} \max_{k \in K} \widehat{f}_k(x).$$

To do this, let us put $\widehat{\phi}(x) := \max_{k \in K} \widehat{f}_k(x)$ and prove that

$$\text{(4.3)} \qquad \widehat{\phi}(\bar{x}) \leq \widehat{\phi}(x) + \epsilon_{\bar{k}} \parallel x - \bar{x} \parallel, \ \forall x \in F_r.$$

Indeed, if (4.3) is not valid, then there exists $x_0 \in F_r$ such that

$$\widehat{\phi}(x_0) + \epsilon_{\bar{k}} \parallel x_0 - \bar{x} \parallel < \widehat{\phi}(\bar{x}).$$

Since $\widehat{\phi}(\bar{x}) = 0$, it holds that

$$\max_{k \in K} \big\{ f_k(x_0) - f_k(\bar{x}) \big\} + \epsilon_{\bar{k}} \parallel x_0 - \bar{x} \parallel < 0.$$

Thus,

$$f(x_0) - f(\bar{x}) + \varepsilon \parallel x_0 - \bar{x} \parallel \in -\mathrm{int}\, \mathbb{R}^l_+,$$

which contradicts the fact that $\bar{x}$ is a weakly quasi ε-Pareto solution of problem (RMOP). So, we can apply (KKT) condition obtained in Theorem 3.3 to problem (4.2). Then we find $\alpha_k \geq 0$, $k \in K$ with $\sum_{k\in K}\alpha_k = 1$, $\lambda_j \geq 0$, $j \in J$, such that

$$\begin{aligned}
&0 \in \sum_{k\in K}\alpha_k \partial \widehat{f}_k(\bar{x}) + \sum_{j\in J}\lambda_j \mathrm{co}\left\{ \cup\, \partial_x g_j(\bar{x}, v_j) \mid v_j \in \mathcal{V}_j(\bar{x})\right\} + \epsilon_{\bar{k}}\mathbb{B},\\
&\alpha_k\big(\widehat{f}_k(\bar{x}) - \max_{k\in K}\widehat{f}_k(\bar{x})\big) = 0,\ k \in K,\\
&\lambda_j \sup_{v_j\in\mathcal{V}_j} g_j(\bar{x}, v_j) = 0,\ j \in J.
\end{aligned} \tag{4.4}$$

It is now clear that (4.4) implies (4.1) and thus, the proof is complete. □

The following theorem describes sufficient optimality conditions for a weakly quasi ε-Pareto solutions of problem (RMOP).

Theorem 4.3. *Let $\varepsilon := (\epsilon_{\bar{k}}, \ldots, \epsilon_{\bar{k}}) \in \mathbb{R}^l_+$, and $\bar{x} \in F_r$ satisfy condition* (4.1). *If (f, g) is generalized convex at $\bar{x}$, then $\bar{x}$ is a weakly quasi ε-Pareto solution of problem* (RMOP).

Proof. Similar to the proof of Theorem 4.2, we put

$$\widehat{f}_k(x) := f_k(x) - f_k(\bar{x}), \quad k \in K,\ x \in \mathbb{R}^n.$$

Now, it is easy to see that $\bar{x}$ satisfies condition (4.4). Let $\widehat{f} := (\widehat{f}_1, \ldots, \widehat{f}_l)$. Since (f, g) is generalized convex on $\mathbb{R}^n$ at $\bar{x}$, it follows that $(\widehat{f}, g)$ is generalized convex on $\mathbb{R}^n$ at this point as well. We apply Theorem 3.5 to conclude that $\bar{x}$ is a quasi $\epsilon_{\bar{k}}$-solution of the robust minimax programming problem

$$\min_{x\in F_r}\max_{k\in K}\widehat{f}_k(x).$$

It means that

$$\widehat{\phi}(\bar{x}) \leq \widehat{\phi}(x) + \epsilon_{\bar{k}} \parallel x - \bar{x} \parallel,\ \forall x \in F_r,$$

where $\widehat{\psi}(x) := \max_{k\in K}\widehat{f}_k(x)$. In other words, we obtain

$$0 \leq \max_{k\in K}\{f_k(x) - f_k(\bar{x})\} + \epsilon_{\bar{k}} \parallel x - \bar{x} \parallel,\ \forall x \in F_r,$$

which entails that

$$f(x) - f(\bar{x}) + \bar{\varepsilon} \parallel x - \bar{x} \parallel \notin -\mathrm{int}\,\mathbb{R}^l_+,\ \forall x \in F_r.$$

Consequently, $\bar{x}$ is a weakly quasi ε-Pareto solution of problem (RMOP). □

5. Conclusions

In this paper, we investigated approximate optimality conditions and duality in robust minimax optimzaion problem under a suitable constraint qualification. Finally, by using the obtained results, we derive necessary and sufficient conditions for weak quasi ε-Pareto solutions to the robust multi-objective optimization problem.

