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FERMAT'S RULE AT INFINITY IN NON-DEGENERATE SEMI-ALGEBRAIC OPTIMIZATION

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This note is dedicated to Professor Do Sang Kim on the occasion of his 70th birthday

Abstract. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a nonconstant polynomial function and S be an unbounded closed semi-algebraic set in $\mathbb{R}^n$ such that $f^* := \inf_{x\in S} f(x) > -\infty$ and that f does not attain its infimum on S. We show that if the restriction of f on S is (Newton) non-degenerate at infinity, then either $f^* = 0$ or there exist a polynomial function $g\colon \mathbb{R}^n \to \mathbb{R}$ and a point $x^* \in (\mathbb{R}\setminus\{0\})^n$ such that $f^* = g(x^*)$ and $0 \in x^* \odot \nabla g(x^*) + C(\infty; S)$, where $x \odot y$ denotes the Hadamard product of two vectors $x, y \in \mathbb{R}^n$ and $C(\infty; S)$ is a cone associated with S at infinity. As an application, we obtain a sufficient condition for the existence of optimal solutions to the optimization problem in question.

1. Introduction

Optimality conditions form the foundations of mathematical programming both theoretically and computationally. There is a large literature on all aspects of optimality conditions. We refer the reader to the comprehensive monographs [2, 4, 12, 13] with the references therein.

In this paper, we are interested in necessary optimality conditions to semi-algebraic optimization problems *whose solution sets are empty.* More precisely, let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a nonconstant polynomial function and S be a nonempty closed semi-algebraic set in $\mathbb{R}^n$. (Definitions and notation will be given in the next section.) Assume that the restriction of f on S is bounded from below and consider the optimization problem

$$\text{(P)} \qquad f^* \ := \ \text{minimize} \ \ f(x) \quad \text{subject to} \quad x \in S.$$

By Fermat's rule, it is well-known that if f attains its infimum on S, i.e., $f^* = f(x^*)$ for some $x^* \in S$, then

$$0 \ \in \ \nabla f(x^*) + N(x^*; S),$$

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where $N(x^*; S)$ stands for the limiting normal cone to S at x^*.

We now assume that f does not attain its infimum on S. Then the main result of this paper is to show that if the restriction of f on S is non-degenerate at infinity, then either $f^* = 0$ or there exist a polynomial function $g\colon \mathbb{R}^n \to \mathbb{R}$ and a point $x^* \in (\mathbb{R} \setminus \{0\})^n$ satisfying a version at infinity of Fermat's rule, which reads as follows:

$$\begin{array}{rcl} f^* & = & g(x^*), \\ 0 & \in & x^* \odot \nabla g(x^*) + C(\infty; S), \end{array}$$

where $x \odot y$ denotes the Hadamard product of two vectors $x, y \in \mathbb{R}^n$ and $C(\infty; S)$ is a certain cone associated with S at infinity; note that if $S = \mathbb{R}^n$, then $C(\infty; S) = \{0\}$ and the second condition is equivalent to the fact that $\nabla g(x^*) = 0$. As an application, we derive a sufficient condition for the existence of optimal solutions of Problem (P).

It should be mentioned that, in the case where S is defined by polynomial inequality constraints, a version at infinity of the Fritz-John (or Karush–Kuhn–Tucker under some qualification constrains) optimality conditions was proposed by the third author in [16] (see also [9, 10, 15]).

The rest of this paper is organized as follows. Definitions and notation are given in Section 2. The main result and its proof are provided in Section 3. The conclusions are presented in Section 4.

2. Preliminaries

We begin by giving some necessary definitions and notational conventions. Let $\mathbb{R}^n$ denote the Euclidean space of dimension n. The corresponding inner product (resp., norm) in $\mathbb{R}^n$ is defined by $\langle x, y\rangle$ for any $x, y \in \mathbb{R}^n$ (resp., $\|x\| := \sqrt{\langle x, x\rangle}$ for any $x \in \mathbb{R}^n$). The closed ball and the sphere centered at the origin $0 \in \mathbb{R}^n$ of radius $R > 0$ will be denoted by $\mathbb{B}_R$ and $\mathbb{S}_R$, respectively. We will adopt the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

For an extended real-valued function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, we denote its *effective domain* and *epigraph* by, respectively,

$$\begin{array}{rcl} \mathrm{dom} f & := & \{x \in \mathbb{R}^n \mid f(x) < \infty\}, \\ \mathrm{epi} f & := & \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \le r\}. \end{array}$$

The function f is *lower semi-continuous* if epi$f$ is closed.

The *indicator function* of a set $S \subset \mathbb{R}^n$, denoted δ_S, is defined by

$$\delta_S(x) := \begin{cases} 0 & \text{if } x \in S, \\ \infty & \text{otherwise.} \end{cases}$$

By definition, δ_S is lower semi-continuous if and only if S is closed.

