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A NEW FULL-NEWTON STEP INFEASIBLE INTERIOR-POINT METHOD FOR LINEAR OPTIMIZATION

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Abstract. Many studies on interior-point methods have utilized kernel functions to find new search directions. Notably, classical kernel functions well-known in this context include self-concordant, self-regular, and eligible kernel functions. In this paper, we introduce a new class of kernel functions that differs from these classical counterparts. The newly defined class of kernel functions is much easier to check than the conditions required for classical kernel functions. Based on this new class of kernel functions, we propose a unified approach for a small-update full-Newton step infeasible interior-point method for linear optimization. Furthermore, we demonstrate that it has the best known worst-case computational complexity in this methodology.

1. Introduction

We consider the following standard form of linear optimization (LO):

$$\min\{c^T x : Ax = b,\ x \geq 0\}, \tag{1.1}$$

and its dual problem is defined as follows:

$$\max\{b^T y : A^T y + s = c,\ s \geq 0\}, \tag{1.2}$$

where $x, s \in \mathbb{R}^n, y \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$, and rank$A = m$.

In 1984, Karmarkar [3] introduced a remarkable method known as the interior-point method (IPM) for solving LO problems within polynomial time, opening a new paradigm. In 1990, Lustig [6] and Tanabe [14] initially introduced the IIPM for LO. Then, Kojima [5] in 1993 and Mizuno [8] in 1994 demonstrated global convergence for the IIPM for LO. In 2008, Salahi et al. [13], using self-regular kernel function, showed that the worst-case iteration bound for a large-update IIPM for LO is $\mathcal{O}(n^{\frac{3}{2}}(\log n)\log(\frac{n}{\epsilon}))$. In 2006, Roos [11] proposed a full-Newton step small-update IIPM for LO, which has the advantage that no line-searches are needed, and achieved the worst-case iteration bound of $\mathcal{O}(n\log(\frac{n}{\epsilon}))$. To date, in the infeasible IPMs (IIPMs) for LO, the best known iteration bound for large-update method is $\mathcal{O}(n^{\frac{3}{2}}(\log n)\log(\frac{n}{\epsilon}))$ and for small-update it is $\mathcal{O}(n\log(\frac{n}{\epsilon}))$.

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In this paper, we propose a unified approach for the complexity analysis of small-update IIPMs for LO, utilizing our new class of kernel functions. We demonstrate that the new algorithm achieves the best known worst-case iteration bound of $\mathcal{O}(n\log(\frac{n}{\epsilon}))$. Our IIPM is motivated by Roos's results [11] in 2006. Classical kernel functions, such as self-concordant, self-regular, and eligible kernel functions, are required to be at least twice continuously differentiable. However, the newly defined kernel functions in this paper are continuously differentiable, making it easier to check the conditions for kernel functions compared to classical ones. Furthermore, our new class of kernel functions includes many other classical kernel functions [2,4,9,10]. In summary, we define a new search direction for the small-update modified full-Newton IIPM using our newly defined class of functions and show the computational complexity which is known to be the best in this methodology.

This paper is organized as follows. In Section 2, we introduce the IIPM for LO. In Section 3, we define a new class of kernel functions. In Section 4, we present some technical lemmas for complexity analysis. In Section 5, we prove the strict feasibility and the crucial inequality for the proximity measure to achieve quadratic convergence, where this process is called the feasibility step. In Section 6, we propose the complexity result for the new algorithm. Finally, Section 7 concludes the paper with conclusions and future research.

Some notations used in this paper are as follows. The nonnegative orthant and positive orthant are denoted as $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$, respectively. xs and $\frac{x}{s}$ represent the componentwise product and division of vectors x and s in $\mathbb{R}^n$, respectively. For $x \in \mathbb{R}^n$, the diagonal matrix with the elements of x on its diagonal is denoted by $\operatorname{diag}(x)$. For $a \in \mathbb{R}$, $\lceil a \rceil$ denotes the least integer greater than or equal to a. For $x \in \mathbb{R}^n$, $\|x\| := \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$ and $\|x\|_\infty := \max_{i=1,\ldots,n} |x_i|$. If $f(x) \leq \gamma g(x)$ for some $\gamma > 0$, we write $f(x) = \mathcal{O}\left(g(x)\right)$.

2. Infeasible interior-point method

In feasible IPM, finding an initial point that satisfies the constraints and lies near the central path is an additional optimization problem. To overcome this difficulty, research on IIPMs has been conducted. Specifically, we utilize the following type of IIPMs, motivated by [11]: As usual, we assume that there exists an optimal solution (x^*, y^*, s^*) for LO (1.1) and (1.2) such that $\|x^* + s^*\|_\infty \leq \zeta$ for some $\zeta > 0$. We define the initial values by $(x^0, y^0, s^0) := \zeta(e, 0, e)$, $\mu^0 e := x^0 s^0$, and $\nu^0 := 1$. Note that the condition $\mu^0 e := x^0 s^0$ signifies that the initial iteration lies on central path. For any $0 < \nu \leq 1$, we define the perturbed primal problem for LO as follows:

$$\min\{\left(c - \nu\left(c - A^T y^0 - s^0\right)\right)^T x : Ax = b - \nu\left(b - Ax^0\right),\ x \geq 0\}, \tag{2.1}$$

and its dual problem as follows:

$$\max\{\left(b - \nu\left(b - Ax^0\right)\right)^T y : A^T y + s = c - \nu\left(c - A^T y^0 - s^0\right),\ s \geq 0\}, \tag{2.2}$$

where $x, s \in \mathbb{R}^n, y \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m\times n}$.

A point (x, y, s) is referred to as an ϵ-approximate solution for the primal-dual problem for LO (1.1) and (1.2) if $\|b - Ax\| \leq \epsilon$, $\|c - A^T y - s\| \leq \epsilon$, and $x^T s \leq \epsilon$. When $\nu = 1$, the point (x^0, y^0, s^0) becomes a feasible solution for (2.1) and (2.2). Therefore, we have obtained a feasible starting point for the perturbed problems

(2.1) and (2.2) when $\nu = 1$. We say that LO satisfies interior-point condition (IPC) if (1.1) has a feasible solution $x > 0$ and (1.2) has a feasible solution (y, s) with $s > 0$. Note that, when $\nu = 1$, the starting point (x^0, y^0, s^0) satisfies the IPC for the problems (2.1) and (2.2).

