Journal of Nonlinear and Convex Analysis
Volume 25, Number 12, 2024, 3033–3049

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

STRONG CONVERGENCE OF ITERATIVE ALGORITHMS FOR ZEROS OF ACCRETIVE OPERATORS IN BANACH SPACES

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This paper is dedicated to Professor Do Sang Kim on the occasion of his 70th Birthday

Abstract. In this paper, as the prox-Tikhonov regularization method, we consider iterative algorithms for finding a zero of an accretive operators in Banach spaces. Under weaker control conditions than previous ones, strong convergence of the sequences generated by proposed algorithms to a zero of an accretive operator is established in a reflexive Banach space having a weakly continuous duality mapping.

1. Introduction

Let E be a real Banach space with the norm $\|\cdot\|$, let E^* be its dual and let C be a nonempty closed convex subset of E. For an operator $A : E \to 2^E$, we define its domain, range and graph as follows:

$$D(A) = \{x \in E : Ax \neq \emptyset\}, \quad R(A) = \cup\{Az : z \in D(A)\}$$

and

$$G(A) = \{(x, y) \in E \times E : x \in D(A),\ y \in A\},$$

respectively. Thus we write $A : E \to 2^E$ as follows: $A \subset E \times E$. The inverse A^{-1} of A is defined by

$$x \in A^{-1}y \ \text{ if and only if } \ y \in Ax.$$

The operator A is said to be accretive if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $i = 1, 2$, there is $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$, where J is the normalized duality mapping form E into 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\} \ \ x \in E,$$

and $\langle\cdot,\cdot\rangle$ denote the generalized duality pairing. I will stand for the identity operator on E. An accretive operator A is said to be *maximal accretive* if there is no proper accretive extension of E and *m-accretive* if $R(I + A) = E$. (It follows that $R(I + rA) = E$ for all $r > 0$) If A is m-accretive, then it is maximal accretive, but the converse is not true in general. If A is accretive, then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda : R(I + \lambda A) \to D(A)$ by

$$J_\lambda = (I + \lambda A)^{-1}.$$

2020 *Mathematics Subject Classification.* 47J20, 49J40, 65J15.

Key words and phrases. Accretive operator, Maximal monotone operator, Proximal point algorithm, Regularization method, Resolvent, Strong convergence, Weakly continuous duality mapping.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2021R1I1A3040289).

It is called the *resolvent* of A. In case of Hilbert space H, accretive operators are also called monotone. It is well-known that a monotone operator A is maximal if and only if $R(I+\lambda A)=H$ ([13]). The best-known example of maximal monotone is the subgradient mapping $\partial\psi$ of a proper lower semicontinuous convex function $\psi : H \to \mathbb{R}\cup\{\infty\}$ ([14]).

The following variational inclusion problem in Banach space E:

$$\text{(P)} \qquad \text{find } \ z\in E \ \text{ such that } \ 0\in Az,$$

plays an essential role in the theory of nonlinear analysis, where $A : E \to 2^E$ is a multi-valued operator acting on E. In fact, the Problem (P) can be regarded as a unified formulation of several important problems and covers a wide range of mathematical applications: for example, variational inequalities, complementarity problems and non-smooth convex optimization. In particular, if $\psi : H \to \mathbb{R}\cup\{\infty\}$ is a proper lower semicontinuous convex function, then its subdifferential $\partial\psi$ is a maximal monotone operator and a point $z \in H$ minimizes ψ if and only if $0\in\partial\psi(z)$.

Iterative algorithms have extensively been studied over the last fifty years for constructions of zeros of accretive operators (see, e.q., [2, 3, 7, 10, 12, 15, 18, 23]). One method for solving zeros of maximal monotone operators is proximal point algorithm. Let A be a maximal monotone operator in a Hilbert space H. The proximal point algorithm generates, for starting $x_1 \in H$, a sequence $\{x_n\}$ in H by

$$x_{n+1} = J^A_{r_n} x_n \quad \text{for all } n\in\mathbb{N}, \tag{1.1}$$

where $J_{r_n} = (I + r_nA)^{-1}$ is the resolvent operator associated with the operator A and $\{r_n\}$ is a regularization sequence in $(0,\infty)$. Note that (1.1) is equivalent to

$$x_n \in x_{n+1} + r_nAx_{n+1} \quad \text{for all } n\in\mathbb{N}.$$

This algorithm was first introduced by Martinet [12]. If $\psi : H \to \mathbb{R}\cup\{\infty\}$ is a proper lower semicontinuous convex function, then the algorithm reduces to

$$x_{n+1} = \operatorname{argmin}_{y\in H}\left\{\psi(y) + \frac{1}{r_n}\|x_n - y\|^2\right\} \quad \text{for all } n\in\mathbb{N}.$$

Rockafellar [15] studied the proximal point algorithm in the framework of Hilbert space and proved that if A is a maximal monotone operator with $A^{-1}0 \neq \emptyset$ and if $\{x_n\}$ is a sequence in H defined by (1.1), where $\{r_n\}$ is a sequence in $(0,\infty)$ such that $\liminf_{n\to\infty} r_n > 0$, then $\{x_n\}$ converges weakly to an element of $A^{-1}0$. Rockafellar [14] has given a more practical method which is an inexact variant of the method:

$$x_n + e_n \in x_{n+1} + r_nAx_{n+1} \quad \text{for all } n\in\mathbb{N}, \tag{1.2}$$

where $\{e_n\}$ is regarded as an error sequence and $\{r_n\}$ is a sequence of positive regularization parameters. Note that the algorithm (1.2) can be rewritten as

$$x_{n+1} = J^A_{r_n}(x_n + e_n) \quad \text{for all } n\in\mathbb{N}.$$

The method is called *inexact proximal point algorithm.* Güler [7] gave an example for which the sequence generated by (1.1) converges weakly but not strong. In order to modify the proximal point algorithm so that strongly convergent sequence is guaranteed, the *Tikhonov method* which generates a sequence $\{\tilde{x_n}\}$ by rule

$$\tilde{x}_n = J^A_{\mu_n} u \quad \text{for all } n \in \mathbb{N},$$

where $u \in H$ and $\mu_n > 0$ such that $\mu_n \to \infty$, is studied by several authors (see, e.q., Wong et al. [22]). The detail of Tikhonov regularization can be found in [4, 20, 21].