References

[1] J. Bram, *The Lagrange multiplier theorem for maxmin with several constraints*, SIAM J. Appl. Math. **14** (1966), 665–667.

[2] C. R. Bector, S. Chandra and V. Kumar, *Duality for a class of minmax and inexact programming problem*, J. Math. Anal. Appl. **186** (1994), 735–746.

[3] T. D. Chuong and D. S. Kim, *Optimality conditions and duality in nonsmooth multiobjective optimization problems*, Ann. Oper. Res. **217** (2014), 117–136.

[4] T. D. Chuong and D. S. Kim, *Nondifferentiable minimax programming problems with applications*, Ann. Oper. Res. **251** (2017), 73–87.

[5] M. Ehrogtt, J. Ide and A. Schobel, *Minmax robustness for multi-objective optimization problems*, Eur. J. Oper. Res. **239** (2014), 17–31.

[6] H. C. Lai and T. Y. Huang, *Nondifferentiable minimax fractional programming in complex spaces with parametric duality*, J. Global Optim. **53** (2012), 243–254.

[7] D. T. V. An, N. H. Hung, D. T. Ngoan and N. V. Tuyen, *Optimality conditions and sensitivity analysis in parametric nonconvex minimax programming*, J. Global Optim. (2024). https://doi.org/10.1007/s10898-024-01388-y.

[8] D. T. V. An, D. T. Ngoan and N. V. Tuyen, *Optimality conditions and sensitivity analysis in parametric convex minimax programming*, Appl. Anal. (2024). https://doi.org/10.1080/00036811.2024.2330512.

[9] D. T. K. Huyen, D. S. Kim and N. D. Yen, *Optimality conditions for nondifferentiable minimax programs and vector optimization problems*, J. Optim. Theory Appl. **200** (2024), 703–723.

[10] A. L. Soyster, *Convex programming with set-inclusive constraints and applications to inexact linear programming*, Oper. Res. **21** (1973), 1154–1157.

[11] J. E. Falk, *Exact solutions to inexact linear programs*, Oper. Res. **24** (1976), 783–787.

[12] C. Singh, *Convex programming with set-inclusive constraints and its applications to generalized linear and fractional programming*, J. Optim. Theory Appl. **38** (1982), 33–42.

[13] P. Kouvelis and G. Yu, *Robust Discrete Optimization and Its Applications*, Kluwer Academic Publishers, London, 1997.

[14] A. Ben-Tal and A. Nemirovski, *Robust convex optimization*, Math. Oper. Res. **23** (1998), 769–805.

[15] L. El Ghaoui and H. Lebret, *Robust solution to least-squares problems with uncertain data*, Siam J Matrix Anal A. **18** (1997), 1035–1064.

[16] L. El Ghaoui, F. Oustry and H. Lebret, *Robust solutions to uncertain semidefinite programs*, SIAM J. Optim. **9** (1998), 33–52.

[17] A. Ben-Tal and A. Nemirovski, *Selected topics in robust convex optimization*, Math. Program. **112** (2008), 125–158.

[18] A. Ben-Tal, L. El Ghaoui and A. Nemirovski, *Robust Optimization*, NJ: Princeton University Press, 2009.

[19] A. Beck and A. Ben-Tal, *Duality in robust optimization: primal worst equals dual best*, Oper. Res. Lett. **37** (2009), 1–6.

[20] J. Chen, E. Kobis and J. C. Yao, *Optimality conditions and duality for robust nonsmooth multiobjective optimization problems with constraints*, J. Optim. Theory Appl. **181** (2019), 411–436.

[21] P. C. Wu, L. G. Jiao and Y. Y. Zhou, *On approximate optimality conditions for robust multiobjective convex optimization problems*, Optimization **72** (2023), 1995–2018.

[22] B. S. Mordukhovich, *Variational Analysis and Applications*, Springer Monographs in Mathematics, 2018.

[23] T. D. Chuong, *Optimality and duality for robust multiobjective optimization problems*, Nonlinear Analysis **134** (2016), 127–143.

[24] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

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