The following Lemma provides a result concerning the IPC for the perturbed problem (2.1) and (2.2) when $0 < \nu \leq 1$.

Lemma 2.1 (Theorem 3.1 in [11]). *The original problems (1.1) and (1.2) are feasible if and only if for each $\nu \in (0, 1]$ the perturbed problems (2.1) and (2.2) satisfy the* IPC.

According to Lemma 2.1, if (1.1) and (1.2) are feasible, then (2.1) and (2.2) satisfy the IPC. That is, the following KKT optimality conditions for (2.1) and (2.2) have a unique solution for every $\mu > 0$:

$$\begin{cases} b - Ax = \nu\left(b - Ax^0\right) \\ c - A^T y - s = \nu\left(c - A^T y^0 - s^0\right) \\ xs = \mu e. \end{cases} \tag{2.3}$$

By applying the Newton's method to the system (2.3) with updated ν as $(1-\theta)\nu$, we have the following system for the search directions $\Delta x, \Delta y$, and Δs:

$$\begin{cases} A\Delta x = \theta\nu r_b^0 \\ A^T\Delta y + \Delta s = \theta\nu r_c^0 \\ s\Delta x + x\Delta s = \mu e - xs, \end{cases} \tag{2.4}$$

where $r_b^0 := b - Ax^0$ and $r_c^0 := c - A^T y^0 - s^0$.

Let us define the following scalings:

$$v := \sqrt{xs/\mu},\ d_x^f := v\Delta x/x,\ d_s^f := v\Delta s/s,\ d_y^f := \Delta y/\mu. \tag{2.5}$$

By applying the scalings in (2.5) to the system (2.4), we obtain the following scaled system:

$$\begin{cases} \overline{A}d_x^f = \theta\nu r_b^0, \\ \overline{A}^T d_y^f + d_s^f = \theta\nu v s^{-1} r_c^0, \\ d_x^f + d_s^f = v^{-1} - v, \end{cases} \tag{2.6}$$

where $\overline{A} := AV^{-1}X$, $V := \operatorname{diag}(v)$, and $X := \operatorname{diag}(x)$.

The right-hand side of the last equation in (2.6) is the negative gradient of the logarithmic kernel function. After replacing the right-hand side of the last equation in (2.6) with the negative gradient of a kernel function $\Psi(v) := \sum_{i=1}^n \psi(v_i)$, for which ψ will be defined in Definition 3.2 in Section 3, we obtain the following:

$$\begin{cases} \overline{A}d_x^f = \theta\nu r_b^0, \\ \overline{A}^T d_y^f + d_s^f = \theta\nu v s^{-1} r_c^0, \\ d_x^f + d_s^f = -\nabla\Psi(v). \end{cases} \tag{2.7}$$

In the feasibility step, we compute the search direction (d_x^f, d_y^f, d_s^f) by solving the system (2.7). Then we update an iteration by modified full-Newton step as follows:

$(x, y, s) \leftarrow (x, y, s) + (\frac{xd_x^f}{v}, \mu d_y^f, \frac{sd_s^f}{v})$. After feasibility step, we need to reduce the proximity measure to the desired threshold value τ. This process is called the centering step. In the centering step, we compute the search direction $(\Delta x, \Delta y, \Delta s)$ by solving the following system with updated μ as $(1-\theta)\mu$:

$$\begin{cases} A\Delta x = 0, \\ A^T \Delta y + \Delta s = 0, \\ s\Delta x + x\Delta s = \mu e - xs. \end{cases} \tag{2.8}$$

Then we update an iteration by full-Newton step as follows: $(x, y, s) \leftarrow (x, y, s) + (\Delta x, \Delta y, \Delta s)$.

Let us define the proximity measure as follows:

$$\delta(x, s; \mu) := \delta(v) := \frac{1}{2}\|v^{-1} - v\|, \tag{2.9}$$

where v is defined in (2.5). Then, the following result is known:

Lemma 2.2 (Corollary 2.4 in [11])**.** *If $\delta := \delta(x, s; \mu) \leq 1/\sqrt{2}$, then $\delta(x^+, s^+; \mu) \leq \delta^2$, where $x^+ > 0$ and $s^+ > 0$ are strictly feasible and denote the new iteration after a full-Newton step.*

In the centering step, we can easily calculate the number of iterations needed to reduce the proximity measure to the desired threshold value τ as follows. Assume that $\delta(x, s; \mu) \leq 1/\sqrt{2}$. Then by Lemma 2.2, after k centering steps, we have iteration (x^+, y^+, s^+) satisfying $\delta(x^+, s^+; \mu) \leq \left(1/\sqrt{2}\right)^{2^k}$. Thus to have $\delta(x^+, s^+, \mu) \leq \tau$, it suffices to show $\left(1/\sqrt{2}\right)^{2^k} \leq \tau$, which equivalent to $\log_2\left(\log_2 \frac{1}{\tau^2}\right) \leq k$. Thus, by iterating the smallest natural number greater than

$$\log_2\left(\log_2(1/\tau^2)\right), \tag{2.10}$$

we can reduce the proximity measure to less than τ.

A formal description of the IIPM for LO is given in Algorithm 1.

3. New class of kernel function

In this section, we propose a new class of such functions.

For the system (2.6), the last term $v^{-1} - v$ is precisely the negative gradient of the logarithmic kernel function. There has been research on modifying this term with another function.

The definition of the kernel function is as follows:

Definition 3.1 (Kernel function [1])**.** We call $\psi : \mathbb{R}_{++} \to \mathbb{R}_+$ a kernel function if ψ is strictly convex and satisfies the following conditions: $\psi(1) = \psi'(1) = 0$, and $\lim_{t\to 0^+} \psi(t) = \lim_{t\to\infty} \psi(t) = \infty$.

Now, we introduce a new class of kernel functions and demonstrate that we obtain the best known worst-case iteration bound for new full-Newton step IIPMs using the newly defined class of kernel functions. This new class of kernel functions is defined in Definition 3.2 below.