In [11], Lehdili and Moudafi introduced the *prox-Tikhonov method* which generates the sequence $\{x_n\}$ by the algorithm

$$x_{n+1} = J^{A_n}_{\lambda_n} x_n \quad \text{for all } n \in \mathbb{N}, \tag{1.3}$$

where $A_n = \mu_n I + A$, $\mu_n > 0$ is viewed as a Tikhonov regularization of A. Note that A_n is strongly monotone, i.e., $\langle x - x', y - y \rangle \geq \mu_n \|x - x'\|^2$ for all $(x, y), (x', y') \in G(A_n)$. Lehdili and Moudafi [11] proved strong convergence of the algorithm (1.3) for solving Problem (P) when A is maximal monotone operator on H under certain conditions imposed upon the sequences $\{\lambda_n\}$ and $\{\mu_n\}$. Note that A_n is a maximal monotone operator and hence (1.3) can be written as

$$x_n \in [(1 + \lambda_n \mu_n) I + \lambda_n A] x_{n+1} \quad \text{for all } n \in \mathbb{N}. \tag{1.4}$$

If $\lambda_n := \frac{r_n}{1-\alpha_n}$ and $\mu_n := \frac{\alpha_n}{r_n}$, then (1.4) reduces to

$$(1 - \alpha_n) x_n \in x_{n+1} + r_n A x_{n+1} \quad \text{for all } n \in \mathbb{N},$$

equivalently,

$$x_{n+1} = J^A_{r_n}((1 - \alpha_n) x_n) \quad \text{for all } n \in \mathbb{N}. \tag{1.5}$$

In [16], Sahu and Yao investigated a prox-Tikhonov method which converges strongly to solution of the Problem (P) in the framework of Banach space. In particular, as a prox-Tikhonov method, they proposed the following algorithm:

$$(1 - \alpha_n) x_n + \alpha_n f x_n \in x_{n+1} + r_n A x_{n+1} \quad \text{for all } n \in \mathbb{N},$$

equivalently,

$$x_{n+1} = J^A_{r_n}(\alpha_n f x_n + (1 - \alpha_n) x_n) \quad \text{for all } n \in \mathbb{N}. \tag{1.6}$$

where $f : C \to C$ is a contractive mapping (i.e., $\|fx - fy\| \leq k\|x - y\|$ for all $x,\ y \in C$ and some $k \in (0, 1)$) and $A \subset E \times E$ is an accretive operator such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \cap_{\lambda>0} R(I + \lambda A)$, and obtained strong convergence of the sequence $\{x_n\}$ generated by (1.6) to a zero of A in a reflexive Banach space with a uniformly Gâteaux differentiable norm. Jung [10] also study the algorithm (1.6) under the different control conditions in a uniformly convex Banach space having a weakly continuous duality mapping $\mathcal{J}_\varphi$ with gauge function φ.

In this paper, as the prox-Tikhonov regularization method for proximal point algorithm, we consider iterative algorithms for finding a zero for an accretive operator A in a reflexive Banach space having a weakly continuous duality mapping $\mathcal{J}_\varphi$ with gauge function φ. Under weaker control conditions than previous ones, we establish strong convergence of the sequence generated by the iterative algorithm (1.6) to a zero of A, which solves a certain variational inequality related to f. As an application, we study an iterative algorithm for inexact variant of the algorithm (1.6) with error sequence. As a continuation of study in this direction, our results can be viewed as improvement, complement and development of the corresponding results in [10, 11, 16, 18, 23] and the references therein.

2. Preliminaries and lemmas

Throughout the paper, we use notations: " $\rightharpoonup$ " for weak convergence, " $\overset{*}{\rightharpoonup}$ " for weak* convergence, and " $\to$ " for strong convergence.

Let E be a real Banach space with the norm $\|\cdot\|$, and let E^* be its dual. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^*\rangle$. The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists for each $x,\ y$ in its unit sphere $S = \{x \in E : \|x\| = 1\}$.

By a gauge function we mean a continuous strictly increasing function φ defined on $\mathbb{R}^+ := [0,\infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty}\varphi(r) = \infty$. The mapping $\mathcal{J}_\varphi : E \to 2^{E^*}$ defined by

$$\mathcal{J}_\varphi(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \varphi(\|x\|)\} \ \text{ for all } x \in E$$

is called the *duality mapping* with gauge function φ. In particular, the duality mapping with gauge function $\varphi(t) = t$, denoted by $\mathcal{J}$, is referred to as the *normalized duality mapping*. It is well known that a Banach space E is smooth if and only if the normalized duality mapping $\mathcal{J}$ is single-valued. The following property of duality mapping is also well-known:

$$\mathcal{J}_\varphi(\lambda x) = \operatorname{sign}\lambda\left(\frac{\varphi(|\lambda|\cdot\|x\|)}{\|x\|}\right)\mathcal{J}(x) \ \text{ for all } x \in E\setminus\{0\}, \quad \lambda\in\mathbb{R}, \tag{2.1}$$

where $\mathbb{R}$ is the set of all real numbers; in particular, $\mathcal{J}(-x) = -\mathcal{J}(x)$ for all $x \in E$ ([1, 6]).

We say that a Banach space E has a *weakly continuous duality mapping* if there exists a gauge function φ such that the duality mapping $\mathcal{J}_\varphi$ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, $\mathcal{J}_\varphi(x_n) \overset{*}{\rightharpoonup} \mathcal{J}_\varphi(x)$. For example, every l^p space $(1 < p < \infty)$ has a weakly continuous duality mapping with gauge function $\varphi(t) = t^{p-1}$ ([1, 6, 8]). Set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau \quad \text{for all } t \in \mathbb{R}^+.$$

Then it is known that $J_\varphi(x)$ is the subdifferential of the convex functional $\Phi(\|\cdot\|)$ at x, that is, $\mathcal{J}_\varphi(x) = \partial\Phi(\|x\|)$.

We need the following lemmas for the proof of our main result.

Lemma 2.1 ([1, 6])**.** *Let E be a Banach space having a weakly continuous duality mapping $\mathcal{J}_\varphi$ with gauge function φ. Define*

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau \ \text{ for all } t \in \mathbb{R}^+.$$

Then (i) *the following inequalities hold:*

$$\Phi(kt) \le k\Phi(t), \quad 0 < k < 1,$$

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, \mathcal{J}_\varphi(x+y)\rangle \text{ for all } x, y \in E.$$

(ii) *Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$. Then the following identity holds:*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|) \quad \text{for all } x,\ y \in E.$$

Lemma 2.2 ([24])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\delta_n + \gamma_n, \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$, $\{\delta_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *or, equivalently,* $\prod_{n=1}^{\infty}(1-\alpha_n) = 0$;
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} \alpha_n|\delta_n| < \infty$,
(iii) $\gamma_n \geq 0$ *for all* $n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

An accretive operator A defined on a Banach space E is said to satisfy the *range condition* if $\overline{D(A)} \subset R(I+\lambda A)$ for all $\lambda > 0$, where $\overline{D(A)}$ denotes the closure of the domain of A. It is well known that for an accretive operator A which satisfies the range condition, $A^{-1}0 = Fix(J_\lambda^A)$ for all $\lambda > 0$ (set of fixed points of J_λ^A). We also know that if A is an m-accretive operator on a Banach space E, then for each $\lambda > 0$, the resolvent $J_\lambda^A = (I+\lambda A)^{-1}$ is a single-valued nonexpansive mapping whose domain is the entire space E. Let C be a closed convex subset of a Banach space E and let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \cap_{\lambda>0} R(I+\lambda A)$. From Takahashi [19], we know that J_r^A is a nonexpansive mapping of C into itself and $\mathrm{Fix}(J_r^A) = A^{-1}0$ for each $r > 0$.

