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STABILITY ANALYSIS OF THE MULTI-SOURCE WEBER PROBLEM

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Dedicated to Professor Do Sang Kim on the occasion of his 70th birthday.

Abstract. Stability analysis of the multi-source Weber problem, where the data set is subject to perturbations, is studied for the first time in this paper. Under suitable conditions, we establish the locally Lipschitz continuity of the optimal value function, the upper semicontinuity of the global solution map, and the inner semicontinuity of the local solution map. Examples are given to show that the lower semicontinuity as well as the inner semicontinuity of the global solution map may not be available under the assumptions guaranteeing the upper semicontinuity of the map.

1. Introduction

Let there be given a finite number of demand points in an Euclidean space. The problem of finding a location for a *source* (a *facility*) with the minimal sum of the weighted Euclidean distances from the demand points is the *single-source Weber problem* (SWP for brevity), which is also called the *Fermat-Weber location problem* or the *Fermat-Torricelli problem*. The basic properties of SWP and methods for solving the problem can be found in the papers of Weiszfeld [23], Kuhn [11], Vardi and Zhang [22], Beck and Sabach [1], Nobakhtian and Raeisi Dehkordi [17], Mordukhovich and Nam [16], Cuong et al. [7], and the references therein.

If one wants to minimize the sum of the weighted minima of the Euclidean distances of finitely many demand points to some facilities, whose locations are to be found, then one has deal with *multi-source Weber problem* (MWP for brevity), which is also known as the *location-allocation problem* and *the clustering problem with Euclidean norms*. The MWP is a fundamental nonsmooth, nonconvex programming problem; see, e.g., Cooper [4,5], Kuenne and Soland [10], Love et al. [13], Brimberg and Mladenović [3], Tuy [20], Tuy et al. [21], Sabach et al. [19], and Raeisi Dehkordi [18].

Very recently, various properties of the global solutions (resp., of the local solutions) of MWP have been obtained in [7] (resp., in [8]).

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In this paper, we are interested in studying the stability of the optimal value function, the global solution map, and the local solution map of the MWP when the data undergoes small perturbations. We will show that, under some conditions, the optimal value function is locally Lipschitz, the global solution map is upper semicontinuous, and the local solution map is inner semicontinous. Several illustrative examples are provided. To obtain these stability properties, we will apply the results of [7] and [8] to the full extent.

The theorems and proofs herein are very different from those given in [6, Section 4], where stability properties of the minimum sum-of-squares clustering problem were investigated. Actually, in comparison with the work done in the just cited paper, the stability analysis of the MWP requires more delicate arguments and tools.

The paper is organized as follows. Some preliminaries are provided in Section 2. Section 3 establishes the locally Lipschitz continuity of the optimal value function of the MWP. Upper semicontinuity of the global solution map is proved in Section 4, while the inner semicontinuity of the local solution map is addressed in Section 5. Concluding remarks and two open questions are given in Section 6.

2. Preliminaries

By $\mathbb{N}$ we denote the set of positive integers. For any $n \in \mathbb{N}$, the Euclidean space $\mathbb{R}^n$ is equipped with the scalar product $\langle x, y\rangle = \sum_{r=1}^n x_r y_r$ and the norm $\|x\| = \big(\sum_{r=1}^n x_r^2\big)^{1/2}$, where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. The open ball (resp., the closed ball) with center $x \in \mathbb{R}^n$ and radius $\rho > 0$ are denoted by $B(x, \rho)$ (resp., $\bar{B}(x, \rho)$). The convex hull of a set $\Omega \subset \mathbb{R}^n$ is abbreviated to $\operatorname{co}\Omega$. For our convenience, vectors of finite-dimensional Euclidean spaces are interpreted as columns of real numbers in matrix calculations, but represented as rows of real numbers in the text.

2.1. **The multi-source Weber problem.** A *data set* $A = \{a^1, \dots, a^m\}$ is a finite subset of $\mathbb{R}^n$. Each element of A is a *demand point*. Let a positive integer k and positive real constants $s_1, \dots, s_m$ be given. It is supposed that $1 \le k \le m$. Put $I = \{1, \dots, m\}$ and $J = \{1, \dots, k\}$. Every constant s_i, $i \in I$, is understood as the *weight* corresponding to the demand point a^i.

