Journal of Nonlinear and Convex Analysis
Volume 25, Number 12, 2024, 3019–3031

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

OPTIMALITY CONDITIONS FOR NONSMOOTH MINIMAX FRACTIONAL OPTIMIZATION PROBLEMS WITH AN INFINITE NUMBER OF CONSTRAINTS

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Dedicated to Professor Do Sang Kim on the occasion of his 70th birthday with respect

Abstract. We study optimality conditions for nonsmooth minimax fractional optimization problems with an infinite number of constraints. Employing some advanced tools of variational analysis and generalized differentiation, we establish necessary conditions for local optimal solutions under the limiting constraint qualification and the constraint qualification. Sufficient conditions for the existence of global solutions to the considered problem are also obtained by means of proposing the use of generalized convex functions. In addition, some of these results are applied to nonsmooth fractional multiobjective optimization problems.

1. Introduction

Let X be the Asplund space (i.e., Banach spaces whose separable subspaces have separable duals), and Ω be a non-empty locally closed subset of X. We consider the following minimax fractional optimization problem of the form:

$$\text{(P)} \qquad \min_{x\in C}\max_{k\in K}\; f_k(x) := \frac{p_k(x)}{q_k(x)},$$

where the set C is defined by

$$(1.1) \qquad x \in C := \{x \in \Omega \mid g_t(x) \leq 0, t \in T\},$$

and p_k, q_k, $k \in K := \{1, \ldots, m\}$ and g_t, $t \in T$ are locally Lipschitz on X and T is an (possibly infinite) index set. For the sake of convenience, we assume that $q_k(x) > 0$, $k \in K$ for all $x \in \Omega$, and that $p_k(\bar{x}) \leq 0$, $k \in K$ for the reference point $\bar{x} \in \Omega$. In what follows, we use the notation $g_T := (g_t)_{t\in T}$. Note that $\mathbb{R}^m_+$ stands for the non-negative orthant of $\mathbb{R}^m$.

The following linear space is used for semi-infinite optimization.

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t\in T} \mid \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

With $\lambda \in \mathbb{R}^{(T)}$, its supporting set, $T(\lambda) := \{t \in T \mid \lambda_t \neq 0\}$, is a finite subset of T. The non-negative cone of $\mathbb{R}^{(T)}$ is denoted by

$$\mathbb{R}^{(T)}_+ := \{\lambda = (\lambda_t)_{t\in T} \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0,\ t \in T\}.$$

2020 *Mathematics Subject Classification.* 49K99, 65K10, 90C29, 90C32, 90C34.

Key words and phrases. Minimax fractional programming, fractional semi-infinite multiobjective programming, limiting subdifferential, optimality condition, generalized convex function.

This work was supported by the National Research Foundation of Korea (NRF) Grant funded by the Korean Government (NRF-2019R1A2C1008672).

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Then, with $\{z_t\}_{t\in T} \subset Z$, Z being a real linear space, we understand that

$$\sum_{t\in T} \lambda_t z_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t z_t & \text{if} \quad T(\lambda) \neq \emptyset, \\ 0 & \text{if} \quad T(\lambda) = \emptyset. \end{cases}$$

For $g_t, t \in T$,

$$\sum_{t\in T} \lambda_t g_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t g_t & \text{if} \quad T(\lambda) \neq \emptyset, \\ 0 & \text{if} \quad T(\lambda) = \emptyset, \end{cases}$$

and for $\{Y_t\}_{t\in T}$, a family of nonempty subsets of $\mathbb{R}^n$,

$$\sum_{t\in T} \lambda_t Y_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t Y_t & \text{if} \quad T(\lambda) \neq \emptyset, \\ 0 & \text{if} \quad T(\lambda) = \emptyset. \end{cases}$$

The framework, which is modeled by problem (P), includes "minimax programming (see e.g., [1, 2, 9, 13, 15, 16, 25, 26])" and "fractional programming (see e.g., [4–6, 8, 14, 23–25])" as special cases. A remarkable feature of a fractional optimization problem is that it usually admits an appealing feature that its objective function is generally not a convex function, even under very restrictive convexity/concavity assumptions. Also, it is worth noting that a great number of ground-breaking results and applications on fractional programming were contributed by Dinkelbach [11] and Schaible [21] (see e.g., [3, 8, 14, 17–19, 22]) and the references therein.