2.1. Normals and subdifferentials.

Here we recall the notions of the normal cones to sets and the subdifferentials of real-valued functions used in this paper. The reader is referred to [13, 14, 17] for more details.

Definition 2.1. Consider a set $S \subset \mathbb{R}^n$ and a point $x \in S$.

(i) The *regular normal cone* (also known as the *prenormal* or *Fréchet normal cone*) $\widehat{N}(x;S)$ to S at x consists of all vectors $v \in \mathbb{R}^n$ satisfying

$$\langle v, x' - x\rangle \quad \leq \quad o(\|x' - x\|) \quad \text{as} \quad x' \to x \quad \text{with} \quad x' \in S.$$

(ii) The *limiting normal cone* (also known as the *basic* or *Mordukhovich normal cone*) $N(x;S)$ to S at x consists of all vectors $v \in \mathbb{R}^n$ such that there are sequences $x^\ell \to x$ with $x^\ell \in S$ and $v^\ell \to v$ with $v^\ell \in \widehat{N}(x^\ell;S)$.

If $x \notin S$, we put $\widehat{N}(x;S) := \emptyset$ and $N(x;S) := \emptyset$.

If S is a manifold of class C^1, then for every point $x \in S$, the normal cones $\widehat{N}(x;S)$ and $N(x;S)$ are equal to the normal space to S at x in the sense of differential geometry; see [17, Example 6.8]. In particular, for all $R > 0$ and all $x \in \mathbb{S}_R$, we have $N(x;\mathbb{S}_R) = \{\mu x \mid \mu \in \mathbb{R}\}$.

Functional counterparts of normal cones are subdifferentials.

Definition 2.2. Consider a function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ and a point $x \in \operatorname{dom} f$. The *limiting* and *singular subdifferentials* of f at x are defined respectively by

$$\begin{aligned}
\partial f(x) \quad &:= \quad \big\{v \in \mathbb{R}^n \;\big|\; (v,-1) \in N\big((x,f(x));\operatorname{epi} f\big)\big\},\\
\partial^\infty f(x) \quad &:= \quad \big\{v \in \mathbb{R}^n \;\big|\; (v,0) \in N\big((x,f(x));\operatorname{epi} f\big)\big\}.
\end{aligned}$$

In [13,14,17], the reader can find equivalent analytic descriptions of the limiting subdifferential $\partial f(x)$ and comprehensive studies of it and related constructions. If the function f is of class C^1, then $\partial f(x) = \{\nabla f(x)\}$ and $\partial^\infty f(x) = \{0\}$.

Lemma 2.3. *For any set $S \subset \mathbb{R}^n$ and point $x \in S$, we have*

$$\partial \delta_S(x) \quad = \quad \partial^\infty \delta_S(x) \quad = \quad N(x;S).$$

Proof. See [14, Proposition 1.19]. □

Theorem 2.4 (Fermat's rule)**.** *Consider a lower semi-continuous function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ and a closed subset S of $\mathbb{R}^n$. If $x \in \operatorname{dom} f \cap S$ is a local minimizer of f on S and the qualification condition*

$$\partial^\infty f(x) \cap \big(-N(x;S)\big) \quad = \quad \{0\}$$

is valid, then $0 \in \partial f(x) + N(x;S)$.

Proof. See [14, Theorem 6.1]. □

2.2. **Semi-algebraic geometry.** Now, we recall some notions and results of semi-algebraic geometry, which can be found in [3] and [7, Chapter 1].

Definition 2.5. A subset S of $\mathbb{R}^n$ is called *semi-algebraic* if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f_i(x) = 0,\ i = 1,\ldots,p; f_i(x) > 0,\ i = p+1,\ldots,q\},$$

where all f_i are polynomials. In other words, S is a union of finitely many sets, each defined by finitely many polynomial equalities and inequalities.

A mapping $f\colon S \to (\mathbb{R} \cup \{\infty\})^m$ is said to be *semi-algebraic* if its graph

$$\{(x,y) \in S \times \mathbb{R}^m \mid y = f(x)\}$$

is a semi-algebraic subset of $\mathbb{R}^n \times \mathbb{R}^m$.

A major fact concerning the class of semi-algebraic sets is its stability under linear projections.

Theorem 2.6 (Tarski–Seidenberg theorem). *The image of any semi-algebraic set $S \subset \mathbb{R}^n$ under a projection to any linear subspace of $\mathbb{R}^n$ is still a semi-algebraic set.*

Remark 2.7. As an immediate consequence of the Tarski–Seidenberg Theorem, we get semi-algebraicity of any set $\{x \in A \mid \exists y \in B, (x, y) \in C\}$, provided that A, B, and C are semi-algebraic sets in the corresponding spaces. Also, $\{x \in A \mid \forall y \in B, (x, y) \in C\}$ is a semi-algebraic set as its complement is the union of the complement of A and the set $\{x \in A \mid \exists y \in B, (x, y) \notin C\}$. Thus, if we have a finite collection of semi-algebraic sets, then any set obtained from them with the help of a finite chain of quantifiers is also semi-algebraic.