The multi-source Weber problem (MWP for brevity) or *the clustering problem with Euclidean norms* is one of minimizing the sum of the weighted minima of the Euclidean distances of the data points to the facilities. To solve the problem, one has to partition A into k disjoint subsets $A^1, \dots, A^k$, called *clusters*, and associate to each cluster A^j a *facility* (also called a *centroid*) $x^j \in \mathbb{R}^n$ so that the number

$$f(x, a) := \sum_{i\in I} \left(s_i \min_{j\in J} \|a^i - x^j\| \right)$$

is as small as possible. Therefore, solving MWP is to resolve the *unconstrained nonsmooth nonconvex optimization problem*

$$\min\Big\{ f(x, a) \;\mid\; x = (x^1, \dots, x^k) \in (\mathbb{R}^n)^k = \mathbb{R}^{nk} \Big\}. \tag{2.1}$$

Let $\mathbb{R}^{nk}$ be equipped with the Euclidean norm, which is denoted by $\|\cdot\|$.

Definition 2.1. A vector $\hat{x} = (\hat{x}^1, \ldots, \hat{x}^k) \in \mathbb{R}^{nk}$ is said to be a *local solution* of (2.1) if there is $\varepsilon > 0$ such that $f(\hat{x}, a) \le f(x, a)$ for every $x = (x^1, \ldots, x^k) \in \mathbb{R}^{nk}$ satisfying $\|x - \hat{x}\| < \varepsilon$.

In what follows, (2.1) is considered as a *parametric optimization problem* with

$$a = \left(a^1, \ldots, a^m\right) \in (\mathbb{R}^n)^m = \mathbb{R}^{nm}$$

being the parameter. The *global solution set* (resp., the *optimal value*, and the *local solution set*) of the problem is denoted by $\mathbf{S}(a)$ (resp., $v(a)$, and $\mathbf{S_1}(a)$). Thus, a vector $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^k)$ belongs to $\mathbf{S}(a)$ if and only if $f(\bar{x}, a) \le f(x, a)$ for every $x = (x^1, \ldots, x^k) \in \mathbb{R}^{nk}$. In addition, $v(a) = f(\bar{x}, a)$ for every $\bar{x} \in \mathbf{S}(a)$. It is clear that $\mathbf{S}(a) \subset \mathbf{S_1}(a)$. Since $\mathbf{S}(a)$ is nonempty by [7, Theorem 3.3], $v(a)$ is a finite real number.

In the case where $k = 1$, (2.1) is known as the *single-source Weber problem*, or the *Fermat-Weber location problem.* In this case, one has the convex minimization problem

$$\min\left\{f(x, a) = \sum_{i\in I} s_i\|a^i - x\| \ \mid \ x \in \mathbb{R}^n\right\}. \tag{2.2}$$

By definition, $\bar{x} \in \mathbb{R}^n$ is a solution of the single-source Weber problem if and only if

$$\sum_{i\in I} s_i\|a^i - \bar{x}\| \le \sum_{i\in I} s_i\|a^i - x\| \quad \forall x \in \mathbb{R}^n.$$

To study the stability of of the multi-source Weber problem, we will need the following notions of natural clustering and attraction set, which were proposed in [6] for the minimum sum-of-squares clustering problem and successfully used in [7] (resp., in [8]) to study of the global solution set (resp., the global solution set) of (2.1).

Definition 2.2. (See [6, 7]) Given a facilities system $x = (x^1, \ldots, x^k) \in \mathbb{R}^{nk}$, we say that the component x^j of x is *attractive* with respect to the data set A if the set

$$A[x^j] := \left\{a^i \in A \ \mid \ \|a^i - x^j\| = \min_{q\in J} \|a^i - x^q\|\right\},$$

called the *attraction set* of x^j, is nonempty.

Given a vector $x = (x^1, \ldots, x^k) \in \mathbb{R}^{nk}$, we construct k disjoint subsets $A^1, \ldots, A^k$ of A in the following way. Put $A^0 = \emptyset$ and let

$$A^j = \left\{a^i \in A \setminus \Big(\bigcup_{p=0}^{j-1} A^p\Big) \ \mid \ \|a^i - x^j\| = \min_{q\in J} \|a^i - x^q\|\right\} \quad (j = 1, \ldots, k). \tag{2.3}$$

Then, for all $i \in I$ and $j \in J$, the data point a^i belongs to the cluster A^j if and only if the distance $\|a^i - x^j\|$ is the minimal one among the distances $\|a^i - x^q\|$, $q \in J$, and $a^i \notin \bigcup_{p=1}^{j-1} A^p$. By (2.3), one has $A^j = A[x^j] \setminus \big(\bigcup_{p=1}^{j-1} A^p\big)$ for every $j \in J$.