In this paper, we focus on the study of the theoretical aspects of problem (P). Using some advanced tools of variational analysis and generalized differentiation (e.g., the nonsmooth version of Fermat's rule, the limiting subdifferential of maximum functions, and the sum rule for the limiting subdifferential), we establish necessary conditions for local optimal solutions of problem (P) under the limiting constraint qualification and the constraint qualification. Also, we propose sufficient conditions for global solutions for problem (P), by means of introducing generalized convex functions defined in terms of the limiting subdifferential for locally Lipschitz functions. In addition, we employ optimality conditions which are obtained for minimax fractional optimization problem to apply the corresponding ones for multiobjective optimization problem.

The rest of the paper is organized as follows. In Section 2, we provide some notations and preliminaries. Section 3 presents some results on minimax fractional optimization problem, including necessary conditions for locally optimal solutions and sufficient conditions for globally optimal solutions under the limiting constraint qualification and constraint qualification. And we perform applications to multiobjective optimization problem in Section 4.

2. Notations and preliminary results

Throughout the paper we use the standard notation of variational analysis; see e.g., [20]. Unless otherwise specified, all spaces under consideration are assumed to be Asplund. The canonical pairing between space X and its topological dual X^* is denoted by $\langle \cdot\,, \cdot\rangle$, while the symbol $\|\cdot\|$ stands for the norm in the considered space. As usual, the *polar cone* of a set $\Omega \subset X$ is defined by

$$\Omega^\circ := \{x^* \in X^* \mid \langle x^*, x\rangle \leqq 0, \ \forall x \in \Omega\}. \tag{2.1}$$

Let $F : X \rightrightarrows X^*$ be a multifunction. The sequential Painlevé–Kuratowski upper/outer limit of F as $x \to \bar{x}$ is defined by

$$\operatorname*{Limsup}_{x\to\bar{x}} F(x) := \Big\{ x^* \in X^* \mid \exists\, x_n \to \bar{x},\, x_n^* \xrightarrow{\omega^*} x^* \text{ with } x_n^* \in F(x_n) \text{ for all } n \in \mathbb{N} := \{1,2,\ldots\} \Big\},$$

where the notation $\xrightarrow{\omega^*}$ indicates the convergence in the weak* topology of X^*.

A set $\Omega \subset X$ is called closed around $\bar{x} \in \Omega$ if there is a neighborhood U of $\bar{x}$ such that $\Omega \cap \operatorname{cl} U$ is closed. Ω is said to be locally closed if Ω is closed around x for every $x \in \Omega$. We assume that sets under consideration are locally closed.

Given $\bar{x} \in \Omega$, define the collection of regular/Fréchet normal cone to Ω at $\bar{x}$ by

$$\widehat{N}(\bar{x};\Omega) := \left\{ x^* \in X^* \;\middle|\; \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leqq 0 \right\},$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. If $x \notin \Omega$, we put $\widehat{N}(x;\Omega) := \emptyset$.

The limiting/Mordukhovich normal cone $N(\bar{x};\Omega)$ to Ω at $\bar{x} \in \Omega \subset X$ is obtained from regular normal cones by taking the sequential Painlevé–Kurotowski upper limits as

$$N(\bar{x};\Omega) := \operatorname*{Limsup}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}(x;\Omega).$$

If $\bar{x} \notin \Omega$, we put $N(\bar{x};\Omega) := \emptyset$.

For an extended real-valued function $\phi : X \to \overline{\mathbb{R}} := [-\infty, \infty]$, its domain is defined by

$$\operatorname{dom} \phi := \{ x \in X \mid \phi(x) < \infty \},$$

and its epigraph is defined by

$$\operatorname{epi} \phi := \{ (x, \mu) \in X \times \mathbb{R} \mid \mu \geqq \phi(x) \}.$$

The limiting/Mordukhovich subdifferential of ϕ at $\bar{x} \in X$ with $|\phi(\bar{x})| < \infty$ is defined by

$$\partial \phi(\bar{x}) := \{ x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \phi(\bar{x})); \operatorname{epi} \phi) \}.$$

If $|\phi(\bar{x})| = \infty$, then one puts $\partial\phi(\bar{x}) := \emptyset$.

Considering the indicator function $\delta(\cdot;\Omega)$ defined by

$$\delta(\cdot;\Omega) = \begin{cases} 0, & \text{if } x \in \Omega, \\ +\infty, & \text{otherwise}, \end{cases}$$

we have a relation between the limiting/Mordukhovich normal cone and the limiting subdifferential of the indicator function as follows [20, Proposition 1.79]:

$$N(\bar{x};\Omega) = \partial\delta(\bar{x};\Omega),\ \forall \bar{x} \in \Omega. \tag{2.2}$$

The generalized Fermat's rule is formulated as follows [20, Proposition 1.114]: Let $\phi\colon X \to \bar{\mathbb{R}}$ be finite at $\bar{x}$. If $\bar{x}$ is a local minimizer of ϕ, then

$$0 \in \partial\phi(\bar{x}). \tag{2.3}$$

For establishing optimality conditions, the following lemmas which are related to the limiting/Mordukhovich subdifferential calculus are quite useful.