The following well-known lemmas will be of great importance for us.

Lemma 2.8 (monotonicity lemma). *Let $f\colon (a, b) \to \mathbb{R}$ be a semi-algebraic function. Then there are finitely many points $a =: t_0 < t_1 < \cdots < t_p := b$ such that for each $i = 0, \ldots, p-1$, the restriction of f to the interval (t_i, t_{i+1}) is analytic, and either constant, or strictly increasing or strictly decreasing.*

Lemma 2.9 (growth dichotomy lemma). *Let $f\colon (0, \epsilon) \to \mathbb{R}$ be a semi-algebraic function with $f(t) \neq 0$ for all $t \in (0, \epsilon)$. Then there exist constants $a \neq 0$ and $\alpha \in \mathbb{Q}$ such that $f(t) = at^{\alpha} + o(t^{\alpha})$ as $t \to 0^+$.*

Lemma 2.10 (curve selection lemma at infinity). *Let $S \subset \mathbb{R}^n$ be a semi-algebraic set, and let $f := (f_1, \ldots, f_m)\colon \mathbb{R}^n \to \mathbb{R}^m$ be a semi-algebraic map. Assume that there exists a sequence $\{x^{\ell}\}_{\ell \geq 1} \subset S$ such that $\lim_{\ell \to \infty} \|x^{\ell}\| = \infty$ and $\lim_{\ell \to \infty} f(x^{\ell}) = y \in (\mathbb{R} \cup \{\pm\infty\})^m$. Then there exists an analytic semi-algebraic curve $\phi\colon (0, \epsilon) \to \mathbb{R}^n$ such that $\phi(t) \in S$ for all $t \in (0, \epsilon), \lim_{t \to 0^+} \|\phi(t)\| = \infty$ and $\lim_{t \to 0^+} f(\phi(t)) = y$.*

We close this subsection with the following fact (see [6, Lemma 2.10]).

Lemma 2.11. *Consider a lower semi-continuous, semi-algebraic function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ and a semi-algebraic curve $\phi\colon (0, \epsilon) \to \mathrm{dom} f$. Then for all $t > 0$ sufficiently small, the maps ϕ and $f \circ \phi$ are analytic at t and satisfy*

$$v(t) \in \partial f(\phi(t)) \implies \left\langle v(t), \frac{d\phi(t)}{dt} \right\rangle = \frac{d}{dt}(f \circ \phi)(t).$$

2.3. **Newton polyhedra and non-degeneracy conditions.** For a nonempty subset J of $\{1, \ldots, n\}$, we define

$$\mathbb{R}^J := \{x \in \mathbb{R}^n \mid x_j = 0, \text{ for all } j \notin J\}.$$

We denote by $\mathbb{Z}_+$ and $\mathbb{R}_+$, respectively, the set of non-negative integers and the set of non-negative real numbers. If $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{Z}_+^n$, we denote by x^{κ} the monomial $x_1^{\kappa_1} \cdots x_n^{\kappa_n}$. We also put $\mathbb{R}^* := \mathbb{R} \backslash \{0\}$.

A subset $\Gamma \subset \mathbb{R}_+^n$ is a *Newton polyhedron* if there exists a finite subset $S \subset \mathbb{Z}_+^n$ such that Γ is the convex hull in $\mathbb{R}^n$ of S. We say that Γ is the *Newton polyhedron determined by* S and write $\Gamma = \Gamma(S)$. A Newton polyhedron Γ is *convenient* if it intersects each coordinate axis at a point different from the origin 0 in $\mathbb{R}^n$, that is, if for any $j \in \{1, \ldots, n\}$ there exists some $\kappa_j > 0$ such that $\kappa_j e^j \in \Gamma$, where $\{e^1, \ldots, e^n\}$ denotes the canonical basis in $\mathbb{R}^n$.

For a Newton polyhedron Γ and a vector $q \in \mathbb{R}^n$, we let

$$\begin{aligned} d(q, \Gamma) &:= \min\{\langle q, \kappa\rangle \mid \kappa \in \Gamma\}, \\ \Delta(q, \Gamma) &:= \{\kappa \in \Gamma \mid \langle q, \kappa\rangle = d(q, \Gamma)\}. \end{aligned}$$

By definition, for each nonzero vector $q \in \mathbb{R}^n$, $\Delta(q, \Gamma)$ is a closed face of Γ. Conversely, if Δ is a closed face of Γ, then there exists a nonzero vector $q \in \mathbb{R}^n$ such that $\Delta = \Delta(q, \Gamma)$, where we can in fact assume that $q \in \mathbb{Q}^n$ since Γ is an integer polyhedron. The *dimension* of a face Δ is the minimum of the dimensions of the affine subspaces containing Δ. The faces of Γ of dimension 0 are the *vertices* of Γ.