Definition 2.3. (See [6, 7]) The family $\{A^1, \ldots, A^k\}$, which is constructed by the rule (2.3), is said to be the *natural clustering* associated with the facilities system $x = (x^1, \ldots, x^k)$.

The inclusion $\mathbf{S}(a) \subset \mathbf{S_1}(a)$ can be strict even for low-dimensional multi-source Weber problems having only a few data points; see, e.g., [8, Examples 2.5–2.7].

The next remark is similar to [6, Remark 2.9], which was given for the Minimum Sum-of-Squares Clustering problem.

Remark 2.4. If $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^k) \in \mathbb{R}^{nk}$ is a global solution (resp., a local solution) of (2.1), then the vector $\bar{x}^\sigma := (\bar{x}^{\sigma(1)}, \ldots, \bar{x}^{\sigma(k)})$ is also a global solution (resp., a local solution) of (2.1) for any permutation σ of J. This observation follows easily from the fact that $f(x) = f(x^\sigma)$, where $x = (x^1, \ldots, x^k) \in \mathbb{R}^{nk}$ and $x^\sigma := (x^{\sigma(1)}, \ldots, x^{\sigma(k)})$.

Remark 2.4 allows us to simply the formulas of $\mathbf{S}(a)$ and $\mathbf{S_1}(a)$. Namely, for a concrete multi-source Weber problem, we need just to describe the sets $\mathbf{S}(a)$ and $\mathbf{S_1}(a)$ up to permutations.

2.2. **Continuity properties of functions and multifunctions.** Let Ω be a nonempty subset of $\mathbb{R}^p$, where $p \in \mathbb{N}$. Equipped with the *relative topology* (see, e.g., [9, p. 51] and [24, Definition 6.1], Ω is a topological space.

One says that a real-valued function $g : \Omega \to \mathbb{R}$ is said to be *locally Lipschitz* at $\bar{\omega} \in \Omega$ if there exist a neighborhood U of $\bar{\omega}$ and a constant $\ell > 0$ such that

$$|g(\omega') - g(\omega)| \leq \ell \|\omega' - \omega\| \quad \forall \omega, \omega' \in U.$$

Clearly, if g is locally Lipschitz at $\bar{\omega} \in \Omega$, then g is continuous at every point in a neighborhood of $\bar{\omega}$.

Given a multifunction $F : \Omega \rightrightarrows \mathbb{R}^q$, where $q \in \mathbb{N}$, one defines the domain and the graph of F respectively by $\text{dom}F := \{x \in \Omega \mid F(x) \neq \emptyset\}$ and

$$\text{gph}F := \{(\omega, y) \in \Omega \times \mathbb{R}^q \mid y \in F(\omega)\}.$$

Definition 2.5. Let $F : \Omega \rightrightarrows \mathbb{R}^q$ be a multifunction.

(a) F is called *upper semicontinuous* [2, p. 109] at $\bar{\omega} \in \Omega$ if for any open set $V \subset \mathbb{R}^q$ with $F(\bar{\omega}) \subset V$ there is a neighborhood U of $\bar{\omega}$ such that $F(\omega) \subset V$ for every $\omega \in U$.
(b) One says that F is *lower semicontinuous* [2, p. 109] at $\bar{\omega} \in \text{dom}F$ if for any open set $V \subset \mathbb{R}^q$ with $F(\bar{\omega}) \cap V \neq \emptyset$ there exists a neighborhood U of $\bar{\omega}$ such that $F(\omega) \cap V \neq \emptyset$ for every $\omega \in U$.
(c) If F is both upper semicontinuous and lower semicontinuous at $\bar{\omega}$, then it is said that F is *continuous* at $\bar{\omega}$.
(d) F is said to be *inner semicontinuous*[1] at $(\bar{\omega}, \bar{y}) \in \text{gph}F$ if for any open set $V \subset \mathbb{R}^q$ with $\bar{y} \in V$ there is a neighborhood U of $\bar{\omega}$ such that $F(\omega) \cap V \neq \emptyset$ for every $\omega \in U$.

Clearly, F is lower semicontinuous at $\bar{\omega} \in \text{dom}F$ if and only if it is inner semicontinuous at every point $(\bar{\omega}, \bar{y})$ with $\bar{y} \in \text{gph}F$.

[1] In the finite-dimensional space setting, this definition is equivalent to the one given in [14, Definition 1.63(i)] whenever $\text{dom}F = \Omega$.