Lemma 2.1. [20, Theorem 3.36] Let $\phi_i \colon X \to \bar{\mathbb{R}}$, $i = 1, 2, \ldots, m$, $m \geqq 2$ be lower semi-continuous around $\bar{x} \in X$, and let all but one of these functions be Lipschitz continuous around $\bar{x}$. Then

$$\partial(\phi_1 + \phi_2 + \ldots + \phi_m)(\bar{x}) \subset \partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x}) + \ldots + \partial\phi_m(\bar{x}). \tag{2.4}$$

Combining this limiting subdifferential sum rule with the quotient rule (cf. [20, Corollary 1.111(ii)]), we get an estimate for the limiting subdifferential of quotients.

Lemma 2.2. Let $\phi_i \colon X \to \bar{\mathbb{R}}$ for $i = 1, 2$ be Lipschitz continuous around $\bar{x}$ with $\phi_1(\bar{x}) \leq 0$ and $\phi_2(\bar{x}) > 0$. Then we have

$$\partial\left(\frac{\phi_1}{\phi_2}\right)(\bar{x}) \subset \frac{\phi_2(\bar{x})\partial\phi_1(\bar{x}) - \phi_1(\bar{x})\partial\phi_2(\bar{x})}{[\phi_2(\bar{x})]^2}. \tag{2.5}$$

3. Optimality conditions

This section is devoted to studying optimality conditions for minimax fractional optimization problems. By using the nonsmooth version of Fermat's rule, the limiting/Mordukhovich subdifferential of maximum functions and the sum rule as well as the quotient rule for the limiting subdifferential, we first establish necessary conditions for (local) optimal solutions of a minimax fractional optimization problem. And then we provide sufficient conditions for the existence of such global solutions by imposing assumptions of generalized convexity.

Definition 3.1. Let $\varphi(x) := \max_{k\in K} f_k(x), x \in X$. A point $\bar{x} \in C$ is a local optimal solution of problem (P) iff there is a neighborhood U of $\bar{x}$ such that

$$\varphi(\bar{x}) \leq \varphi(x),\ \forall x \in U \cap C. \tag{3.1}$$

If the inequality in (3.1) holds for every $x \in C$, then $\bar{x}$ is said to be a global optimal solution of problem (P).

To obtain the necessary optimality condition of the Karush–Kuhn–Tucker (KKT) type for a local optimal solution to problem (P), the following constraint qualifications are needed.

Definition 3.2. (see [7, 10]) Let $\bar{x} \in C$. We call that the limiting constraint qualification (LCQ) is satisfied at $\bar{x}$ iff

$$N(\bar{x}; C) \subset \bigcup_{\lambda\in A(\bar{x})} \left[\sum_{t\in T} \lambda_t \partial g_t(\bar{x})\right] + N(\bar{x}; \Omega), \tag{3.2}$$

where $A(\bar{x}) := \{\lambda \in \mathbb{R}_+^{(T)} \mid \lambda_t g_t(\bar{x}) = 0 \text{ for all } t \in T\}$.

Definition 3.3. We say that the constraint qualification (CQ) is satisfied at $\bar{x} \in C$ if $\lambda \in \mathbb{R}_+^{(T)}$ such that

$$0 \in \sum_{t\in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega),$$

then $\lambda_t = 0$ for all $t \in T(\lambda)$, which is equivalent to $T(\lambda) = \emptyset$.

It is worth to mention here that when considering $\bar{x} \in C$ defined in (1.1) with $\Omega = X$ and $T(\bar{x}) := \{t \in T \mid g_t(\bar{x}) = 0\}$, T is finite in the smooth setting, the above-defined (CQ) is guaranteed by the Mangasarian-Fromovitz constraint qualification (see e.g., [20] for more details).

Theorem 3.4. Let the (LCQ) be satisfied at $\bar{x} \in C$. If $\bar{x} \in C$ is a local optimal solution to problem (P), then there exist multipliers $\alpha \in \mathbb{R}^m_+ \backslash \{0\}$ and $\lambda \in A(\bar{x})$ such that the inclusion

$$0 \in \sum_{k \in K} \alpha_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega) \tag{3.3}$$

holds.