Lemma 2.3 (The Resolvent Identity ([6]))**.** *For $\lambda,\ \mu > 0$,*

$$J_\lambda = J_\mu\left(\frac{\mu}{\lambda}I + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda\right).$$

Let LIM be a continuous linear functional on l^∞ and $(a_1, a_2, \dots) \in l^\infty$. We write $LIM_n(a_n)$ instead of $LIM((a_1, a_2, \dots))$. LIM is said to be *Banach limit* if LIM satisfies $\|LIM\| = LIM_n(1) = 1$ and $LIM_n(a_{n+1}) = LIM_n(a_n)$ for all $(a_1, a_2, \dots) \in l^\infty$. If LIM is a Banach limit, the following are well-known ([1]):

(i) for all $n \geq 1, a_n \leq c_n$ implies $LIM_n(a_n) \leq LIM_n(c_n)$,
(ii) $LIM_n(a_{n+N}) = LIM_n(a_n)$ for any fixed positive integer N,
(iii) $\liminf_{n\to\infty} a_n \leq LIM_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $(a_1, a_2, \dots) \in l^\infty$.

Lemma 2.4 ([17])**.** *Let $a \in \mathbb{R}$ be a real number and a sequence $\{a_n\} \in l^\infty$ satisfy the condition $LIM_n(a_n) \leq a$ for all Banach limit LIM. If* $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, *then* $\limsup_{n\to\infty} a_n \leq a$.

Finally, we recall that the sequence $\{x_n\}$ in E is said to be *weakly asymptotically regular* if

$$w - \lim_{n\to\infty}(x_{n+1} - x_n) = 0, \quad \text{that is,} \quad x_{n+1} - x_n \rightharpoonup 0$$

and *asymptotically regular* if

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

respectively.

3. Main results

In our first result, we utilize contractive mappings as the Tikhonov regularization of resolvent J_r^A of accretive operator A. Define an operator $Q_t : C \to C$ by

$$Q_t v := J_r^A(tfv + (1-t)v) \quad \text{for all } v \in C,$$

where $t \in (0,1)$, $r > 0$ is a fixed constant and $f : C \to C$ is a contractive mapping with contractive constant $k \in (0,1)$. Then Q_t is a contractive mapping with contractive constant $1-(1-k)t$. By the Banach contraction principle, for every $t \in (0,1)$, there exists a unique fixed point $v_t \in C$ of the contractive mapping Q_t defined by

$$v_t = J_r^A(tfv_t + (1-t)v_t) \tag{3.1}$$

Theorem 3.1. *Let E be a reflexive Banach space having a weakly continuous duality mapping $\mathcal{J}_\varphi$ with gauge function φ and let C be a closed convex subset of E. Let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \cap_{\lambda>0} R(I+\lambda A)$ and let $f : C \to C$ be a contractive mapping with contractive constant $k \in (0,1)$. Let $\{v_t\}$ be a path defined by (3.1). Then $\{v_t\}$ converges strongly as $t \to 0$ to $\tilde{x} \in A^{-1}0$, which is the unique solution of the variational inequality:*

$$\langle (I-f)\tilde{x}, \mathcal{J}_\varphi(\tilde{x}-z)\rangle \le 0 \quad \text{for all } z \in A^{-1}0. \tag{3.2}$$

Proof. We first show the uniqueness of a solution of the variational inequality (3.2). Suppose that both $\tilde{x}$ and $x^* \in A^{-1}0$ are solution to (3.2). Then

$$\langle (I-f)\tilde{x}, \mathcal{J}_\varphi(\tilde{x}-x^*)\rangle \le 0 \tag{3.3}$$

and

$$\langle (I-f)x^*, \mathcal{J}_\varphi(x^*-\tilde{x})\rangle \le 0 \tag{3.4}$$

Adding up (3.3) and (3.4), we obtain

$$\langle (I-f)\tilde{x} - (I-f)x^*, \mathcal{J}_\varphi(\tilde{x}-x^*)\rangle \le 0.$$

Noticing that for any $x, y \in E$

$$\begin{aligned}\langle (I-f)x-(I-f)y, \mathcal{J}_\varphi(x-y)\rangle &= \langle x-y, \mathcal{J}_\varphi(x-y)\rangle - \langle fx-fy, \mathcal{J}_\varphi(x-y)\rangle \\ &\ge \|x-y\|\varphi(\|x-y\|) - k\|x-y\|\varphi(\|x-y\|) \\ &= \Phi(\|x-y\|) - k\Phi(\|x-y\|) \\ &= (1-k)\Phi(\|x-y\|) \ge 0,\end{aligned}$$

we have $\tilde{x} = x^*$ and uniqueness is proved. Below we use $\tilde{x}$ to denote the unique solution of the variational inequality (3.2).

Next, from (3.1), we obtain $\frac{t(fv_t - v_t)}{r} \in Av_t$ and hence

$$\langle v_t - fv_t, \mathcal{J}_\varphi(v_t - z)\rangle \le 0 \quad \text{for all } z \in A^{-1}0 \text{ and } t \in (0,1). \tag{3.5}$$

Take $p \in A^{-1}0$. Using (3.5), we induce

$$\begin{aligned}\|v_t - p\|\varphi(\|v_t - p\|) &= \langle v_t - p, \mathcal{J}_\varphi(v_t - p)\rangle \\ &= \langle v_t - fv_t + fv_t - p, \mathcal{J}_\varphi(v_t - p)\rangle \\ &= \langle v_t - fv_t, \mathcal{J}_\varphi(v_t - p)\rangle + \langle fv_t - p, \mathcal{J}_\varphi(v_t - p)\rangle \\ &\le \langle fv_t - p, \mathcal{J}_\varphi(v_t - p)\rangle \le \|fv_t - p\|\varphi(\|v_t - p\|),\end{aligned}$$

and so

$$\begin{aligned}\|v_t - p\| \le \|fv_t - p\| &\le \|fv_t - fp\| + \|fp - p\| \\ &\le k\|v_t - p\| + \|fp - p\|,\end{aligned}$$

which implies

$$\|v_t - p\| \le \frac{1}{1-k}\|fp - p\|.$$

Thus $\{v_t\}$ is bounded and $\{fv_t\}$ is bounded. By boundedness of $\{v_t\}$ and$\{fv_t\}$, one can easily see

$$\begin{aligned}\|v_t - J_r^A v_t\| =&\|J_r^A(tfv_t + (1-t)v_t) - J_c^A v_t\| \\ \le&\|tfv_t + (1-t)v_t - v_t\| = t\|fv_t - v_t\| \to 0 \quad \text{as } t \to 0.\end{aligned}$$

It follows from reflexivity of E and the boundedness of sequence $\{v_t\}$ that there exists $\{v_{t_n}\}$ which is a subsequence of $\{v_t\}$ converges weakly to $w \in C$ as $n \to \infty$. Since $\mathcal{J}_\varphi$ is weakly continuous, we have by Lemma 2.1(ii) that

$$\limsup_{n\to\infty} \Phi(\|v_{t_n} - x\|) = \limsup_{n\to\infty} \Phi(\|v_{t_n} - w\|) + \Phi(\|x - w\|) \quad \text{for all } x \in C.$$

Let

$$H(x) = \limsup_{n\to\infty} \Phi(\|v_{t_n} - x\|) \quad \text{for all } x \in C.$$

Then it follows that

$$H(x) = H(w) + \Phi(\|x - w\|) \quad \text{for all } x \in C.$$

Since $\|v_{t_n} - J_r^A v_{t_n}\| \le t_n\|fv_{t_n} - v_{t_n}\| \to 0$ as $n \to \infty$, we obtain

$$\begin{aligned}H(J_c^H w) =& \limsup_{n\to\infty} \Phi(\|v_{t_n} - J_r^A w\|) \\ =& \limsup_{n\to\infty} \Phi(\|J_r^A v_{t_n} - J_r^A w\|) \\ \le& \limsup_{n\to\infty} \Phi(\|v_{t_n} - w\|) = H(w).\end{aligned} \tag{3.6}$$

On the other hand, however

$$H(J_r^A w) = H(w) + \Phi(\|J_r^A w - w\|). \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\Phi(\|J_r^A w - w\|) = H(J_r^A w) - H(w) \le 0.$$

This implies that $J_r^A w = w$, that is, $w \in A^{-1}0$.