Algorithm 1 IIPM for LO

inputs:
  accuracy parameter $\epsilon > 0$
  barrier update parameter θ, $0 < \theta < 1$
  threshold parameter $\tau > 0$
  upper bound of optimal solution $\zeta > 0$
initialize:
  $x = \zeta e,\ y = 0,\ s = \zeta e,\ \mu = \zeta^2,\ \nu = 1$
while $\max\left\{x^T s,\ \|b - Ax\|,\ \|c - A^T y - s\|\right\} > \epsilon$ **do**
  feasibility step:
    solve the system (2.7) to get (d_x^f, d_y^f, d_s^f)
    update: $(x, y, s) \leftarrow (x, y, s) + (\Delta x, \Delta y, \Delta s)$ using (2.5)
  $\nu \leftarrow (1-\theta)\nu$ and $\mu \leftarrow (1-\theta)\mu$
  centering step:
  while $\delta(x, s; \mu) > \tau$ **do**
    solve the system (2.8) to get $(\Delta x, \Delta y, \Delta s)$
    update: $(x, y, s) \leftarrow (x, y, s) + (\Delta x, \Delta y, \Delta s)$
  end
end

Definition 3.2. We call a kernel function $\psi \in \mathcal{C}^1 : (0, \infty) \to [0, \infty)$ a $(1/t^2)$-bounded kernel function if it is defined by $\psi(t) := \frac{t^2-1}{2} - \varphi(t)$, where the barrier term φ satisfies the following conditions:

$$1 \le \varphi'(t) \le \frac{1}{t^2}, \qquad \text{for } 0 < t \le 1, \tag{3.1}$$

$$\frac{1}{t^2} \le \varphi'(t) \le t, \qquad \text{for } t > 1, \tag{3.2}$$

$$\inf_{t>0}\ t\varphi'(t) > \frac{3}{10}. \tag{3.3}$$

Roos's result [11] in 2006, employing the logarithmic barrier function, adheres to the conditions in Definition 3.2. Additionally, our new class of kernel functions includes many other classical kernel functions in [2, 4, 9, 10].

Let us define the new $(1/t^2)$-kernel function denoted by $\hat{\psi}$, for example, as follows:

$$\hat{\psi}(t) := \begin{cases} \dfrac{t^2-1}{2} - \left(1 - \dfrac{1}{t}\right), & \text{for } 0 < t < 1, \\[2ex] \dfrac{t^2-1}{2} - (t-1), & \text{for } t \ge 1. \end{cases}$$

Then $\hat{\psi}$ does not belong to the class of self-concordant functions, self-regular functions, and eligible functions. This observation illustrates the distinctiveness of our newly defined class of kernel functions. Furthermore, the newly introduced kernel function is continuously differentiable, simplifying the process of checking the conditions compared to classical ones.

4. Technical lemmas

In this section, we present several technical lemmas that are essential for the complexity analysis.

For the use in the proof of Lemma 5.6, let us define $a(\xi)$, $b(n,\xi)$, and $c(\xi)$ as follows: For $n \geq 20$ and $\xi \in (0.3, 0.39)$,

$$\begin{aligned} a(n,\xi) &:= 1 - 1.6n + \frac{16}{1-\xi},\ b(n,\xi) := 15 + 16.8n - \frac{16}{1-\xi}, \text{ and} \\ c(\xi) &:= -31 + 16\xi^2 + \frac{16}{1-\xi}. \end{aligned} \tag{4.1}$$

Lemma 4.1. *Let $a(n,\xi)$, $b(n,\xi)$, and $c(\xi)$ be defined in (4.1). Then, for $n \geq 20$ and $\xi \in (0.3, 0.39)$, we have the following: (i) $a(n,\xi) < 0$, (ii) $b(n,\xi) > 0$, and (iii) $c(\xi) < 0$.*

Proof. (*i*) $a(n,\xi)$ is decreasing with respect to n and increasing with respect to ξ. Since $a(20, 0.39) \approx -4.77$, it follows that $a(n,\xi) \leq a(20, 0.39) < 0$ for $n \geq 20$ and $\xi \in (0.3, 0.39)$.
(*ii*) For $n \geq 20$ and $\xi \in (0.3, 0.39)$, it is clear that $b(n,\xi) = 15 + 16.8n - \frac{16}{1-\xi} > 0$.
(*iii*) For $\xi \in (0.3, 0.39)$, we have $\frac{d}{d\xi}c(\xi) = 32\xi + \frac{16}{(1-\xi)^2} > 0$. Thus $c(n,\xi)$ is increasing with respect to ξ. Since $c(0.39) \approx -2.336$, it follows that $c(\xi) < 0$. Thus the proof is complete. □

Let us define $T(n,\xi)$ as follows: For $\xi \in (0.3, 0.39)$ and $n \geq 20$,

$$\begin{aligned} T(n,\xi) :=& 17.64(1-\xi)n^2 + 2\Big((1-\xi)(14.8 + 0.8\xi^2) - 16\Big)n \\ &+ (\xi^3 - 17\xi^2 - 16\xi + 16). \end{aligned} \tag{4.2}$$

Then we have the following lemma.

Lemma 4.2. *Let $T(n,\xi)$ be defined in (4.2) and let $a(n,\xi)$, $b(n,\xi)$, and $c(\xi)$ be defined in (4.1). Then, for $n \geq 20$ and $\xi \in (0.3, 0.39)$, we have*

$$b(n,\xi)^2 - a(n,\xi)c(\xi) > 0.$$

Proof. From (4.1), it is easy to show that $b(n,\xi)^2 - a(n,\xi)c(\xi) = \frac{16}{1-\xi}T(n,\xi)$. Since $1 - \xi > 0$, it suffices to show that $T(n,\xi) > 0$, for $n \geq 20$ and $\xi \in (0.3, 0.39)$. The partial derivative of $T(n,\xi)$ with respect to ξ is given by

$$\frac{\partial T(n,\xi)}{\partial \xi} = (3 - 4.8n)\xi^2 + \xi(3.2n - 34) - (17.64n^2 + 28.4n + 16). \tag{4.3}$$