3. Locally Lipschitz continuity of the function v(.)

In this section, we will study the *optimal value function* $v : \mathbb{R}^{nm} \to \mathbb{R}$ of (2.1) which puts every data tube $a = \big(a^1, \ldots, a^m\big) \in \mathbb{R}^{nm}$ in correspondence with a nonnegative real value $v(a)$. Thus, the multi-source Weber problem is perturbed, and a plays the role of a perturbation parameter.

Using some theorems from [7] we will be able to prove the following result.

Theorem 3.1. *The optimal value function $v(.)$ of the multi-source Weber problem (2.1) is locally Lipschitz at every point $\bar{a} = \big(\bar{a}^1, \ldots, \bar{a}^m\big) \in \mathbb{R}^{nm}$ satisfying the condition $\bar{a}^i \neq \bar{a}^p$ for any $i, p \in I$ with $i \neq p$. Moreover, the value*

$$\sqrt{2(km+1)}\left(\sum_{i \in I} s_i\right),$$

which does not depend on $\bar{a}$, can be chosen as the Lipschitz constant for the locally Lipschitz property. In particular, the function $v(.)$ is continuous at every point $\bar{a} = \big(\bar{a}^1, \ldots, \bar{a}^m\big) \in \mathbb{R}^{nm}$ satisfying the condition $\bar{a}^i \neq \bar{a}^p$ for any $i, p \in I$ with $i \neq p$.

Proof. By definition, we have

$$v(a) = \min\left\{ f(x,a) = \sum_{i \in I}\left(s_i \min_{j \in J} \|a^i - x^j\|\right) \mid x = (x^1, \ldots, x^k) \in \mathbb{R}^{nk}\right\} \tag{3.1}$$

for all $a = \big(a^1, \ldots, a^m\big) \in \mathbb{R}^{nm}$. For each pair $(i,j) \in I \times J$, set $\varphi_{i,j}(x,a) = \|a^i - x^j\|$ for every $(x,a) \in \mathbb{R}^{nk} \times \mathbb{R}^{nm}$, where $x = (x^1, \ldots, x^k)$ and $a = \big(a^1, \ldots, a^m\big)$.

Claim 1. *The function $\varphi_{i,j}$ is Lipschitz on $\mathbb{R}^{nk} \times \mathbb{R}^{nm}$ with the Lipschitz constant $\sqrt{2}$, i.e., $\big|\varphi_{i,j}(\tilde{x},\tilde{a}) - \varphi_{i,j}(x,a)\big| \leq \sqrt{2}\,\|(\tilde{x},\tilde{a}) - (x,a)\|$ for all $(\tilde{x},\tilde{a})$ and (x,a) from $\mathbb{R}^{nk} \times \mathbb{R}^{nm}$.*

Indeed, to prove the desired inequality for any $(\tilde{x},\tilde{a}), (x,a) \in \mathbb{R}^{nk} \times \mathbb{R}^{nm}$, it suffices to use the Cauchy–Schwarz inequality and notice that

$$\begin{aligned}
\big|\varphi_{i,j}(\tilde{x},\tilde{a}) - \varphi_{i,j}(x,a)\big| &= \big|\|\tilde{a}^i - \tilde{x}^j\| - \|a^i - x^j\|\big| \\
&\leq \|(\tilde{a}^i - \tilde{x}^j) - (a^i - x^j)\| \\
&\leq 1.\|\tilde{a}^i - a^i\| + 1.\|\tilde{x}^j - x^j\| \\
&\leq (1+1)^{1/2}\Big(\|\tilde{a}^i - a^i\|^2 + \|\tilde{x}^j - x^j\|^2\Big)^{1/2} \\
&\leq \sqrt{2}\,\|(\tilde{x},\tilde{a}) - (x,a)\|.
\end{aligned}$$

Claim 2. *If $\varphi_1, \ldots, \varphi_s : \mathbb{R}^r \to \mathbb{R}$ are Lipschitz functions with the Lipschitz constants $\ell_1, \ldots, \ell_s$, then $\varphi(z) := \min_{p \in P} \varphi_p(z)$ for $z \in \mathbb{R}^r$ with $P := \{1, \ldots, s\}$ is a Lipschitz function on $\mathbb{R}^r$ with the Lipschitz constant $\ell := \max_{p \in P} \ell_p$.*