Next, we show that $v_{t_n} \to w$ as $n \to \infty$. In fact using Lemma 2.1 (i) and (3.5), we derive

$$\begin{aligned}\Phi(\|v_{t_n} - w\|) &= \Phi(\|v_{t_n} - fv_{t_n} + fv_{t_n} - fw + fw - w\|)\\ &\le \Phi(\|fv_{t_n} - fw\|) + \langle v_{t_n} - fv_{t_n} + fw - w, \mathcal{J}_\varphi(v_{t_n} - w)\rangle\\ &\le \Phi(k\|v_{t_n} - w\|) + \langle v_{t_n} - fv_{t_n}, \mathcal{J}_\varphi(v_{t_n} - w)\rangle + \langle fw - w, \mathcal{J}_\varphi(v_{t_n} - w)\rangle\\ &\le k\Phi(\|v_{t_n} - w\|) + \langle fw - w, \mathcal{J}_\varphi(v_{t_n} - w)\rangle.\end{aligned}$$

This implies that

$$\Phi(\|v_{t_n} - w\|) \le \frac{1}{1-k}\langle fw - w, \mathcal{J}_\varphi(v_{t_n} - w)\rangle. \tag{3.8}$$

Now observing that $v_{t_n} \rightharpoonup w$ implies $\mathcal{J}_\varphi(v_{t_n} - w) \overset{*}{\rightharpoonup} 0$, we conclude from the last inequality (3.8) that

$$\Phi(\|v_{t_n} - w\|) \to 0 \quad \text{as } n \to \infty.$$

Hence $v_{t_n} \to w$ as $n \to \infty$.

Moreover, we have from (3.5)

$$\langle (I - f)w, \mathcal{J}_\varphi(w - z)\rangle \le 0 \quad \text{for all } z \in A^{-1}0.$$

So, $w \in A^{-1}0$ is a solution of the variational inequality (3.2) and hence $w = \tilde{x}$ by the uniqueness.

In a summary, we have shown that each cluster point of $\{v_t\}$ as $t \to 0$ equals $\tilde{x}$. This completes the proof. □

In the proofs of the next theorems, we need the following result for the existence of solutions of a certain variational inequality, which can be obtained by the similar argument as in [5, 9] and the argument of proof of Theorem 3.1. We omit its proof.

Theorem 3.2. *Let E be a reflexive Banach space having a weakly continuous duality mapping $\mathcal{J}_\varphi$ with gauge function φ and let C be a closed convex subset of E. Let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \cap_{\lambda>0} R(I + \lambda A)$ and let $f : C \to C$ be a contractive mapping with contractive constant $k \in (0, 1)$. Then for fixed $r > 0$ and every $t \in (0, 1)$, there exists the unique fixed point $z_t \in C$ of the contractive mapping $C \ni z \mapsto tfz + (1-t)J_r^A z$ defined by*

$$z_t = tfz_t + (1-t)J_r^A z_t, \tag{3.9}$$

and the path $\{z_t\}$ defined by (3.9) converges strongly as $t \to 0$ to $q \in A^{-1}0$, which is the unique solution of the variational inequality:

$$\langle (I - f)q, \mathcal{J}_\varphi(q - p)\rangle \le 0 \quad \text{for all } p \in A^{-1}0.$$

Motivated by algorithm (1.5) and Theorem 3.1, we present the prox-Tikhonov method for solving Problem (P) in the Banach space setting. As in [16], our prox-Tikhonov method is defined to generate a sequence $\{x_n\}$ in C as follows: for $x_1 \in C$,

$$x_{n+1} = J_{r_n}^A((1 - \alpha_n)x_n + \alpha_n f x_n) \quad \text{for all } n \in \mathbb{N}, \tag{3.10}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$ and $\{r_n\}$ is a regularization sequence in $(0, \infty)$.

We consider our prox-Tikhonov method under the conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\lim_{n\to\infty} r_n = r > 0$.

First of all, using Theorem 3.2, we give the following result.

Theorem 3.3. *Let E be a reflexive Banach space having a weakly continuous duality mapping $\mathcal{J}_\varphi$ with gauge function φ and let C be a closed convex subset of E. Let $A \subset E\times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \cap_{\lambda>0} R(I+\lambda A)$ and let $f : C \to C$ be a contractive mapping with contractive constant $k \in (0,1)$. Let $\{x_n\}$ be a sequence in C generated by* (3.10)*, where $\{\alpha_n\}$ is a sequence in* $(0,1)$ *and $\{r_n\}$ is a regularization sequence in* $(0,\infty)$ *satisfying conditions* (C1) *and* (C3). *Let LIM be a Banach limit. Then*

$$LIM_n(\langle (I-f)q, \mathcal{J}_\varphi(q-x_n)\rangle) \le 0,$$

where $q := \lim_{t\to 0^+} z_t$ with z_t being defined by $z_t = tfz_t + (1-t)J_r^A z_t$ for $r = \lim_{n\to\infty} r_n$.

Proof. Let $z_t = tfz_t + (1-t)J_r^A z_t$ for any $t \in (0,1)$. Then, by Theorem 3.2, we know that $\{z_t\}$ is bounded and $\{z_t\}$ converges strongly to a point in $A^{-1}0$ as $t \to 0$, which is denoted by $q := \lim_{t\to 0} z_t$.

From now, let $y_n = \alpha_n f x_n + (1-\alpha_n)x_n$ for all $n \in \mathbb{N}$.

First, we show that $\{x_n\}$ is bounded. In fact, since $A^{-1}0 \neq \emptyset$, we take $p \in A^{-1}0 = F(J_\lambda^A)$ for all $\lambda > 0$. Noting that

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n(fx_n - p) + (1-\alpha_n)(x_n - p)\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|fx_n - p\|, \end{aligned} \tag{3.11}$$

from (3.10), (3.11) and the nonexpansivity of $J_{r_n}^A$ for all n, we obtain by induction

$$\begin{aligned} \|x_{n+1} - p\| &= \|J_{r_n}^A y_n - p\| \\ &\le \|y_n - p\| \\ &\le [\alpha_n\|fx_n - fp\| + (1-\alpha_n)\|x_n - p\| + \alpha_n\|fp - p\|] \\ &\le [\alpha_n k\|x_n - p\| + (1-\alpha_n)\|x_n - p\| + \alpha_n\|fp - p\|] \\ &= (1-(1-k)\alpha_n)\|x_n - p\| + (1-k)\alpha_n \frac{\|fp-p\|}{1-k} \\ &\le \max\left\{\|x_n - p\|, \frac{\|fp-p\|}{1-k}\right\} \\ &\quad \cdots\cdots\cdots \\ &\le \max\left\{\|x_1 - p\|, \frac{\|fp-p\|}{1-k}\right\}. \end{aligned} \tag{3.12}$$

Hence $\{x_n\}$ is bounded. From (3.12), it also follows that $\{y_n\}$ and $\{fx_n\}$ are bounded.