Then the discriminant of (4.3) is $-338.688n^3 - 323.36n^2 - 184n + 1348$ and it is negative for $n \geq 20$. Since the leading coefficient of (4.3) is negative for $n \geq 20$, $T(n,\xi)$ is decreasing on $\xi \in (0.3, 0.39)$ for each $n \geq 20$. For $n \geq 20$, we have

$$T(n,\xi) \geq T(n, 0.39) = 10.76n^2 - 14.527n + 7.233. \tag{4.4}$$

The discriminant of the right-hand side of (4.4) is negative. Since the leading coefficient of (4.4) is positive, it follows that $T(n,\xi) \geq T(n, 0.39) > 0$. Thus the proof is complete. □

Lemma 4.3. *Let $a(n,\xi)$, $b(n,\xi)$, and $c(\xi)$ be defined in (4.1). Then, for $n \geq 20$ and $\xi \in (0.3, 0.39)$, $a(n,\xi)\,(1/16n)^2 + 2b(n,\xi)\,(1/16n) + c(\xi) \leq 0$.*

Proof. By the definition of $a(n,\xi)$ and $b(n,\xi)$ in (4.1), $a(n,\xi)$ is increasing with respect to ξ and $b(n,\xi)$ is decreasing with respect to ξ. From the proof of (*iii*) in Lemma 4.1, $c(\xi)$ is increasing with respect to ξ. For $n \geq 20$, we have

$$\begin{aligned} a(n,\xi)\,(1/16n)^2 + 2b(n,\xi)\,(1/16n) + c(\xi) &\leq a(n,0.39)(1/16n)^2 \\ &\quad + 2b(n,0.3)(1/16n) + c(0.39) \\ &< -0.236 - 0.988/n + 0.107/n^2 < 0. \end{aligned}$$

Thus the proof is complete. □

Lemma 4.4. *Let $a(n,\xi)$ and $b(n,\xi)$ be defined in (4.1). Then, for $n \geq 20$ and $\xi \in (0.3, 0.39)$, it holds that $-\,(b(n,\xi)/a(n,\xi)) > 1/16n$.*

Proof. From Lemma 4.1, $-a(n,\xi) > 0$ and $b(n,\xi) > 0$. For $n \geq 20$, we have

$$-\frac{b(n,\xi)}{a(n,\xi)} \geq \frac{b(n,0.39)}{-a(n,0.3)} > \frac{16.8 - 12/n}{1.6 - 23/n} > \frac{1}{16n}.$$

Thus the proof is complete. □

The following two lemmas are used to prove Lemma 4.7.

Lemma 4.5. *Let $n \geq 20$ and $\delta(v) \leq 1/24$, where δ is defined in Definition 2.9. Then, $0.959 \leq v_i \leq 1.043$, $i = 1, \ldots, n$.*

Proof. From $\frac{d}{dt}(t - 1/t)^2 = 2(t - 1/t^3)$ and $\frac{d^2}{dt^2}(t - 1/t)^2 = 2 + 6/t^4$, it follows that $(t-1/t)^2$ has minimum at $t = 1$ and it is strictly convex. From $\delta(v) = \frac{1}{2}\,\|v - 1/v\| \leq 1/24$, if v_i is the possible minimum or maximum for some $i = 1, \ldots, n$, then v_i satisfies the equation $(v_i - 1/v_i)^2 = 1/144$, which is equivalent to $v_i^4 - (289/144)v_i^2 + 1 = 0$. Then the positive solutions are $v_i = \sqrt{\left(289/144 \pm \sqrt{(289/144)^2 - 4}\right)/2}$, which are approximately 0.9592, and 1.0425. Thus the proof is complete. □

The following lemma is used to prove Lemma 4.7.

Lemma 4.6. *Let $f_\pm(w,z) := \left(1 + \frac{w}{z}\right) \pm \sqrt{\left(1 + \frac{w}{z}\right)^2 - 1}$ and $g_\pm(w,z) := \sqrt{f_\pm(w,z)} - \frac{1}{f_\pm(w,z)}$, for $w, z \in \mathbb{R}_{++}$. Then the following holds:*

(*i*) *$f_+(w,z)$ is increasing and $f_-(w,z)$ is decreasing with respect to $w \in \mathbb{R}_{++}$,*
(*ii*) *For $0 < w < \frac{1}{50}$, $z\,(g_+(w,z))^2$ is increasing with respect to $z \geq 1$,*
(*iii*) *For $0 < w < \frac{1}{50}$, $z\,(g_-(w,z))^2$ is decreasing with respect to $z \geq 1$.*

Proof. (*i*) Since $\frac{\partial}{\partial w}\,(f_\pm(w,z)) = \frac{1}{z}\left(1 \pm 1/\sqrt{1 - 1/\left(1 + w/z\right)^2}\right)$, $f_+(w,z)$ is increasing and $f_-(w,z)$ is decreasing with respect to $w \in \mathbb{R}_{++}$.

(ii) By taking the partial derivative with respect to z, we obtain

$$\frac{\partial}{\partial z}\Big(z\,(g_\pm(w,z))^2\Big)=\Big(g_\pm(w,z)\Big)\Big(g_\pm(w,z)+2z\frac{\partial}{\partial z}g_\pm(w,z)\Big). \tag{4.5}$$

By the definition of f_+, if follows that $f_+(w,z)>1$ for any $w,z\in\mathbb{R}_{++}$. Then by the definition of g_+, if follows that $g_+(w,z)>0$ for any $w,z\in\mathbb{R}_{++}$. By (4.5), to prove (ii), it suffices to show that

$$g_+(w,z)+2z\frac{\partial}{\partial z}g_+(w,z)>0. \tag{4.6}$$

The partial derivative $g_+(w,z)$ with respect to z is as follows:

$$\frac{\partial}{\partial z}g_+(w,z)=-\frac{w}{z^2}\left(1+\frac{(1+w/z)}{\sqrt{(1+w/z)^2-1}}\right)\left(\frac{1}{2\sqrt{f_+(w,z)}}+\frac{1}{f_+(w,z)^2}\right)<0.$$

Let $t:=\left(1+\frac{w}{z}\right)^2-1$. Since $0<w<\frac{1}{50}$ and $z\geq 1$, $0<t<\left(1+\frac{1/50}{1}\right)^2-1\leq 0.1$. Then, consider the following term in (4.5). Since $f_+(w,z)=\sqrt{t}+\sqrt{t+1}$,

$$\begin{aligned}&g_+(w,z)+2z\frac{\partial}{\partial z}g_+(w,z)\\&=\sqrt{f_+(w,z)}-\frac{1}{f_+(w,z)}-\frac{\sqrt{t+1}-1}{\sqrt{t}}\left(\sqrt{f_+(w,z)}+2f_+(w,z)^{-1}\right).\end{aligned} \tag{4.7}$$