Proof. Suppose that $\bar{x} \in C$ is a local optimal solution of problem (P). Then $\bar{x}$ is a local minimizer of the following problem

$$\min_{x \in C} \varphi(x),$$

where $\varphi(x) := \max_{k \in K} f_k(x)$. Thus, $\bar{x}$ is a local minimizer of the following unconstrained optimization problem

$$\min_{x \in X} \varphi(x) + \delta(x; C). \tag{3.4}$$

Applying the nonsmooth version of Fermat's rule (2.3) to problem (3.4), we have

$$0 \in \partial\big(\varphi + \delta(\cdot; C)\big)(\bar{x}). \tag{3.5}$$

Since the function φ is Lipschitz continuous around $\bar{x}$ and the function $\delta(\cdot; C)$ is lower semi-continuous around this point, it follows from the sum rule (2.4) applied to (3.5) and from the relation in (2.2) that

$$0 \in \partial\varphi(\bar{x}) + N(\bar{x}; C). \tag{3.6}$$

Employing the formula for the limiting subdifferential of maximum functions (see [20, Theorem 3.46(ii)]) and the sum rule (2.4), we obtain

$$\partial\varphi(\bar{x}) = \partial \left(\max_{k \in K} f_k \right)(\bar{x}) \subset \left\{ \sum_{k \in K(\bar{x})} \beta_k \partial f_k(\bar{x}) \;\middle|\; \beta_k \geq 0, k \in K(\bar{x}), \sum_{k \in K(\bar{x})} \beta_k = 1 \right\},$$

where $K(\bar{x}) := \{k \in K \mid f_k(\bar{x}) = \varphi(\bar{x})\} \neq \emptyset$.

Taking (2.5) into account, we arrive at

$$\partial\varphi(\bar{x}) \subset \left\{ \sum_{k \in K(\bar{x})} \beta_k \frac{q_k(\bar{x})\partial p_k(\bar{x}) - p_k(\bar{x})\partial q_k(\bar{x})}{[q_k(\bar{x})]^2} \;\middle|\; \beta_k \geq 0, k \in K(\bar{x}), \sum_{k \in K(\bar{x})} \beta_k = 1 \right\}. \tag{3.7}$$

The (LCQ) being satisfied at $\bar{x}$ entails that

$$N(\bar{x}; C) \subset \bigcup_{\lambda \in A(\bar{x})} \left[\sum_{t \in T} \mu_t \partial g_t(\bar{x}) \right] + N(\bar{x}; \Omega), \tag{3.8}$$

where the set $A(\bar{x})$ was defined in (3.2). It follows from (3.6)-(3.8) that

$$0 \in \left\{ \sum_{k\in K(\bar{x})} \frac{\beta_k}{q_k(\bar{x})} \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) \;\middle|\; \beta_k \geq 0, k \in K(\bar{x}), \sum_{k\in K(\bar{x})} \beta_k = 1 \right\}$$
$$+ \bigcup_{\lambda \in A(\bar{x})} \left[\sum_{t\in T} \lambda_t \partial g_t(\bar{x}) \right] + N(\bar{x};\Omega).$$

Now, by letting $\alpha_k := \frac{\beta_k}{q_k(\bar{x})}$ for $k \in K(\bar{x})$ and $\beta_k := 0$ for $k \in K\backslash K(\bar{x})$. It is clear that the above inclusion implies

$$0 \in \sum_{k\in K} \alpha_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t\in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x};\Omega),$$

and so the proof is complete. □

Theorem 3.5. Let the (CQ) be satisfied at $\bar{x} \in C$. If $\bar{x} \in C$ is a local optimal solution to problem (P), then there exist multipliers $\alpha \in \mathbb{R}^m_+ \backslash \{0\}$ and $\lambda \in \mathbb{R}^{(T)}_+$ such that the inclusion

$$0 \in \sum_{k\in K} \alpha_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t\in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x};\Omega) \tag{3.9}$$

holds.

Proof. Suppose that $\bar{x} \in C$ is a local optimal solution to problem (P). Then $\bar{x}$ is a local minimizer of problem (3.4).