Now, we show that $LIM_n(\langle (I-f)q, \mathcal{J}_\varphi(q-x_n)\rangle) \le 0$, where $q = \lim_{t\to 0^+} z_t$ with z_t being defined by $z_t = tfz_t + (1-t)J_r^A z_t$ for $r > 0$. Indeed, since

$$\begin{aligned} z_t - x_{n+1} &= (1-t)(J_r^A z_t - x_{n+1}) + t(fx_t - x_{n+1}) \\ &= (1-t)(J_r^A z_t - J_r^A y_n) + (1-t)(J_r^A y_n - J_{r_n}^A y_n) \\ &\quad + t(fz_t - x_{n+1}), \end{aligned} \tag{3.13}$$

applying Lemma 2.1 (i) with (3.13), we have

$$\begin{aligned} \Phi(\|z_t - x_{n+1}\|) &\le \Phi((1-t)\|J_r^A z_t - x_{n+1}\|) \\ &\quad + t\langle fx_t - x_{n+1}, \mathcal{J}_\varphi(x_t - x_{n+1})\rangle \\ &\le \Phi((1-t)(\|J_r^A z_t - J_r^A y_n\| + \|J_r^A y_n - J_{r_n}^A y_n\|)) \\ &\quad + t\langle fx_t - x_{n+1}, \mathcal{J}_\varphi(x_t - x_{n+1})\rangle. \end{aligned} \tag{3.14}$$

We note from Lemma 2.3 that

$$\begin{aligned} \|J_r^A y_n - J_{r_n}^A y_n\| &= \left\| J_r^A\left(\frac{r}{r_n} y_n + \left(1 - \frac{r}{r_n}\right) J_{r_n}^A y_n\right) - J_r^A y_n \right\| \\ &\le \left\| \left(\frac{r}{r_n} y_n + \left(1 - \frac{r}{r_n}\right) J_{r_n}^A y_n\right) - y_n \right\| \\ &\le \left|1 - \frac{r}{r_n}\right| \|y_n - J_{r_n}^A y_n\| \\ &\le \frac{|r_n - r|}{\varepsilon}(\|y_n\| + \|J_{r_n}^A y_n\|) \\ &\le \frac{|r_n - r|}{\varepsilon} M_1, \end{aligned} \tag{3.15}$$

where $r_n \ge \varepsilon$ for some $\varepsilon > 0$ and some constant $M_1 > 0$. Also we observe that

$$\begin{aligned} \|z_t - y_n\| &= \|z_t - (\alpha_n fx_n + (1-\alpha_n)x_n)\| \\ &\le \|z_t - x_n\| + \alpha_n\|x_n - fx_n\| \\ &\le \|z_t - x_n\| + \alpha_n(\|x_n\| + \|fx_n\|) \\ &\le \|z_t - x_n\| + \alpha_n M_2, \end{aligned} \tag{3.16}$$

where some constant $M_2 > 0$. From (3.15) and (3.16), we derive

$$\begin{aligned} \|J_r^A z_t - x_{n+1}\| &= \|J_r^A z_t - J_{r_n}^A y_n\| \\ &\le \|J_r^A z_t - J_r^A y_n\| + \|J_r^A y_n - J_{r_n}^A y_n\| \\ &\le \|z_t - y_n\| + \|J_r^A y_n - J_{r_n}^A y_n\| \\ &\le \|z_t - x_n\| + \alpha_n M_2 + \frac{|r_n - r|}{\varepsilon} M_1 \\ &= \|z_t - x_n\| + \varepsilon_n, \end{aligned} \tag{3.17}$$

where $\varepsilon_n = \alpha_n M_2 + \frac{|r_n - r|}{\varepsilon} M_1 \to 0$ as $n \to \infty$ (by conditions (C1) and (C3)). Moreover, we know that

$$\begin{aligned} \langle fz_t - x_{n+1}, \mathcal{J}_\varphi(z_t - x_{n+1})\rangle &= \langle fz_t - z_t, \mathcal{J}_\varphi(z_t - x_{n+1})\rangle \\ &\quad + \|z_t - x_{n+1}\|\varphi(\|z_t - x_{n+1}\|). \end{aligned} \tag{3.18}$$

Hence, by (3.14), (3.17) and (3.18), we derive

$$
\begin{aligned}
\Phi(\|z_t - x_{n+1}\|) &\le \Phi((1-t)(\|z_t - x_n\| + \varepsilon_n)) + t\langle fz_t - z_t, \mathcal{J}_\varphi(z_t - x_{n+1})\rangle \\
&\quad + t\|z_t - x_{n+1}\|\varphi(\|z_t - x_{n+1}\|) \\
&\le \Phi((1-t)\|z_t - x_n\|) + \varepsilon_n\varphi(\|z_t - x_n\| + \varepsilon_n) \\
&\quad + t\langle fz_t - z_t, \mathcal{J}_\varphi(z_t - x_{n+1})\rangle \\
&\quad + t\|z_t - x_{n+1}\|\varphi(\|z_t - x_{n+1}\|).
\end{aligned} \tag{3.19}
$$

Applying the Banach limit LIM to (3.19) with $\lim_{n\to\infty}\varepsilon_n\varphi(\|z_t - x_n\| + \varepsilon_n) = 0$, we have

$$
\begin{aligned}
&LIM_n(\Phi(\|z_t - x_{n+1}\|)) \\
&\le LIM_n(\Phi((1-t)\|z_t - x_n\|) + LIM_n(\varepsilon_n\varphi(\|z_t - x_n\| + \varepsilon_n)) \\
&\quad + tLIM_n(\langle fz_t - z_t, \mathcal{J}_\varphi(z_t - x_{n+1})\rangle) \\
&\quad + tLIM_n(\|z_t - x_{n+1}\|\varphi(\|z_t - x_{n+1}\|)) \\
&\le LIM_n(\Phi((1-t)\|z_t - x_n\|) + tLIM_n(\langle fz_t - z_t, \mathcal{J}_\varphi(z_t - x_{n+1})\rangle) \\
&\quad + tLIM_n(\|z_t - x_{n+1}\|\varphi(\|z_t - x_{n+1}\|)).
\end{aligned} \tag{3.20}
$$