By (4.7) and since $f_+(w,z)=\sqrt{t}+\sqrt{t+1}>0$ and $\sqrt{t}-\sqrt{t+1}+1>0$ for $t>0$, the following equivalences hold:

$$\begin{aligned}&g_+(w,z)+2z\frac{\partial}{\partial z}g_+(w,z)>0\\&\Leftrightarrow h(t):=\left(\sqrt{t}+\sqrt{t+1}\right)^{1.5}-\frac{\sqrt{t}+2\sqrt{t+1}-2}{\sqrt{t}-\sqrt{t+1}+1}>0.\end{aligned} \tag{4.8}$$

By multiplying $(\sqrt{t}-\sqrt{t+1}+1)^2$ to the derivative of h, we have

$$h'(t)(\sqrt{t}-\sqrt{t+1}+1)^2=\frac{1.5(\sqrt{t+1}-1)}{\sqrt{t}\sqrt{t+1}}\left((\sqrt{t}+\sqrt{t+1})^{0.5}-1\right)>0,$$

for $t>0$. Since $(\sqrt{t}-\sqrt{t+1}+1)^2>0$, it follows that $h'(t)>0$ for $t>0$. By the L'Hôpital's rule, $\lim_{t\to 0+}\frac{\sqrt{t}+2\sqrt{t+1}-2}{\sqrt{t}-\sqrt{t+1}+1}=1$. Then, by the definition of h in (4.8), $\lim_{t\to 0+}h(t)=0$. Since h is continuous and differentiable on $t>0$, $h(t)>0$ for $t>0$. Indeed, let us assume that there exists a point $\alpha>0$ such that $h(\alpha)<0$. Then, from $\lim_{t\to 0+}h(t)=0$, there exists $0<\beta<\alpha$ such that $h(\alpha)<h(\beta)$. Then, by the mean value theorem, there exists $\gamma\in(\beta,\alpha)$ such that $h'(\gamma)=\frac{h(\alpha)-h(\beta)}{\alpha-\beta}<0$, which contradicts to the fact that $h'(t)>0$ for $t>0$. Therefore, by (4.8), $g_+(w,z)+2z\frac{\partial}{\partial z}g_+(w,z)>0$ for $w\in(0,\frac{1}{50})$ and $z\geq 1$. This implies that (4.6) holds.

(iii) It can be derived in a similar way as in case (ii). Thus the proof is complete. □

Let us define

$$\sigma(v) := \frac{1}{2} \left\| v - \varphi'(v) \right\|, \tag{4.9}$$

where $\varphi(t)$ is defined in Definition 3.2.

We use the following lemma to prove Theorem 6.4.

Lemma 4.7. *Suppose that $\delta(v) \leq 1/24$, where δ is defined in (2.9). Then $\sigma(v) \leq 0.1$, where v is defined in (2.5) and σ is defined in (4.9).*

Proof. By the condition (3.1) in Definition 3.2, for $0 < t \leq 1$, we have

$$0 < 1 - t < \varphi'(t) - t < 1/t^2 - t. \tag{4.10}$$

By the condition (3.2) in Definition 3.2, for $1 < t$, we have

$$1/t^2 - t \leq \varphi'(t) - t \leq 0. \tag{4.11}$$

From (4.10) and (4.11), we have:

$$4\sigma(v)^2 \leq \left(v_1 - 1/v_1^2\right)^2 + \cdots + \left(v_n - 1/v_n^2\right)^2 = \left\| v - 1/v^2 \right\|. \tag{4.12}$$

Without loss of generality, we assume that $\delta := \delta(v) \neq 0$. From the definition of δ, $4\delta^2 = \sum_{i=1}^n \left(v_i - 1/v_i\right)^2$. To derive the upper bound for $\sigma(v)$, consider the following problem

$$\text{minimize } -\sum_{i=1}^n \left(v_i - 1/v_i^2\right)^2 \text{ subject to } \sum_{i=1}^n \left(v_i - 1/v_i\right)^2 = 4\delta^2. \tag{4.13}$$

Then, by the first-order optimality condition, the optimal solution v of (4.13) satisfies the following:

$$\left(-v_i - 1/v_i^2 + 2/v_i^5\right) + \lambda \left(v_i - 1/v_i^3\right) = 0, \; i = 1, \ldots, n, \tag{4.14}$$

where λ is the Lagrange multiplier.

Let I be the collection of indices such that $v_i \neq 1$. Then from (4.14),

$$\lambda = \frac{v_i^6 + v_i^3 - 2}{v_i^6 - v_i^2} = \frac{(v_i^3 + 2)(v_i^2 + v_i + 1)}{v_i^2(v_i^2 + 1)(v_i + 1)}, \; i \in I. \tag{4.15}$$

Combining (4.15) with Lemma 4.5, we have $\lambda > 0$. The derivative of right-hand side of (4.15) with respect to v_i is

$$-\frac{3v_i^5 + 10v_i^4 + 17v_i^3 + 12v_i^2 + 8v_i + 4}{v_i^3(1 + v_i)^2(1 + v_i^2)^2}. \tag{4.16}$$

Since $v > 0$, it is clear that (4.16) is negative. This implies that (4.15) is strictly decreasing with respect to $\{v_i\}_{i\in I}$. Since λ is constant, there are no $v_i \neq v_j$ satisfying (4.15). So, all elements $\{v_i\}_{i\in I}$ are identical. Let us denote this same value as $\hat{v} \in \mathbb{R}$. Then the objective function of (4.13) becomes

$$-\sum_{i=1}^n \left(v_i - 1/v_i^2\right)^2 = -|I| \left(\hat{v} - 1/\hat{v}^2\right)^2 \tag{4.17}$$

and the constraints of (4.13) becomes

$$\sum_{i=1}^{n} (v_i - 1/v_i)^2 = |I|\,(\hat{v} - 1/\hat{v})^2 = 4\delta^2. \tag{4.18}$$