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial function. Suppose that f is written as $f = \sum_\kappa c_\kappa x^\kappa$. The *support* of f, denoted by $\mathrm{supp}(f)$, is the set of $\kappa \in \mathbb{Z}^n_+$ such that $c_\kappa \neq 0$. The *Newton polyhedron (at infinity)* of f, denoted by $\Gamma(f)$, is the convex hull in $\mathbb{R}^n$ of the set $\mathrm{supp}(f)$, i.e., $\Gamma(f) := \Gamma(\mathrm{supp}(f))$. The polynomial f is *convenient* if $\Gamma(f)$ is convenient. For each (closed) face Δ of $\Gamma(f)$, we will denote

$$f_\Delta(x) \quad := \quad \sum_{\kappa \in \Delta} c_\kappa x^\kappa.$$

Remark 2.12. (i) We have $\Gamma(f) \cap \mathbb{R}^J = \Gamma(f|_{\mathbb{R}^J})$ for all nonempty subset J of $\{1, \ldots, n\}$.

(ii) Let $\Delta := \Delta(q, \Gamma(f))$ for some nonzero vector $q := (q_1, \ldots, q_n) \in \mathbb{R}^n$. Then $f_\Delta(x)$ is a weighted homogeneous polynomial of type $(q, d := d(q, \Gamma(f)))$, i.e., we have for all $t > 0$ and all $x \in \mathbb{R}^n$,

$$f_\Delta(t^{q_1} x_1, \ldots, t^{q_n} x_n) \quad = \quad t^d f_\Delta(x_1, \ldots, x_n).$$

This implies the Euler relation

$$\sum_{j=1}^n q_j x_j \frac{\partial f_\Delta}{\partial x_j}(x) \quad = \quad d \cdot f_\Delta(x).$$

In particular, if $d \neq 0$ and $\nabla f_\Delta(x) = 0$, then $f_\Delta(x) = 0$.

For any $x := (x_1, \ldots, x_n)$ and $v := (v_1, \ldots, v_n)$ we put

$$x \odot v \quad := \quad (x_1 v_1, \ldots, x_n v_n)$$

and call it the *Hadamard product* of x and y.

Let S be an unbounded closed semi-algebraic set in $\mathbb{R}^n$ and let $C(\infty; S)$ denote the set of all vectors $w \in \mathbb{R}^n$ such that there are sequences $x^\ell \in S$ and $v^\ell \in N(x^\ell; S)$ satisfying $\|x^\ell\| \to \infty$ and $x^\ell \odot v^\ell \to w$. By definition, $C(\infty; S)$ is a closed cone.

Example 2.13. Consider the unbounded closed semi-algebraic set

$$S \quad := \quad \left\{x := (x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_2 - 1)^2 + x_3^2 \leq 1\right\}.$$

We have

$$N(x; S) \quad = \quad \begin{cases} \{(0, 0, 0)\} & \text{if } (x_2 - 1)^2 + x_3^2 < 1, \\ \{(0, t(x_2 - 1), t x_3) \mid t \geq 0\} & \text{if } (x_2 - 1)^2 + x_3^2 = 1. \end{cases}$$

Then a direct calculation shows that

$$C(\infty; S) \quad = \quad \big\{\big(0, t(s-1), t(2-s)\big) \mid 0 \le t \ \text{ and } \ 0 \le s \le 2\big\},$$

which is a closed cone.

The following definition is inspired by those of Kouchnirenko [11] and Khovanskii [8].

Definition 2.14. We say that *the restriction of f on S is (Newton) non-degenerate at infinity* if for any vector[1] $q \in \mathbb{R}^n$ with $d(q, \Gamma(f)) < 0$, there is no $x \in (\mathbb{R}^*)^n$ satisfying the following two constraints

$$\begin{aligned} 0 &= f_\Delta(x), \\ 0 &\in x \odot \nabla f_\Delta(x) + C(\infty; S), \end{aligned}$$

where $\Delta := \Delta(q, \Gamma(f))$.

Remark 2.15. By definition, if $S = \mathbb{R}^n$, then $C(\infty; S) = \{0\}$, and so the above two constraints are equivalent to the fact that $\nabla f_\Delta(x) = 0$.

3. The main result and its proof

In what follows, we let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a nonconstant polynomial function and S be an unbounded closed semi-algebraic set in $\mathbb{R}^n$ such that f is bounded from below on S. Consider the optimization problem formulated in the introduction section:

$$\text{(P)} \qquad f^* \ := \ \text{minimize} \ \ f(x) \quad \text{subject to} \quad x \in S.$$

The main result of this paper is as follows (see also [16]).