Hence, using the property $LIM_n(a_{n+1}) = LIM_n(a_n)$ of Banach limit LIM to (3.20), we obtain

$$
\begin{aligned}
&LIM_n(\langle z_t - fz_t, \mathcal{J}_\varphi(z_t - x_n)\rangle) \\
&\le \frac{1}{t}LIM_n(\Phi((1-t)\|z_t - x_n\|) - \Phi(\|z_t - x_n\|)) \\
&\quad + LIM_n(\|z_t - x_n\|\varphi(\|z_t - x_n\|)) \\
&= -\frac{1}{t}LIM_n\left(\int_{(1-t)\|z_t - x_n\|}^{\|z_t - x_n\|}\varphi(\tau)d\tau\right) + LIM_n(\|z_t - x_n\|\varphi(\|z_t - x_n\|)) \\
&= LIM_n(\|z_t - x_n\|(\varphi(\|z_t - x_n\|) - \varphi(\theta_n)))
\end{aligned} \tag{3.21}
$$

for some θ_n satisfying $(1-t)\|z_t - x_n\| \le \theta_n \le \|z_t - x_n\|$. Since φ is uniformly continuous on compact intervals on $\mathbb{R}^+$ and

$$
\begin{aligned}
\|z_t - x_n\| - \theta_n &\le t\|z_t - x_n\| \\
&\le t\left(\frac{2}{1-k}\|fp - p\| + \|x_1 - p\|\right) \to 0 \quad \text{as } t \to 0,
\end{aligned}
$$

we conclude from (3.21) and $q = \lim_{t\to 0^+} z_t$ that

$$
\begin{aligned}
&LIM_n(\langle (I-f)q, \mathcal{J}_\varphi(q - x_n)\rangle) \\
&\qquad \le \limsup_{t\to 0} LIM_n(\langle z_t - fz_t, \mathcal{J}_\varphi(z_t - x_n)\rangle) \\
&\qquad \le \limsup_{t\to 0} LIM_n(\|z_t - x_n\|(\varphi(\|z_t - x_n\|) - \varphi(\theta_n))) \le 0.
\end{aligned}
$$

This completes the proof. □

Using Theorem 3.3, we establish the main theorem for solving the Problem (P) in the Banach space setting.

Theorem 3.4. *Let E be a reflexive Banach space having a weakly continuous duality mapping $\mathcal{J}_\varphi$ with gauge function φ and let C be a closed convex subset of E. Let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \cap_{\lambda>0} R(I+\lambda A)$ and let $f : C \to C$ be a contractive mapping with contractive constant $k \in (0,1)$. Let $\{x_n\}$ be a sequence in C generated by (3.10), where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{r_n\}$ is a regularization sequence in $(0,\infty)$ satisfying conditions (C1), (C2) and (C3). If $\{x_n\}$ is weakly asymptotically regular, then $\{x_n\}$ converges strongly to $q \in A^{-1}0$, which is the unique solution of the variational inequality:*

$$\langle (I-f)q, \mathcal{J}_\varphi(q-p)\rangle \leq 0 \quad \text{for all } p \in A^{-1}0. \tag{3.22}$$

Proof. First, we note that by Theorem 3.2, there exists the unique solution q of the variational inequality (3.22), where $q := \lim_{t\to\infty} z_t \in A^{-1}0$ and z_t is defined by $z_t = tfz_t + (1-t)J_r^A z_t$ for $t \in (0,1)$. Let $y_n = \alpha_n f x_n + (1-\alpha_n)x_n$ for all $n \in \mathbb{N}$.

Now, we divide the proof into four steps.

Step 1. We obtain $\|x_n - p\| \leq \max\{\|x_1 - p\|, \|fp - p\|/(1-k)\}$ for all $n \geq 1$ and $p \in A^{-1}0$ as in the proof of Theorem 3.3, and hence $\{x_n\}$, $\{y_n\}$ and $\{fx_n\}$ are bounded. As a consequence, it follows from condition (C1) that

$$\|y_n - x_n\| = \alpha_n\|fx_n - x_n\| \leq \alpha_n(\|fx_n\| + \|x_n\|) \to 0 \quad \text{as } n \to \infty. \tag{3.23}$$

Step 2. We show that $\limsup_{n\to\infty}\langle (I-f)q, \mathcal{J}_\varphi(q-x_n)\rangle \leq 0$. To this end, put $a_n := \langle (I-f)q, J_\varphi(q-x_n)\rangle$ for all $n \geq 1$. Then, Theorem 3.3 implies that $LIM_n(a_n) \leq 0$ for any Banach limit LIM. Since $\{x_n\}$ is bounded, Since $\{x_n\}$ bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \lim_{j\to\infty}(a_{n_j+1} - a_{n_j})$$

and $x_{n_j} \rightharpoonup z \in E$. Since $\{x_n\}$ is weakly asymptotically regular, this implies that $x_{n_j+1} \rightharpoonup z$. Moreover, from the weak continuity of duality mapping $\mathcal{J}_\varphi$, we have

$$w - \lim_{j\to\infty}\mathcal{J}_\varphi(q - x_{n_j+1}) = w - \lim_{j\to\infty}\mathcal{J}_\varphi(q - x_{n_j}) = \mathcal{J}_\varphi(q-z),$$

and so

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \lim_{j\to\infty}\langle (I-f)q, \mathcal{J}_\varphi(q - x_{n_j+1}) - \mathcal{J}_\varphi(q - x_{n_j})\rangle = 0.$$

Thus, by Lemma 2.4, we obtain $\limsup_{n\to\infty} a_n \leq 0$, that is,

$$\limsup_{n\to\infty}\langle (I-f)q, \mathcal{J}_\varphi(q-x_n)\rangle \leq 0.$$

Step 3. We show that $\limsup_{n\to\infty}\langle (I-f)q, \mathcal{J}_\varphi(q-y_n)\rangle \leq 0$. In fact, let $\{y_{n_i}\}$ be a subsequence of $\{y_n\}$ such that $y_{n_i} \rightharpoonup v \in E$ and

$$\limsup_{n\to\infty}\langle (I-f)q, \mathcal{J}_\varphi(q-y_n)\rangle = \lim_{i\to\infty}\langle (I-f)q, \mathcal{J}_\varphi(q-y_{n_i})\rangle.$$

Since $\lim_{n\to\infty}\|x_n - y_n\| = 0$ by (3.23), we have also $x_{n_i} \rightharpoonup v$. From the weak continuity of J_φ, it follows that

$$w - \lim_{i\to\infty}\mathcal{J}_\varphi(q - y_{n_i}) = w - \lim_{i\to\infty}\mathcal{J}_\varphi(q - x_{n_i}) = J_\varphi(q-v).$$

Hence, by Step 2, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle (I-f)q, \mathcal{J}_\varphi(q-y_n)\rangle &= \lim_{i\to\infty}\langle (I-f)q, \mathcal{J}_\varphi(q-y_{n_i})-\mathcal{J}_\varphi(q-x_{n_i})\rangle \\ &\quad + \lim_{i\to\infty}\langle (I-f)q, \mathcal{J}_\varphi(q-x_{n_i})\rangle \\ &= \lim_{i\to\infty}\langle (I-f)q, \mathcal{J}_\varphi(q-x_{n_i})\rangle \\ &\le \limsup_{n\to\infty}\langle (I-f)q, \mathcal{J}_\varphi(q-x_n)\rangle \le 0.\end{aligned}$$