The first-order and second-order derivatives of (4.17) with respect to $\hat{v}$ are

$$-2|I|\left(\hat{v} + 1/\hat{v}^2 - 2/\hat{v}^5\right) \text{ and } -2|I|\left(1 - 2/\hat{v}^3 - 10/\hat{v}^6\right), \tag{4.19}$$

respectively. Then (4.17) has maximum at $\hat{v} = 1$ and it is strictly concave. The two positive solutions of (4.18) are

$$\hat{v}_\pm := \sqrt{1 + 2\delta^2/|I| \pm \sqrt{(1 + 2\delta^2/|I|)^2 - 1}}. \tag{4.20}$$

We divide the proof into two cases for $\hat{v}_+$ and $\hat{v}_-$. To apply Lemma 4.6, we regard $|I|$ as z and $2\delta^2$ as w in Lemma 4.6. Then, $\hat{v}_+^2$ corresponds to $f_+(w, z)$ in Lemma 4.6. By Lemma 4.6 (i), $\hat{v}_+$ is increasing with respect to δ. From the definition of $\hat{v}_+$ in (4.20), $\hat{v}_+ > 1$. Since (4.17) has maximum at $\hat{v} = 1$ and it is strictly concave, (4.17) is decreasing with respect to δ. By Lemma 4.6 (ii), (4.17) decreases as $|I|$ increases. Thus (4.17) obtains its minimum when both δ and $|I|$ are at their maximum, that is, $\delta = 1/24$ and $|I| = n$. Then the objective function of (4.13) becomes

$$\begin{aligned} &-\sum_{i=1}^{n} \left(v_i - 1/v_i^2\right)^2 \\ &= -\frac{1}{\hat{v}_+^4}\left(\frac{\left(1/(288\sqrt{n}) + \sqrt{1/144 + 1/(288\sqrt{n})^2}\right)\left(\hat{v}_+^2 + \hat{v}_+ + 1\right)}{\sqrt{1 + 1/(288n) + \sqrt{1/(144n) + 1/\left(288n\right)^2} + 1}}\right)^2. \end{aligned} \tag{4.21}$$

Since $\lim_{n\to\infty} \hat{v}_+ = 1$, the last expression in (4.21) converges to -0.015625 as $n \to \infty$.

For case $\hat{v}_-$, in a similar way to case for $\hat{v}_+$, we can show that (4.17) obtains its minimum when δ is at its maximum and $|I|$ is at its minimum, that is, $\delta = 1/24$ and $|I| = 1$. Then the objective function of (4.13) becomes

$$-\sum_{i=1}^{n} \left(v_i - \frac{1}{v_i^2}\right)^2 = -\left(\hat{v}_- - \frac{1}{(\hat{v}_-)^2}\right)^2 > -0.017.$$

Therefore, the upper bound for the minimum of (4.17) in both cases for $\bar{v}_+$ and $\bar{v}_-$ is as follows:

$$\sigma(\hat{v}_\pm) = \frac{1}{2}\left\|\hat{v}_\pm - \varphi'(\hat{v}_\pm)\right\| \le \frac{1}{2}\left\|\hat{v}_\pm - 1/(\hat{v}_\pm)^2\right\| \le \frac{1}{2}\sqrt{0.017} < 0.1,$$

where the first inequality follows from (4.12) and second inequality follows from (4.21) and (4.22). Thus the proof is complete. □

To estimate the upper bound of the proximity measures in Lemma 5.4, we need the following lemma.

Lemma 4.8. *Let φ be any function satisfying* (3.1) *and* (3.2) *of Definition 3.2. Then we have*

$$\left|\Big(t\varphi'(t)\Big)^{\frac{1}{2}} - \Big(t\varphi'(t)\Big)^{-\frac{1}{2}}\right| \leq |t - 1/t|\,,\ t > 0.$$

Proof. Since

$$\left|t - \frac{1}{t}\right|^2 - \left|\big(t\varphi'(t)\big)^{\frac{1}{2}} - \big(t\varphi'(t)\big)^{-\frac{1}{2}}\right|^2 = t\left(1 - \frac{1}{t^3\varphi'(t)}\right)\big(t - \varphi'(t)\big)\,,$$

it is enough to show that $t\left(1 - \frac{1}{t^3\varphi'(t)}\right)(t - \varphi'(t)) > 0$. For $0 \leq t \leq 1$, by (3.1) of Definition 3.2, we have $t \leq 1 \leq \varphi'(t) \leq \frac{1}{t} \leq \frac{1}{t^3}$. So we have $1 - \frac{1}{t^3\varphi'(t)} \leq 0$ and $t - \varphi'(t) \leq 0$. Therefore $t\left(1 - \frac{1}{t^3\varphi'(t)}\right)(t - \varphi'(t)) \geq 0,$ for $0 < t \leq 1$. For $1 < t$, by (3.2) of Definition 3.2, we have $\frac{1}{t^3} < \varphi'(t) \leq t$. So we have $1 - \frac{1}{t^3\varphi'(t)} \geq 0$ and $t - \varphi'(t) \geq 0$. Therefore $t\left(1 - \frac{1}{t^3\varphi'(t)}\right)(t - \varphi'(t)) \geq 0,$ for $1 < t$. Thus the proof is complete. □

5. Feasibility step

In this section, we address the feasibility step, ensuring that the new iterations are strictly positive and that the proximity measure is less than or equal to $1/\sqrt{2}$, which is the assumption in Lemma 2.2.

The following lemma can be derived using a similar approach to Lemma II.45 in [12] and Lemma 4.1 in [11].

Lemma 5.1. *In the feasibility step, the modified full-Newton step is strictly feasible if and only if $v\varphi'(v) + d_x^f d_s^f > 0$, where v, d_x^f, and d_x^f are defined in (2.5) and φ is defined in Definition 3.2.*

In the following, we define $\omega_i(v)$ as follows: $\omega_i := \omega_i(v) := \frac{1}{2}\sqrt{(d_x^f)_i^2 + (d_s^f)_i^2}$, and

$$\omega := \omega(v) := \|(\omega_1, \ldots, \omega_n)\|\,. \tag{5.1}$$

This implies that $\|d_x^f\| \leq 2\omega$ and $\|d_s^f\| \leq 2\omega$. Moreover,

$$|(d_x^f d_s^f)_i| = |(d_x^f)_i||(d_s^f)_i| \leq \frac{1}{2}\left((d_x^f)_i^2 + (d_s^f)_i^2\right) \leq 2\omega_i^2 \leq 2\omega^2. \tag{5.2}$$

The following lemma gives the conditions ensuring that the new iteration, after feasibility step, is strictly feasible.