Theorem 3.1 (Fermat's rule at infinity)**.** *Assume that the restriction of f on S is non-degenerate at infinity. If f does not attain its infimum f^* on S, then either $f^* = 0$ or there exist a point $x^* \in (\mathbb{R}^*)^n$, a nonempty set $J \subset \{1, \ldots, n\}$, and a vector $q \in \mathbb{R}^n$ with $\min_{j \in J} q_j < 0$ and $d(q, \Gamma(f)) = 0$ such that the following conditions hold:*

(i) $f^* = f_\Delta(x^*)$, *where* $\Delta := \Delta(q, \Gamma(f)) \subset \mathbb{R}^J$;
(ii) $0 \in x^* \odot \nabla f_\Delta(x^*) + C(\infty; S) \cap \mathbb{R}^J$.

Remark 3.2. By definition, if $S = \mathbb{R}^n$, then $C(\infty; S) = \{0\}$, and so Condition (ii) is equivalent to the fact that $\nabla f_\Delta(x^*) = 0$.

The proof of the above theorem will make use of the following lemma.

Lemma 3.3. *For all R sufficiently large and all $x \in S \cap \mathbb{S}_R$ the following inclusion holds*

$$N(x; S \cap \mathbb{S}_R) \quad \subset \quad N(x; S) + N(x; \mathbb{S}_R).$$

[1] The number of vectors $q \in \mathbb{R}^n$ is infinite; however, there exists a finite number of faces Δ of the Newton polyhedron $\Gamma(f)$.

Proof. In view of [14, Theorem 2.16], it suffices to show that a version of normal qualification condition holds, i.e., for all R sufficiently large and all $x \in S \cap \mathbb{S}_R$ we have

$$N(x; S) \cap \big(-N(x; \mathbb{S}_R)\big) \quad = \quad \{0\}.$$

Suppose to the contrary that this fact does not hold: there exist sequences $x^\ell \in S$, with $\lim_{\ell\to\infty} \|x^\ell\| = \infty$, and $\mu^\ell \in \mathbb{R}^*$ such that $-\mu^\ell x^\ell \in N(x^\ell; S)$. Applying the Curve Selection Lemma at infinity (see Lemma 2.10) to the semi-algebraic set

$$\{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} \mid x \in S, \mu \neq 0, -\mu x \in N(x; S)\}$$

and the semi-algebraic map

$$\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, \quad (x, \mu) \mapsto \|x\|,$$

we get an analytic semi-algebraic curve $(\phi, \mu)\colon (0, \epsilon) \to \mathbb{R}^n \times \mathbb{R}$ with $\lim_{t\to 0^+} \|\phi(t)\| = \infty$ such that for all $t \in (0, \epsilon)$, we have

$$\phi(t) \in S, \quad \mu(t) \neq 0, \quad \text{and} \quad -\mu(t)\phi(t) \in N(\phi(t); S).$$

In view of Lemma 2.8, we may assume that the function

$$(0, \epsilon) \to \mathbb{R}, \quad t \mapsto \|\phi(t)\|^2,$$

is strictly increasing (perhaps after decreasing ϵ); in particular, $\frac{d}{dt}\|\phi(t)\|^2 > 0$ for all $t \in (0, \epsilon)$.

On the other hand, since $(\delta_S \circ \phi)(t) = 0$ and $\partial\delta_S(\phi(t)) = N(\phi(t); S)$, it follows from Lemma 2.11 that for all $t > 0$ small enough,

$$0 \quad = \quad \frac{d}{dt}(\delta_S \circ \phi)(t) \quad = \quad \left\langle -\mu(t)\phi(t), \frac{d\phi(t)}{dt} \right\rangle \quad = \quad -\frac{\mu(t)}{2}\frac{d}{dt}\|\phi(t)\|^2.$$

which yields $\mu(t) = 0$, a contradiction. □

Based on Lemma 3.3 we now give a proof of the main result.

Proof of Theorem 3.1. Since f does not attain its infimum f^* on S, there exists a sequence $\{a^\ell\}_{\ell \geq 1} \subset S$ such that

$$\lim_{\ell\to\infty} \|a^\ell\| = \infty \quad \text{and} \quad \lim_{\ell\to\infty} f(a^\ell) = f^*.$$

For each $\ell \geq 1$, we consider the problem

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && x \in S \ \text{ and } \ \|x\|^2 = \|a^\ell\|^2. \end{aligned}$$

Since the objective function f is continuous and the constraint set is nonempty compact, by the Weierstrass theorem, an optimal solution $x^\ell \in S$ of the problem exists. By Theorem 2.4 and Lemma 3.3, we have for all ℓ large enough,

$$\begin{aligned} 0 \ \in \ \nabla f(x^\ell) + N(x^\ell; S \cap \mathbb{S}_{\|a^\ell\|}) \ &\subset \ \nabla f(x^\ell) + N(x^\ell; S) + N(x^\ell; \mathbb{S}_{\|a^\ell\|}) \\ &= \ \nabla f(x^\ell) + N(x^\ell; S) + \{\mu x^\ell \mid \mu \in \mathbb{R}\}, \end{aligned}$$

and so there are $\mu^\ell \in \mathbb{R}$ and $v^\ell \in N(x^\ell; S)$ satisfying

$$\nabla f(x^\ell) + \mu^\ell x^\ell + v^\ell \quad = \quad 0.$$

We also note that

$$\|x^\ell\|^2 = \|a^\ell\|^2 \quad \text{and} \quad f^* < f(x^\ell) \le f(a^\ell).$$