Step 4. We show that $\lim_{n\to\infty}\|x_n-q\|=0$. Indeed, we derive from Lemma 2.1 (i) that

$$\begin{aligned}\Phi(\|y_n-q\|) &= \Phi(\|(1-\alpha_n)(x_n-q)+\alpha_n(fx_n-fq)+\alpha_n(fq-q)\|) \\ &\le \Phi(\|(1-\alpha_n)(x_n-q)+\alpha_n(fx_n-fq)\|)+\alpha_n\langle fq-q, \mathcal{J}_\varphi(y_n-q)\rangle \\ &\le \Phi((1-\alpha_n)\|x_n-q\|+\alpha_n k\|x_n-q\|)+\alpha_n\langle fq-q, \mathcal{J}_\varphi(y_n-q)\rangle \\ &= \Phi((1-(1-k)\alpha_n)\|x_n-q\|)+\alpha_n\langle fq-q, \mathcal{J}_\varphi(y_n-q)\rangle \\ &\le (1-(1-k)\alpha_n)\Phi(\|x_n-q\|)+\alpha_n\langle (I-f)q, \mathcal{J}_\varphi(q-y_n)\rangle.\end{aligned}$$

Hence, from (3.10), we obtain

$$\begin{aligned}\Phi(\|x_{n+1}-q\|) &= \Phi(\|J^A_{r_n}y_n-q\|) \\ &\le \Phi(\|y_n-q\|) \\ &\le (1-(1-k)\alpha_n)\Phi(\|x_n-q\|)+\alpha_n\sigma_n,\end{aligned}$$

where $\sigma_n=\langle (I-f)q, \mathcal{J}_\varphi(q-y_n)\rangle$. Note $\sum_{n=1}^{\infty}\alpha_n=\infty$ by condition (C2) and $\limsup_{n\to\infty}\sigma_n\le 0$ by Step 3. Therefore, we conclude from Lemma 2.2 with $\gamma_n=0$ that $\lim_{n\to\infty}\Phi(\|x_n-q\|)=0$, and hence $\lim_{n\to\infty}\|x_n-q\|=0$. This complete the proof. □

Corollary 3.5. *Let E, $\mathcal{J}_\varphi$, C, A and f be as in Theorem 3.4. Let $\{x_n\}$ be a sequence in C generated by* (3.10)*, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{r_n\}$ is a regularization sequence in $(0,\infty)$ satisfying conditions* (C1)*,* (C2) *and* (C3)*. If $\{x_n\}$ is asymptotically regular, then $\{x_n\}$ converges strongly to $q\in A^{-1}0$, which is the unique solution of the variational inequality* (3.22)*.*

Remark 3.6. If $\{\alpha_n\}$ and $\{r_n\}$ in Corollary 3.5 satisfy the following additional conditions:

(C4) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$; or $\lim_{n\to\infty}\frac{|\alpha_{n+1}-\alpha_n|}{\alpha_{n+1}}=0$; or
(C5) $|\alpha_{n+1}-\alpha_n|\le o(\alpha_{n+1})+\sigma_n$, $\sum_{n=1}^{\infty}\sigma_n<\infty$ (the perturbed control condition);
(C6) $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$,

then the sequence $\{x_n\}$ generated by (3.10) in Theorem 3.4 is asymptotically regular.

Now we give only the proof in case when $\{\alpha_n\}$ and $\{r_n\}$ satisfy the conditions (C2), (C3), (C5) and (C6). First, let $y_n=\alpha_n fx_n+(1-\alpha_n)x_n$ for all $n\ge 1$. Then, from Theorem 3.4, we note that $\{x_n\}$, $\{y_n\}$ and $\{fx_n\}$ are bounded. In order to show that $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$, we observe that

$$
\begin{aligned}
&\|y_n - y_{n-1}\| \\
&= \|(1-\alpha_n)x_n + \alpha_n f x_n - (1-\alpha_{n-1})x_{n-1} - \alpha_{n-1} f x_{n-1}\| \\
&= \|(1-\alpha_n)x_n - (1-\alpha_n)x_{n-1} - (1-\alpha_n)x_{n-1} + \alpha_n(f x_n - f x_{n-1}) \\
&\qquad + \alpha_n f x_{n-1} - (1-\alpha_{n-1})x_{n-1} - \alpha_{n-1} f x_{n-1}\| \\
&\le (1-\alpha_n)\|x_n - x_{n-1}\| + \alpha_n k\|x_n - x_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}|(\|x_{n-1}\| + \|f x_{n-1}\|) \\
&\le (1-(1-k)\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|K_1 \\
&\le (1-(1-k)\alpha_n)\|x_n - x_{n-1}\| + (o(\alpha_n) + \sigma_{n-1})K_1
\end{aligned} \tag{3.24}
$$

for some constant $K_1 > 0$. Let $K_2 > 0$ be a constant such that $\|J^A_{r_n} y_n - y_n\| \le K_2$ for all $n \in \mathbb{N}$. Without loss of generality, we may assume that $r_n \ge \varepsilon$ for all $n \in \mathbb{N}$ and for some $\varepsilon > 0$. If $r_n \le r_{n+1}$, then by Lemma 2.3, we obtain

$$J^A_{r_n} y_n = J^A_{r_{n-1}}\left(\frac{r_{n-1}}{r_n} y_n + \left(1 - \frac{r_{n-1}}{r_n}\right) J^A_{r_n} y_n\right)$$

and hence

$$
\begin{aligned}
\|J^A_{r_n} y_n - J^A_{r_{n-1}} y_{n-1}\| &\le \left\|\frac{r_{n-1}}{r_n}(y_n - y_{n-1}) + \left(1 - \frac{r_{n-1}}{r_n}\right)(J^A_{r_n} y_n - y_{n-1})\right\| \\
&= \left\|\frac{r_{n-1}}{r_n}(y_n - y_{n-1}) + \left(1 - \frac{r_{n-1}}{r_n}\right)(y_n - y_{n-1}) \right. \\
&\qquad \left. + \left(1 - \frac{r_{n-1}}{r_n}\right)(J^A_{r_n} y_n - y_n)\right\| \\
&\le \|y_n - y_{n-1}\| + \frac{r_n - r_{n-1}}{\varepsilon} K_2.
\end{aligned}
$$

If $r_n > r_{n+1}$, then again by Lemma 2.3,

$$J^A_{r_{n-1}} y_{n-1} = J^A_{r_n}\left(\frac{r_n}{r_{n-1}} y_{n-1} + \left(1 - \frac{r_n}{r_{n-1}}\right) J^A_{r_{n-1}} y_{n-1}\right),$$

and hence

$$
\begin{aligned}
\|J^A_{r_n} y_n - J^A_{r_{n-1}} y_{n-1}\| &\le \left\|\frac{r_n}{r_{n-1}}(y_{n-1} - y_n) + \left(1 - \frac{r_n}{r_{n-1}}\right)(J^A_{r_{n-1}} y_{n-1} - y_n)\right\| \\
&= \left\|\frac{r_n}{r_{n-1}}(y_{n-1} - y_n) + \left(1 - \frac{r_n}{r_{n-1}}\right)(y_{n-1} - y_n) \right. \\
&\qquad \left. + \left(1 - \frac{r_n}{r_{n-1}}\right)(J^A_{r_{n-1}} y_{n-1} - y_{n-1})\right\| \\
&\le \|y_n - y_{n-1}\| + \frac{r_{n-1} - r_n}{\varepsilon} K_2.
\end{aligned}
$$

Hence, from the above inequality, we have

$$\|J^A_{r_n} y_n - J^A_{r_{n-1}} y_{n-1}\| \le \|y_n - y_{n-1}\| + \frac{|r_n - r_{n-1}|}{\varepsilon} K_2.$$

It follows from (3.10) and (3.24) that

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|J_{r_n}^A y_n - J_{r_{n-1}}^A y_{n-1}\|\\ &\le \|y_n - y_{n-1}\| + \frac{|r_n - r_{n-1}|}{\varepsilon}K_2\\ &\le (1-(1-k)\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|K_1 + \frac{|r_n - r_{n-1}|}{\varepsilon}K_2\\ &\le (1-(1-k)\alpha_n)\|x_n - x_{n-1}\| + (o(\alpha_n) + \sigma_{n-1})K_1 + \frac{|r_n - r_{n-1}|}{\varepsilon}K_2.\end{aligned}$$

Using conditions (C2), (C3), (C5), (C6) and Lemma 2.2, we obtain $\|x_{n+1}-x_n\| \to 0$.