Lemma 5.2. *Let $n \geq 20$ and $\xi \in (0.3,\ \min\{0.39,\ \inf_{t>0} t\varphi'(t)\})$. Then for any $\omega \leq \frac{\xi}{\sqrt{2}}$, the new iteration after feasibility step is strictly feasible for* (2.1) *and* (2.2) *with $\nu = \nu^+$.*

Proof. Suppose that $\omega \leq \frac{\xi}{\sqrt{2}}$. Then from (5.2), we have $|d_x^f d_s^f| \leq 2\omega^2 \leq \xi^2$. Since $0 < \xi < 1$, $\xi^2 < \xi$. Thus $|d_x^f d_s^f| < \xi$. Then by the condition (3.3) of Definition 3.2, $\xi < v\varphi'(v)$. This implies that $|d_x^f d_s^f| < \xi < v\varphi'(v)$. By Lemma 5.1, the proof is complete. □

For the notational convenience, we define u as follows:

$$u := v\varphi'(v). \tag{5.3}$$

Lemma 5.3. *Let $n \geq 20$. Suppose that $x^T s = n\mu$. Then $\|u^{\frac{1}{2}}\|^2 \leq 2.05n$*

Proof. By the conditions (3.1) and (3.2) of Definition 3.2,

$$\|u^{\frac{1}{2}}\|^2 \leq \|v^{-1}\| + \|v\| \leq 1.05n + \|v\|^2 = 2.05n,$$

where the second inequality follows from the triangle inequality and Lemma 4.5. Thus the proof is complete. □

Lemma 5.4. *Let $n \geq 20$. Suppose that $x^T s = n\mu$. Then we have*

$$4\delta\left(\frac{u^{\frac{1}{2}}}{\sqrt{1-\theta}}\right)^2 \leq 4(1-\theta)\delta(v)^2 + \frac{\theta^2 n}{1-\theta} + 2.1\theta n.$$

Proof. By the definition of $\delta(v)$,

$$\begin{aligned} 4\delta\left(\frac{u^{\frac{1}{2}}}{\sqrt{1-\theta}}\right)^2 &= (1-\theta)\left\|u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right\|^2 + \frac{\theta^2}{1-\theta}\left\|u^{\frac{1}{2}}\right\|^2 - 2\theta\left\langle u^{-\frac{1}{2}} - u^{\frac{1}{2}}, u^{\frac{1}{2}}\right\rangle \\ &\leq 4(1-\theta)\delta(v)^2 + \frac{\theta^2 n}{1-\theta} + 2.1\theta n, \end{aligned}$$

where the inequality follows from Lemma 4.8 and Lemma 5.3. Thus the proof is complete. □

In the sequel, we denote

$$\delta(v^f) := \delta(x^f, s^f; \mu^+), \tag{5.4}$$

where x^f and s^f are the new iteration after a feasibility step and $\mu^+ := (1-\theta)\mu$.

The following lemma can be derived by a similar way to Lemma 2.3 in [7] and Lemma 4.4 in [11]. Note that to prove following lemma, we need Lemma 5.4 above.

Lemma 5.5. *Let $n \geq 20$. Assume that $\omega \leq \frac{\xi}{\sqrt{2}}$, where ω is defined in (5.1). Then for $\theta \in (0, 1)$, we have*

$$4\delta(v^f)^2 \leq 4(1-\theta)\delta(v)^2 + \frac{\theta^2 n}{1-\theta} + 2.1\theta n + \frac{2\omega^2}{1-\theta} + \frac{(1-\theta)2\omega^2}{\xi(\xi - 2\omega^2)},$$

where $\delta(v)$ is defined in (2.9), and $\delta(v^f)$ is defined in (5.4).

Lemma 5.6. *Let $n \geq 20$. Assume that $\xi \in \Big(0.3,\ \min\{0.39,\ \inf_{t>0} t\varphi'(t)\}\Big)$. Then for $\theta \leq 1/16n$, $\delta(v) \leq 1/24$ and $\omega \leq \frac{\xi}{\sqrt{2}}$, we obtain*

$$\delta(v^f) \leq \frac{1}{\sqrt{2}},$$

where $\delta(v)$ is defined in (2.9), and $\delta(v^f)$ is defined in (5.4).

Proof. By Lemma 5.5,

$$\begin{aligned} 4\delta(v^f)^2 &\le 4(1-\theta)\delta(v)^2 + \frac{2\theta^2 n}{1-\theta} + 2.1\theta n + \frac{2\omega^2}{1-\theta} + \frac{2(1-\theta)\omega^2}{\xi(\xi-2\omega^2)} \\ &\le \frac{1-\theta}{16} + \frac{2\theta^2 n}{1-\theta} + 2.1\theta n + \frac{\xi^2}{1-\theta} + \frac{1-\theta}{1-\xi} \qquad (5.5) \\ &= \frac{1}{16(1-\theta)}\Big(a(n,\xi)\theta^2 + 2b(n,\xi)\theta + c(\xi)\Big) + 2 \end{aligned}$$

where the second inequality holds by choosing $\delta(v)$ and ω as maximum values, and $a(n,\xi)$, $b(n,\xi)$, and $c(\xi)$ are defined in (4.1). By Lemma 4.2, for $\xi \in (0.3, 0.39)$ and for $n \ge 20$, the equation

$$a(n,\xi)\theta^2 + 2b(n,\xi)\theta + c(\xi) = 0 \tag{5.6}$$

has two solutions. By Lemma 4.3, for any $\xi \in (0.3, 0.39)$, $a(n,\xi)\,(1/16n)^2 + 2b(n,\xi)\,(1/16n) + c(\xi) < 0$ for $n \ge 20$. By Lemma 4.4, for any $\xi \in (0.3, 0.39)$, the value $-a(n,\xi)/b(n,\xi)$, which is an axis of symmetry, is greater than $1/16n$, for $n \ge 20$. These imply that the smaller root of the equation (5.6) is greater than $1/16n$. Since $a(n,\xi) \le 0$, as given in Lemma 4.1, it follows that the equation is negative for $\theta \le 1/16n$. Thus from (5.5), $4\delta(V)^2 \le 2$. Thus the proof is complete. ☐