Similar to the proof of Theorem 3.4, we have

$$\partial\varphi(\bar{x}) \subset \left\{ \sum_{k\in K(\bar{x})} \beta_k \frac{q_k(\bar{x})\partial p_k(\bar{x}) - p_k(\bar{x})\partial q_k(\bar{x})}{[q_k(\bar{x})]^2} \;\middle|\; \beta_k \geq 0, k \in K(\bar{x}), \sum_{k\in K(\bar{x})} \beta_k = 1 \right\}.$$

The (CQ) being satisfied at $\bar{x}$ entails that

$$0 \in \sum_{t\in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x};\Omega)$$

has $\lambda_t = 0$ for all $t \in T(\lambda)$. Thus, we obtain

$$0 \in \left\{ \sum_{k\in K(\bar{x})} \frac{\beta_k}{q_k(\bar{x})} \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) \;\middle|\; \beta_k \geq 0, k \in K(\bar{x}), \sum_{k\in K(\bar{x})} \beta_k = 1 \right\}$$
$$+ \sum_{t\in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x};\Omega).$$

Then, by letting $\alpha_k := \frac{\beta_k}{q_k(\bar{x})}$ for $k \in K(\bar{x})$ and $\beta_k := 0$ for $k \in K\backslash K(\bar{x})$. It is clear that the above inclusion implies

$$0 \in \sum_{k\in K} \alpha_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t\in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x};\Omega).$$

Thus, the proof is complete. □

The following example shows that the conclusions of Theorem 3.4 and Theorem 3.5 may fail if the (LCQ) and the (CQ) are not satisfied at the point under consideration, respectively.

Example 3.6. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) := \left(\frac{p_1(x)}{q_1(x)}, \frac{p_2(x)}{q_2(x)}\right)$, where $p_1(x) := p_2(x) := x$, $q_1(x) := q_2(x) := 2x^2 + 1$, $x \in \mathbb{R}$, and let $g_t : \mathbb{R} \to \mathbb{R}$ be given by

$$g_t(x) = tx^2, \quad x \in \mathbb{R}, \quad t \in T := [0, +\infty).$$

We consider problem (P) with $m := 2$ and $\Omega := (-\infty, 0] \subset \mathbb{R}$. Then $C = \{0\}$ and thus $\bar{x} := 0$ is a local optimal solution of problem (P). Since $N(\bar{x}; \Omega) = [0, +\infty)$ and $\partial g_t(\bar{x}) = \{0\}$ for all $t \in T$, we have

$$\bigcup_{\lambda \in A(\bar{x})} \left[\sum_{t \in T} \lambda_t \partial g_t(\bar{x})\right] + N(\bar{x}; \Omega) = [0, +\infty).$$

Hence, the (LCQ) in Theorem 3.4 is not satisfied at $\bar{x}$ due to $N(\bar{x}; C) = \mathbb{R}$. Actually, condition (3.3) fails to hold.

On the other hand, since $\partial g_t(\bar{x}) = 0$ for all $t \in T$, there exist $\lambda_t > 0$ for $t \in T(\lambda)$ such that

$$0 \in \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega) = [0, +\infty).$$

So, the (CQ) is not satisfied at $\bar{x}$, which means that (3.9) fails to hold.

In order to obtain sufficient condition for the existence of (global) optimal solutions of problem (P) presented in the next theorem, we recall [6] the notion of generalized convexity for a family of locally Lipschitz functions.

Definition 3.7. (i) We say that (f, g_T) is generalized convex on Ω at $\bar{x} \in \Omega$ if for any $x \in \Omega$, $\xi_k \in \partial p_k(\bar{x})$, $\zeta_k \in \partial q_k(\bar{x})$, $k \in K$ and any $\eta_t \in \partial g_t(\bar{x})$, $t \in T$, there exists $\nu \in N(\bar{x}; \Omega)^\circ$ such that

$$p_k(x) - p_k(\bar{x}) \geq \langle \xi_k, \nu \rangle, \ q_k(x) - q_k(\bar{x}) \geq \langle \zeta_k, \nu \rangle, \ k \in K,$$
$$g_t(x) - g_t(\bar{x}) \geq \langle \eta_t, \nu \rangle, \ t \in T.$$

(ii) We say that (f, g_T) is strictly generalized convex on Ω at $\bar{x} \in \Omega \setminus \{\bar{x}\}$ if for any $x \in \Omega$, $\xi_k \in \partial p_k(\bar{x})$, $\zeta_k \in \partial q_k(\bar{x})$, $k \in K$ and any $\eta_t \in \partial g_t(\bar{x})$, $t \in T$, there exists $\nu \in N(\bar{x}; \Omega)^\circ$ such that

$$p_k(x) - p_k(\bar{x}) > \langle \xi_k, \nu \rangle, \ q_k(x) - q_k(\bar{x}) \geq \langle \zeta_k, \nu \rangle, \ k \in K,$$
$$g_t(x) - g_t(\bar{x}) \geq \langle \eta_t, \nu \rangle, \ t \in T.$$

We are now ready to provide sufficient condition for a feasible point of problem (P) to be a global optimal solution.