Hence

$$\lim_{\ell\to\infty} \|x^\ell\| = \infty \quad \text{and} \quad \lim_{\ell\to\infty} f(x^\ell) = f^*.$$

Let

$$\mathscr{A} := \{(x, v, \mu) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mid x \in S,\ v \in N(x; S),\ \nabla f(x) + \mu x + v = 0\}.$$

Then the sequence $(x^\ell, v^\ell, \mu^\ell) \in \mathscr{A}$ tends to infinity in the sense that $\|(x^\ell, v^\ell, \mu^\ell)\| \to \infty$ as $\ell \to \infty$. Moreover, in view of Theorem 2.6, it is not hard to see that $\mathscr{A}$ is a semi-algebraic set. Applying Lemma 2.10 to the semi-algebraic function $\mathscr{A} \to \mathbb{R}^2, (x, v, \mu) \mapsto (\|x\|, f(x))$, we get an analytic semi-algebraic curve

$$(\phi, v, \mu)\colon (0, \epsilon) \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \quad t \mapsto \big(\phi(t), v(t), \mu(t)\big),$$

satisfying the following conditions

(c1) $\phi(t) \in S$;
(c2) $v(t) \in N(\phi(t); S)$;
(c3) $\lim_{t\to 0^+} \|\phi(t)\| = \infty$;
(c4) $\lim_{t\to 0^+} f(\phi(t)) = f^*$;
(c5) $\nabla f(\phi(t)) + \mu(t)\phi(t) + v(t) \equiv 0$.

Since the function $f \circ \phi$ is semi-algebraic, by shrinking ϵ if necessary, we can assume that this function is strictly decreasing (see Lemma 2.8). Then, by Condition (c4), $f \circ \phi(t) \searrow f^*$ as $t \searrow 0^+$. This, together with Lemma 2.9, yields

$$f \circ \phi(t) = f^* + a t^\alpha + \cdots$$

for some $a > 0$ and $\alpha > 0$. Here and in the following, the dots stand for the higher order terms in t.

On the other hand, from Condition (c5), we deduce that

$$\begin{aligned} \frac{d}{dt}(f \circ \phi)(t) &= \left\langle \nabla f(\phi(t)), \frac{d\phi(t)}{dt} \right\rangle \\ &= -\mu(t) \left\langle \phi(t), \frac{d\phi(t)}{dt} \right\rangle - \left\langle v(t), \frac{d\phi(t)}{dt} \right\rangle. \end{aligned}$$

Note that $(\delta_S \circ \phi)(t) = 0$ and $\partial\delta_S(\phi(t)) = N(\phi(t); S)$ (see Lemma 2.3). By Lemma 2.11 and by shrinking ϵ somewhat if necessary, we have for all $t \in (0, \epsilon)$,

$$\left\langle v(t), \frac{d\phi(t)}{dt} \right\rangle = \frac{d}{dt}(\delta_S \circ \phi)(t) = 0.$$

Therefore,

$$\frac{d}{dt}(f \circ \phi)(t) = -\frac{\mu(t)}{2} \frac{d\|\phi(t)\|^2}{dt}. \tag{3.1}$$

Consequently, for all $t > 0$ small, $\mu(t) \neq 0$, which, together with Lemma 2.9, yields

$$\mu(t) = \mu^0 t^\beta + \cdots,$$

where $\mu^0 \neq 0$ and $\beta \in \mathbb{Q}$. By Condition (c3), the set $J := \{j \mid \phi_j \not\equiv 0\}$ is nonempty. In view of Lemma 2.9, for each $j \in J$, we can expand the coordinate ϕ_j as follows

$$\phi_j(t) \quad = \quad x_j^0 t^{q_j} + \cdots ,$$

where $x_j^0 \neq 0$ and $q_j \in \mathbb{Q}$. From Condition (c3), we get $\min_{j \in J} q_j < 0$. Moreover, it follows from (3.1) that

$$\beta + 2 \min_{j \in J} q_j \quad = \quad \alpha \ > \ 0. \tag{3.2}$$

Let $q_j := M$ for $j \notin J$ with M being sufficiently large and satisfying

$$M \quad > \quad \max \left\{ \sum_{j \in J} q_j \kappa_j \mid \kappa \in \Gamma(f) \right\}.$$