In view of these observations, by Corollary 3.5, we have following results:

Corollary 3.7. *Let E, $\mathcal{J}_\varphi$, C, A and f be as in Theorem 3.4. Let $\{x_n\}$ be a sequence in C generated by* (3.10)*, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{r_n\}$ is a regularization sequence in $(0,\infty)$ satisfying conditions* (C1), (C2), (C3), (C5) *and* (C6) *(or, the conditions* (C1), (C2), (C3), (C4) *and* (C6)*). Then $\{x_n\}$ converges strongly to $q \in A^{-1}0$, which is the unique solution of the variational inequality* (3.22).

Corollary 3.8. *Let E be a reflexive Banach space having a weakly continuous duality mapping $\mathcal{J}_\varphi$ with gauge function φ. Let $A \subset E \times E$ be an m-accretive operator with $A^{-1}0 \neq \emptyset$ and let $f : E \to E$ be a contractive mapping with contractive constant $k \in (0,1)$. Let $\{x_n\}$ be a sequence in E generated by* (3.10)*, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{r_n\}$ is a regularization sequence in $(0,\infty)$ satisfying conditions* (C1), (C2), (C3), (C5) *and* (C6) *(or, the conditions* (C1), (C2), (C3), (C4) *and* (C6)*). Then $\{x_n\}$ converges strongly to $q \in A^{-1}0$, which is the unique solution of the variational inequality* (3.22).

Now, in order to study the pro-Tikhonov regularization for inexact proximal point algorithm (1.2) in the Banach space setting, we define our inexact iterative algorithm to generate a sequence $\{z_n\}$ as follows: for $z_1 \in E$,

$$(1-\alpha_n)z_n + \alpha_n f z_n + e_n \in z_{n+1} + r_n A z_{n+1} \quad \text{for all } n \in \mathbb{N}, \tag{3.25}$$

where $A \in E \times E$ is an m-accretive operator with $A^{-1}0 \neq \emptyset$, $\{\alpha_n\}$ is a relaxation parameter in $(0,1]$, $\{r_n\}$ is a regularization sequence in $(0,\infty)$ and $\{e_n\}$ is a sequence of errors in E satisfying the condition:

(C7) $\sum_{n=1}^{\infty}\|e_n\| < \infty$, or $\lim_{n\to\infty}\|e_n\|/\alpha_n = 0$.

Note that the algorithm (3.25) can be rewritten as

$$z_{n+1} = J_{r_n}^A((1-\alpha_n)z_n + \alpha_n f z_n + e_n) \quad \text{for all } n \in \mathbb{N}. \tag{3.26}$$

In case of Hilbert space H, if $fx = u$, then (3.26) reduces to the proximal point algorithm studied by Xu [23] and Song and Yang [18].

Utilizing Corollary 3.8, we obtain the following result for inexact variant of algorithm (3.10) with error sequence.

Theorem 3.9. *Let E, $\mathcal{J}_\varphi$, A and f be as in Corollary 3.8. Let $\{z_n\}$ be a sequence in E generated by* (3.26)*, where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{r_n\}$ is a regularization sequence in $(0,\infty)$, and $\{e_n\}$ is a sequence of errors in E satisfying conditions* (C1), (C2), (C3), (C5), (C6) *and* (C7) *(or, the conditions* (C1), (C2), (C3), (C4), (C6) *and* (C7)*), respectively. Then* $\{z_n\}$ *converges strongly to* $q \in A^{-1}0$*, which is the unique solution of the variational inequality* (3.22)*.*

Proof. For $x_1 = z_1 \in E$, let $\{x_n\}$ be a iterative sequence in E defined by (3.10). It follows from Corollary 3.8 that $\{x_n\}$ converges strongly to $q \in A^{-1}0$, where q is the unique solution of the variational inequality (3.22). From (3.10) and (3.26), we derive

$$\begin{aligned}\|z_{n+1} - x_{n+1}\| &= \|J^A_{r_n}((1-\alpha_n)z_n + \alpha_n f z_n + e_n) - J^A_{r_n}((1-\alpha_n)x_n + \alpha_n f x_n)\| \\ &\le \|(1-\alpha_n)(z_n - x_n) + \alpha_n(fz_n - fx_n) + e_n\| \\ &\le (1-(1-k)\alpha_n)\|z_n - x_n\| + \|e_n\| \quad \text{for all } n \in \mathbb{N}.\end{aligned}$$

By Lemma 2.2, we obtain $\|z_n - x_n\| \to 0$. Therefore, $\{z_n\}$ converges strongly to q. □

Remark 3.10. (1) Theorem 3.1 is a new result for solving Problem (P) in this direction in framework of Banach space. In particular, Theorem 3.1 develops Theorem 3.1 of Xu [23] in the Banach space setting different from ones in Sahu and Yao [16].

(2) Theorem 3.3 improves Theorem 3.1 of Jung [10]. In particular, the convergence condition $\lim_{n\to\infty} \|y_n - J^A_{r_n} y_n\| = 0$ in Theorem 3.1 of Jung [10] was dispensed.

(3) Theorem 3.4 develops Theorem 3.5 in Sahu and Yao [16] to the Banach space having a weakly continuous duality mapping $\mathcal{J}_\varphi$ with gauge function φ together with using weaker control conditions than ones in [16]. Theorem 3.4 also improves Theorem 3.2 and Corollary 3.1 of Jung [10] by assuming only reflexivity instead of uniformly convexity on the space in [10].

(4) Corollary 3.8 extends the convergence result in Lehdili and Moudafi [11] to the Banach space setting without using the variational distance between two maximal monotone operator.

(5) Theorem 3.1, Theorem 3.2, Corollary 3.7 and Corollary 3.8 develop and supplement the corresponding results of Sahu and Yao [16] to the Banach space having a weakly continuous duality mapping $\mathcal{J}_\varphi$ with gauge function φ.

(6) Theorem 3.9 is an extension of Theorem 2 of Song and Yang [18] and Theorem 3.3 of Xu [23] to the Banach space having a weakly continuous duality mapping $\mathcal{J}_\varphi$ with gauge function φ together with error sequence $\{e_n\}$ which doesn't necessarily satisfy the convergence condition $\sum_{n=1}^{\infty} \|e_n\| < \infty$.

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Manuscript received December 23, 2023
revised September 13, 2024

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