Theorem 3.8. Let $\bar{x} \in C$. Assume that $\bar{x}$ satisfies condition (3.3). If (f, g_T) is generalized convex at $\bar{x}$, then $\bar{x}$ is a global optimal solution of problem (P).

Proof. Since $\bar{x}$ satisfies condition (3.3), there exist multipliers $\alpha \in \mathbb{R}^m_+ \backslash \{0\}$, $\lambda \in A(\bar{x})$ and $\xi_k \in \partial p_k(\bar{x})$, $\zeta_k \in \partial q_k(\bar{x})$, $k \in K$, $\eta_t \in \partial g_t(\bar{x})$, $t \in T$ such that

$$-\left[\sum_{k\in K} \alpha_k \left(\xi_k - \frac{p_k(\bar{x})}{q_k(\bar{x})}\zeta_k\right) + \sum_{t\in T} \lambda_t \eta_t\right] \in N(\bar{x}; \Omega). \tag{3.10}$$

Assume that $\bar{x}$ is not a global optimal solution of (P), then there exist $\hat{x} \in C$ such that

$$\varphi(\hat{x}) < \varphi(\bar{x}), \tag{3.11}$$

where $\varphi(x) := \max_{k\in K} f_k(x)$.

Since (f, g_T) is generalized convex on Ω at $\bar{x}$, for $\hat{x}$ above, there exists $\nu \in N(\bar{x}; \Omega)^\circ$ such that

$$\begin{aligned}
&\sum_{k\in K} \alpha_k \left(\langle \xi_k, \nu\rangle - \frac{p_k(\bar{x})}{q_k(\bar{x})}\langle \zeta_k, \nu\rangle\right) + \sum_{t\in T} \lambda_t \langle \eta_t, \nu\rangle \\
&\le \sum_{k\in K} \alpha_k \left[p_k(\hat{x}) - p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})}(q_k(\hat{x}) - q_k(\bar{x}))\right] + \sum_{t\in T} \lambda_t (g_t(\hat{x}) - g_t(\bar{x})) \\
&= \sum_{k\in K} \alpha_k \left(p_k(\hat{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} q_k(\hat{x})\right) + \sum_{t\in T} \lambda_t (g_t(\hat{x}) - g_t(\bar{x})).
\end{aligned}$$

By definition of polar cone (2.1), it follows from (3.10) and the relation $\nu \in N(\bar{x}; \Omega)^\circ$ that

$$0 \le \sum_{k\in K} \alpha_k \left(\langle \xi_k, \nu\rangle - \frac{p_k(\bar{x})}{q_k(\bar{x})}\langle \zeta_k, \nu\rangle\right) + \sum_{t\in T} \lambda_t \langle \eta_t, \nu\rangle.$$

Thus,

$$0 \le \sum_{k\in K} \alpha_k \left(p_k(\hat{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} q_k(\hat{x})\right) + \sum_{t\in T} \lambda_t (g_t(\hat{x}) - g_t(\bar{x})). \tag{3.12}$$

In addition, due to the fact that $\lambda_t g_t(\bar{x}) = 0$ and $\lambda_t g_t(\hat{x}) \le 0$ for $t \in T$, we get by (3.12) that

$$0 \le \sum_{k\in K} \alpha_k \left(p_k(\hat{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} q_k(\hat{x})\right). \tag{3.13}$$

By (3.13), we have

$$\sum_{k\in K} \alpha_k \frac{p_k(\bar{x})}{q_k(\bar{x})} \le \sum_{k\in K} \alpha_k \frac{p_k(\hat{x})}{q_k(\hat{x})}. \tag{3.14}$$

According to the fact that $\alpha_k \ge 0$ for $k \in K(\bar{x})$ with $\sum_{k\in K(\bar{x})} \alpha_k = 1$ and $\alpha_k = 0$ for $k \in K \backslash K(\bar{x})$, the inequality in (3.14) is equivalent to

$$\varphi(\hat{x}) \ge \varphi(\bar{x}),$$

which contradicts (3.11) and, therefore, the proof is complete. □

4. Applications to multiobjective optimization problem

We consider the following constrained multiobjective fractional optimization problem

$$\text{(MOP)} \qquad \min_{\mathbb{R}^m_+} \left\{ f(x) := \left(\frac{p_1(x)}{q_1(x)}, \cdots, \frac{p_m(x)}{q_m(x)} \right) \middle| x \in C \right\},$$

where the constraint set C is defined by (1.1), and "$\min_{\mathbb{R}^m_+}$" is understood with respect to the ordering cone $\mathbb{R}^m_+$.