Let d be the minimal value of the linear function $\sum_{j=1}^n q_j \kappa_j$ on $\Gamma(f)$ and let Δ be the maximal face of $\Gamma(f)$ (maximal with respect to the inclusion of faces) where the linear function takes this value, i.e.,

$$d \ := \ d(q, \Gamma(f)) \quad \text{and} \quad \Delta \ := \ \Delta(q, \Gamma(f)).$$

Recall that $\mathbb{R}^J := \{x := (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_j = 0 \text{ for } j \notin J\}$, which contains the curve ϕ. Since the function $f \circ \phi$ is strictly decreasing, then the restriction of f on $\mathbb{R}^J$ is not constant, and so $\Gamma(f) \cap \mathbb{R}^J = \Gamma(f|_{\mathbb{R}^J})$ is nonempty and different from $\{0\}$. Furthermore, by definition of the vector q, one has

$$d \ = \ d(q, \Gamma(f|_{\mathbb{R}^J})) \quad \text{and} \quad \Delta \ = \ \Delta(q, \Gamma(f|_{\mathbb{R}^J})) \ \subset \ \mathbb{R}^J.$$

Consequently, for each $j \notin J$, the polynomial f_Δ does not depend on the variable x_j. Now suppose that f is written as $f(x) = \sum_\kappa a_\kappa x^\kappa$. Then

$$\begin{aligned} f(\phi(t)) \quad &= \quad \sum_{\kappa \in \Gamma(f) \cap \mathbb{R}^J} a_\kappa (\phi(t))^\kappa \\ &= \quad \sum_{\kappa \in \Gamma(f) \cap \mathbb{R}^J} \Big(a_\kappa \prod_{j=1}^n \phi_j(t)^{\kappa_j} \Big) \\ &= \quad \sum_{\kappa \in \Gamma(f) \cap \mathbb{R}^J} \Big(a_\kappa \prod_{j \in J} (x_j^0 t^{q_j})^{\kappa_j} + \cdots \Big) \\ &= \quad \sum_{\kappa \in \Gamma(f) \cap \mathbb{R}^J} \Big(a_\kappa (x^*)^\kappa t^{\sum_{j \in J} q_j \kappa_j} + \cdots \Big) \\ &= \quad \sum_{\kappa \in \Delta} a_\kappa (x^*)^\kappa t^d + \cdots , \end{aligned}$$

where $x^* := (x_1^*, \ldots, x_n^*) \in (\mathbb{R}^*)^n$ with[2]

$$x_j^* \quad := \quad \begin{cases} 1 & \text{if } j \notin J, \\ x_j^0 & \text{otherwise.} \end{cases}$$

[2]For each $j \notin J$ the polynomial f_Δ does not depend on x_j, and so x_j^* can be arbitrary real number.

Recall that $f_\Delta(x) = \sum_{\kappa\in\Delta} a_\kappa x^\kappa$. Hence

$$f(\phi(t)) \quad = \quad f_\Delta(x^*)t^d + \cdots . \tag{3.3}$$

If $d > 0$, then it follows from Condition (c4) that $f^* = 0$ and the theorem is proved. So, in the rest of the proof, we assume that $d \le 0$. Note that if $d < 0$, then $f_\Delta(x^*) = 0$, which follows directly from Condition (c4) and (3.3).

For $j = 1, \ldots, n$, by some similar calculations as with $f(\phi(t))$, we have

$$\frac{\partial f}{\partial x_j}(\phi(t)) \quad = \quad \frac{\partial f_\Delta}{\partial x_j}(x^*)t^{d-q_j} + \cdots .$$

Then Condition (c5) reads as follows

$$0 \quad = \quad \frac{\partial f_\Delta}{\partial x_j}(x^*)t^{d-q_j} + \cdots + \left(\mu^0 x_j^0 t^{\beta+q_j} + \cdots\right) + v_j(t), \tag{3.4}$$

where the expression in the bracket is dropped when $j \notin J$. Note that if $v_j(t) \not\equiv 0$, then we can write

$$v_j(t) \quad = \quad v_j^0 t^{p_j} + \cdots ,$$

where $v_j^0 \neq 0$ and $p_j \in \mathbb{Q}$. If $v_j(t) \equiv 0$ we put $p_j := \infty$. From (3.2) and the definition of q_j we have for all $j = 1, \ldots, n$,

$$\beta + q_j \quad \geq \quad \alpha - q_j \quad > \quad -q_j \quad \geq \quad d - q_j,$$

which, together with (3.4), yields $p_j \geq d - q_j$. Moreover, for $j \notin J$, we have $\frac{\partial f_\Delta}{\partial x_j}(x^*) = 0$ (because the polynomial f_Δ does not depend on the variable x_j) and so $p_j > d - q_j$, while for $j \in J$, we have

$$\frac{\partial f_\Delta}{\partial x_j}(x^*) \quad = \quad \begin{cases} 0 & \text{if } p_j > d - q_j, \\ -v_j^0 & \text{otherwise.} \end{cases}$$