6. The complexity analysis

Let us define $\mathcal{L} := \{\xi \in \mathbf{R}^n : \bar{A}\xi = 0\}$. Then by the first equation of (2.7), $\{\xi \in \mathbf{R}^n : \bar{A}\xi = \theta\nu r_b^0\}$ is equal to $d_x^f + \mathcal{L}$. We know that affine space $\{\bar{A}^T\zeta : \zeta \in \mathbf{R}^m\}$ is the orthogonal complement of $\mathcal{L}$, denoted by $\mathcal{L}^\perp$. By the second equation of (2.7), we conclude that $\{\theta\nu v s^{-1} r_c^0 + \bar{A}^T\zeta : \zeta \in \mathbf{R}^n\} = d_s^f + \mathcal{L}^\perp$. Since $\mathcal{L} \cap \mathcal{L}^\perp = \{0\}$, the spaces $d_x^f + \mathcal{L}$ and $d_s^f + \mathcal{L}$ meet in a unique vector q.

To compute the number of outer iterations, we need the following lemma.

Lemma 6.1 (Lemma II.17 in [12]). *If the barrier parameter μ has the initial value μ^0 and is repeatedly multiplied by $1-\theta$, with $0 < \theta < 1$, then after ate most $\lceil \frac{1}{\theta} \log \frac{n\mu^0}{\epsilon} \rceil$ iterations we have $n\mu \le \epsilon$.*

We have the following lemma which is derived in a similar way to proving Lemma 4.6 in [11].

Lemma 6.2. *Let q be the unique vector in the intersection of the affine spaces $d_x^f + \mathcal{L}$ and $d_s^f + \mathcal{L}$. Then*

$$2\omega \le \sqrt{\|q\|^2 + \big(\|q\| + 2\sigma(v)\big)^2},$$

where v is defined in (2.5), *ω is defined in (5.1), and $\sigma(v)$ is defined in (4.9).*

The following lemma is for analyzing the feasibility step in the following Theorem 6.4.

Lemma 6.3 (Lemma 4.7 in [11]). *One has*

$$\sqrt{\mu}\|q\| \le \theta\nu\zeta\sqrt{e^T\left(\frac{x}{s} + \frac{s}{x}\right)}.$$

In the following, we give complexity result of the Algorithm 1 for IIPMs for LO. This is the best known complexity result for such the method.

Theorem 6.4. *Let $n \geq 20$, $\theta = 1/16n$, and $\tau = 1/24$. Assume that there is an optimal solution (x^*, y^*, s^*) such that $\|x^* + s^*\|_\infty < \zeta$ for some $\zeta > 0$. Let us define the initial iteration by $(x^0, y^0, s^0) := \zeta(e, 0, e)$, $\mu^0 e := x^0 s^0$, and $\nu^0 := 1$. Then after at most*

$$\left\lceil 80n \;\log \frac{\max\left\{n\zeta^2, \left\|r_b^0\right\|, \left\|r_c^0\right\|\right\}}{\epsilon} \right\rceil$$

iterations, the Algorithm 1 finds an ϵ-approximate solution for LO, where $r_b^0 := b - Ax^0$, and $r_c^0 := c - A^T y^0 - s^0$.

Proof. Let us assume that $\theta = 1/16n$ and $\xi \in (0.3,\ \min\{0.39,\ \inf_{t>0} t\varphi'(t)\})$. Since $\delta(v) \leq \tau = 1/24$, by Corollary A.10 in [11], $\sqrt{x/s} \leq \sqrt{2}\; x(\mu, \nu)/\sqrt{\mu}$ and $\sqrt{s/x} \leq \sqrt{2}\; s(\mu, \nu)/\sqrt{\mu}$, where $x(\mu, \nu)$ and $s(\mu, \nu)$ denote that they are μ-centers of the perturbed problems (2.1) and (2.2) with respect to ν. For reasons similar to those in section 4.5 of [11], we can deduce the following based on Lemma 6.3,

$$\|q\| \leq \frac{\theta\nu\zeta}{\sqrt{\mu}} \sqrt{e^T\left(\frac{x}{s} + \frac{s}{x}\right)} \leq \frac{\sqrt{2}\theta}{\zeta} \sqrt{\|x(\mu, \nu)\|^2 + \|s(\mu, \nu)\|^2}. \tag{6.1}$$

By the same argument in the section 4.6 of [11], we have

$$\sqrt{\|x(\mu, \nu)\|^2 + \|s(\mu, \nu)\|^2} \leq 2\zeta n. \tag{6.2}$$

Then by (6.1) and (6.2), we have $\|q\| \leq 2\sqrt{2}\theta n$. Then by the Lemma 6.2 and Lemma 4.7,

$$4\omega^2 \leq 8\theta^2 n^2 + \left(2\sqrt{2}\theta n + 0.2\right)^2 < 0.1733 < 2\xi^2,$$

where $\theta = 1/16n$. From (2.10) and $\tau = 1/24$, at each centering step, we need 4 iterations in the centering step. Then, by Lemma 6.1, the total number of iterations is at most $\left\lceil 80n \;\log \frac{\max\left\{n\zeta^2, \left\|r_b^0\right\|, \left\|r_c^0\right\|\right\}}{\epsilon} \right\rceil$. Thus the proof is complete. ☐

7. Conclusion

In this paper, we propose new full-Newton step IIPMs for LO. A search direction is determined by computing the newly defined class of kernel functions and we employ a full-Newton step method. The class of kernel functions defined in this paper has the advantage of weaker conditions compared to existing kernel functions. We propose a unified approach for complexity analysis with the best known worst-case iteration bounds. Therefore, we have developed new small-update full-Newton step IIPMs for LO using the newly defined class of kernel functions and shown that this method has the most efficient computational complexity known to date. As part of our future work, our primary expectation is to improve the constants in computational complexity. Additionally, we want to extend this method to more general optimization problems.

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