More clearly, one says that $\bar{x} \in C$ is a local weak Pareto solution of problem (MOP) iff there exists a neighborhood U of $\bar{x}$ such that

$$f(x) - f(\bar{x}) \notin -\text{int}\mathbb{R}^m_+, \ \forall x \in U \cap C, \tag{4.1}$$

and $\bar{x} \in C$ is a local Pareto solution of problem (MOP) iff there exists a neighborhood U of $\bar{x}$ such that

$$f(x) - f(\bar{x}) \notin -\mathbb{R}^m_+ \backslash \{0\}, \ \forall x \in U \cap C, \tag{4.2}$$

where $\text{int}\mathbb{R}^m_+$ stands for the topological interior of $\mathbb{R}^m_+$.

If the inclusion in (4.1) and (4.2) holds for every $x \in C$, then $\bar{x}$ is said to be a weak Pareto solution and Pareto solution of problem (MOP) (see [12]), respectively.

The following results are the KKT necessary conditions for local weak Pareto solutions of problem (MOP).

Theorem 4.1. Let the (LCQ) be satisfied at $\bar{x} \in C$. If $\bar{x}$ is a local weak Pareto solution of (MOP), then there exist multipliers $\alpha \in \mathbb{R}^m_+ \backslash \{0\}$ and $\lambda \in A(\bar{x})$ such that the inclusion

$$0 \in \sum_{k \in K} \alpha_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})} \partial q_k(\bar{x}) \right) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega) \tag{4.3}$$

holds.

Proof. Let $\bar{x}$ be a local weak Pareto solution of (MOP), and let

$$\hat{f}_k(x) := f_k(x) - f_k(\bar{x}), \ k \in K, \ x \in X.$$

We will show that $\bar{x}$ is a local optimal solution of the following problem

$$(\hat{\text{P}}) \qquad \min_{x \in C} \max_{k \in K} \hat{f}_k(x).$$

To do this, let us put $\hat{\varphi}(x) := \max_{k \in K} \hat{f}_x(x)$ and prove that

$$\hat{\varphi}(\bar{x}) \le \hat{\varphi}(x), \ \forall x \in U \cap C. \tag{4.4}$$

Indeed, if (4.4) is not valid, then there exists $x_0 \in U \cap C$ such that $\hat{\varphi}(\bar{x}) > \hat{\varphi}(x_0)$. Since $\hat{\varphi}(\bar{x}) = 0$, it holds that

$$\max_{k \in K} \{f_k(x_0) - f_k(\bar{x})\} < 0.$$

Thus,

$$f(x_0) - f(\bar{x}) \in -\text{int}\mathbb{R}^m_+,$$

which contradicts the fact that $\bar{x}$ is a local weak Pareto solution of (MOP). Then, we can employ the KKT condition in Theorem 3.4, but applied to problem $(\hat{\text{P}})$. Since

$K(\bar{x}) = K$ due to $\hat{f}_k(\bar{x}) := \max_{k\in K} \hat{f}_k(\bar{x})$, then we find multipliers $\alpha \in \mathbb{R}^m_+\backslash\{0\}$ and $\lambda \in A(\bar{x})$ such that

$$0 \in \sum_{k\in K} \alpha_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})}\partial q_k(\bar{x})\right) + \sum_{t\in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x};\Omega).$$

Thus, we complete the proof. □

Theorem 4.2. Let the (CQ) be satisfied at $\bar{x} \in C$. If $\bar{x}$ is a local weak Pareto solution of (MOP), then there exist $\alpha \in \mathbb{R}^m_+\backslash\{0\}$ and $\lambda \in \mathbb{R}^{(T)}_+$ such that the inclusion

$$0 \in \sum_{k\in K} \alpha_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})}\partial q_k(\bar{x})\right) + \sum_{t\in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x};\Omega) \tag{4.5}$$

holds.

Proof. Suppose that $\bar{x}$ is a local weak Pareto solution of (MOP). Then we will show that $\bar{x}$ is a local optimal solution of $(\hat{\text{P}})$.