Next we put $w := (w_1, \ldots, w_n) \in \mathbb{R}^n$, where

$$w_j \quad := \quad \begin{cases} 0 & \text{if } p_j > d - q_j, \\ x_j^0 v_j^0 & \text{otherwise.} \end{cases}$$

Clearly $w \in \mathbb{R}^J$ and $x^* \odot \nabla f_\Delta(x^*) + w = 0$. Observe that if $w \neq 0$, then $\phi(t) \odot v(t) \not\equiv 0$ and

$$\lim_{t\to 0^+} \frac{\phi(t) \odot v(t)}{\|\phi(t) \odot v(t)\|} \quad = \quad \frac{w}{\|w\|},$$

which can be checked by simple calculation. By definition, $w \in C(\infty; S)$.

We know that if $d < 0$, then $f_\Delta(x^*) = 0$. Therefore, $d = 0$ by our assumption that the restriction of f on S is non-degenerate at infinity. This, together with Condition (c4) and (3.3), yields $f_\Delta(x^*) = f^*$, and the proof is complete. □

The next example, which is inspired by [16, Example 1] (see also [1, page 47]), illustrates Theorem 3.1.

Example 3.4. Consider the following semi-algebraic optimization problem

$$\begin{aligned} &\text{minimize} \ \ f(x_1, x_2, x_3) := x_1^2 x_2 - 2x_1 x_3 \\ &\text{subject to} \ \ (x_1, x_2, x_3) \in S, \end{aligned}$$

where $S := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_2 - 1)^2 + x_3^2 \le 1\right\}$. It is known that the infimum of f over the constraint set S is -2. However, this infimum cannot be attained by any feasible point.

By definition, the Newton polyhedron of f is $\Gamma(f) = \operatorname{conv}\{(2, 1, 0), (1, 0, 1)\}$-the convex hull of the two points $(2, 1, 0)$ and $(1, 0, 1)$, and so $\Gamma(f)$ consists of two vertices and one edge. Furthermore, we know from Example 2.13 that

$$C(\infty; S) \quad = \quad \left\{\big(0, t(s-1), t(2-s)\big) \mid 0 \le t \ \text{ and } \ 0 \le s \le 2\right\}.$$

By a simple calculation, we can see that for any face Δ of $\Gamma(f)$ there is no $x \in (\mathbb{R}^*)^3$ satisfying the following two constraints

$$\begin{aligned} 0 &= f_\Delta(x), \\ 0 &\in x \odot \nabla f_\Delta(x) + C(\infty; S). \end{aligned}$$

Consequently, the restriction of f on S is non-degenerate at infinity.

Let $J := \{1, 2, 3\}$ and $q := (-1, 2, 1) \in \mathbb{R}^3$. We have $\min_{j \in J} q_j = -1 < 0$ and

- $d := d(q, \Gamma(f)) = 0$;
- $\Delta := \Delta(q, \Gamma(f)) = \operatorname{conv}\{(2, 1, 0), (1, 0, 1)\}$;
- $f_\Delta(x_1, x_2, x_3) = x_1^2 x_2 - 2x_1 x_3$.

Therefore, the system

$$\begin{aligned} -2 &= f_\Delta(x), \\ 0 &\in x \odot \nabla f_\Delta(x) + C(\infty; S) \end{aligned}$$

is equivalent to the system

$$\begin{aligned} -2 &= x_1^2 x_2 - 2x_1 x_3, \\ 0 &= 2x_1^2 x_2 - 2x_1 x_3, \\ 0 &= x_1^2 x_2 + t(s-1), \\ 0 &= -2x_1 x_3 + t(2-s), \\ 0 &\le t \ \text{ and } \ 0 \le s \le 2. \end{aligned}$$

Finally, we easily see that $x^* := (c^{-1}, 2c^2, 2c) \in (\mathbb{R}^*)^3$ with $c \neq 0$ is a solution of the system.

We finally propose a sufficient condition for the existence of optimal solutions of Problem (P).

Corollary 3.5 (compare [5, Theorem 1.1])**.** *Assume that f is convenient and the restriction of f on S is non-degenerate at infinity. Then f attains its infimum on S.*

Proof. Since the polynomial f is convenient, $d(q, \Gamma(f)) < 0$ for all $q \in \mathbb{R}^n$ with $\min_j q_j < 0$. This, together with the assumption that the restriction of f on S is non-degenerate at infinity, implies that there is no analytic semi-algebraic curve

$(\phi, v, \mu)\colon (0, \epsilon) \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ satisfying Conditions (c1)–(c5) in the proof of Theorem 3.1. The desired conclusion is derived easily. □

4. Conclusion

In this paper, we propose a version at infinity of Fermat's rule for non-degenerate semi-algebraic optimization problems, in which the solution sets are empty. It would be interesting to have a similar result for vector optimization problems with semi-algebraic data.

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