Similar to the proof of Theorem 4.1, let us prove that the inequality in (4.4) holds for every $x \in U \cap C$. If (4.4) is not valid, then there exists $x_0 \in U \cap C$ such that $\hat{\varphi}(\bar{x}) > \hat{\varphi}(x_0)$. Since $\hat{\varphi}(\bar{x}) = 0$, we obtain

$$\max_{k\in K}\{f_k(x_0) - f_k(\bar{x})\} < 0.$$

Thus,

$$f(x_0) - f(\bar{x}) \in -\text{int}\mathbb{R}^m_+,$$

which contradicts the assumption at the beginning of the proof. Let us employ the KKT condition in Theorem 3.5, but applied to problem $(\hat{\text{P}})$. Then we can find multipliers $\alpha \in \mathbb{R}^m_+\backslash\{0\}$ and $\lambda \in \mathbb{R}^{(T)}_+$ such that

$$0 \in \sum_{k\in K} \alpha_k \left(\partial p_k(\bar{x}) - \frac{p_k(\bar{x})}{q_k(\bar{x})}\partial q_k(\bar{x})\right) + \sum_{t\in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x};\Omega),$$

which completes the proof of the theorem. □

Remark 4.3. We consider problem (MOP) with $\Omega := (-\infty, 0]$ in Example 3.6. Then $C = \{0\}$ and thus $\bar{x} := 0$ is a local weak Pareto solution of problem (MOP). Hence, the (LCQ) in Theorem 4.1 is not satisfied at $\bar{x}$, and (4.3) fails to holds. Also, the (CQ) in Theorem 4.2 is not satisfied at $\bar{x}$, which means that (4.5) fails to hold.

The next result is a sufficient condition for the existence of a weak Pareto/or Pareto solution of problem (MOP).

Theorem 4.4. Let $\bar{x} \in C$. Assume that $\bar{x}$ satisfies condition (4.3).

(i) If (f, g_T) is generalized convex at $\bar{x}$, then $\bar{x}$ is a weak Pareto solution of (MOP).
(ii) If (f, g_T) is strictly generalized convex at $\bar{x}$, then $\bar{x}$ is a Pareto solution of (MOP).

Proof. We put

$$\hat{f}_k(x) := f_k(x) - f_k(\bar{x}),\ k \in K,\ x \in X.$$

Let $\hat{f} := (\hat{f}_1, \ldots, \hat{f}_m)$. Since (f, g_T) is generalized convex on Ω at $\bar{x}$, it follows that $(\hat{f}, g_T)$ is generalized convex at this point as well.

We first prove (i). We apply the sufficient criteria in Theorem 3.8 to conclude that $\bar{x}$ is a global optimal solution of the following problem

$$(\hat{\mathrm{P}}) \qquad \min_{x \in C} \max_{k \in K} \hat{f}_k(x).$$

It means that

$$\hat{\varphi}(\bar{x}) \leq \hat{\varphi}(x),\ \forall x \in C,$$

where $\hat{\varphi}(x) := \max_{k \in K} \hat{f}_k(x)$. In other words, we obtain

$$0 \leq \max_{k \in K} \{f_k(x) - f_k(\bar{x})\},\ \forall x \in C,$$

which entails that

$$f(x) - f(\bar{x}) \notin -\mathrm{int}\mathbb{R}^m_+,\ \forall x \in C.$$

Consequently, $\bar{x}$ is a weak Pareto solution of problem (MOP). Therefore, the proof of (i) is complete.

Then, we prove (ii). Similar to the proof of (i), we conclude that

$$\hat{\varphi}(\bar{x}) < \hat{\varphi}(x),\ \forall x \in C,$$

So, we have

$$0 < \max_{k \in K} \{f_k(x) - f_k(\bar{x})\},\ \forall x \in C.$$

Thus,

$$f(x) - f(\bar{x}) \notin -\mathbb{R}^m_+ \backslash \{0\},\ \forall x \in C.$$

Hence, $\bar{x}$ is a Pareto solution of problem (MOP), which completes the proof of (ii). $\square$

5. Conclusions

In this paper, we investigated optimality conditions for nonsmooth minimax fractional optimization problems with an infinite number of constraints. Employing the Fermat's rule, the limiting subdifferential sum rule and the limiting subdifferential quotient rule, we establish necessary optimality conditions for local optimal solutions under the limiting constraint qualification and the constraint qualification. Sufficient conditions for the existence of global solutions to the considered problem are also provided by means of introducing the concepts of generalized convex functions. In addition, some optimality results are applied to nonsmooth fractional multiobjective optimization problems.

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Manuscript received February 28, 2024
revised September 9, 2024

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