This standard fact can be proved easily if one puts $P(z) = \big\{p \in P \mid \varphi_p(z) = \varphi(z)\big\}$ and observes that the following holds for any $\tilde{z}, z \in \mathbb{R}^r$ and $\bar{p} \in P(z)$:

$$\begin{aligned}
\varphi(\tilde{z}) - \varphi(z) = (\min_{p \in P} \varphi_p(\tilde{z})) - \varphi_{\bar{p}}(z) &\leq \varphi_{\bar{p}}(\tilde{z}) - \varphi_{\bar{p}}(z) \\
&\leq \ell_{\bar{p}}\|\tilde{z} - z\| \\
&\leq \ell\|\tilde{z} - z\|.
\end{aligned}$$

Indeed, changing the roles of $\tilde{z}$ and z, one gets $\varphi(z) - \varphi(\tilde{z}) \leq \ell\|z - \tilde{z}\|$. Hence, $|\varphi(\tilde{z}) - \varphi(z)| \leq \ell\|\tilde{z} - z\|$.

Applying Claim 1 with Claim 2 to the function $f(x,a)$ in (3.1) yields the next result.

Claim 3. *The objective function* $f(x,a) = \sum_{i\in I}\big(s_i \min_{j\in J}\|a^i - x^j\|\big)$ *of the optimization problem in* (3.1) *is Lipschitz on* $\mathbb{R}^{nk}\times\mathbb{R}^{nm}$ *with the Lipschitz constant* $\ell_0 := \sqrt{2}\,\big(\sum_{i\in I} s_i\big)$.

Let $\bar a = \big(\bar a^1,\dots,\bar a^m\big) \in \mathbb{R}^{nm}$ be such that $\bar a^i \neq \bar a^p$ for any $i,p\in I$ with $i\neq p$. Clearly, there exists a neighborhood U of $\bar a$ in $\mathbb{R}^{nm}$ with the property that for every $a = \big(a^1,\dots,a^m\big)\in U$ we have $a^i\neq a^p$ for any $i,p\in I$ with $i\neq p$. Hence, for each $a = \big(a^1,\dots,a^m\big)\in U$, by [7, Theorem 3.7] we can infer that if $x = (x^1,\dots,x^k)\in \mathbf{S}(a)$ then $x^j\in \operatorname{co} A$ for all $j\in J$, where $A := \{a^1,\dots,a^m\}$. In other words,

$$\mathbf{S}(a)\subset (\operatorname{co} A)^k \qquad \forall a\in U. \tag{3.2}$$

Moreover, by [7, Theorem 3.3], $\mathbf{S}(a)$ is nonempty. Therefore, from (3.1) and (3.2) we can deduce that

$$v(a) = \min\Big\{ f(x,a)\ \mid\ x\in (\operatorname{co} A)^k\Big\} \qquad \forall a\in U. \tag{3.3}$$

Claim 4. *For any* $\tilde a = \big(\tilde a^1,\dots,\tilde a^m\big)$ *and* $a = \big(a^1,\dots,a^m\big)$ *from* $\mathbb{R}^{nm}$, *one has*

$$\operatorname{co}\widetilde{A}\subset \operatorname{co} A + \big(\|\tilde a^1 - a^1\| + \cdots + \|\tilde a^m - a^m\|\big)\,\bar B_{\mathbb{R}^n} \tag{3.4}$$

with $\widetilde{A} := \{\tilde a^1,\dots,\tilde a^m\}$ *and* $\bar B_{\mathbb{R}^n}$ *denoting the closed unit ball in* $\mathbb{R}^n$.

To justify the claim, take any $\tilde x\in\operatorname{co}\widetilde{A}$. Let $\lambda_1\geq 0,\dots,\lambda_m\geq 0$ be such that $\sum_{i=1}^m\lambda_i = 1$ and $\tilde x = \sum_{i=1}^m \lambda_i\tilde a^i$. Setting $x = \sum_{i=1}^m\lambda_i a^i$, we have $x\in\operatorname{co} A$. Since

$$\begin{aligned}\|\tilde x - x\| = \Big\|\sum_{i=1}^m\lambda_i\tilde a^i - \sum_{i=1}^m\lambda_i a^i\Big\| &= \Big\|\sum_{i=1}^m\lambda_i(\tilde a^i - a^i)\Big\|\\ &\leq \sum_{i=1}^m\Big\|\lambda_i(\tilde a^i - a^i)\Big\|\\ &\leq \sum_{i=1}^m\Big\|\tilde a^i - a^i\Big\|.\end{aligned}$$

So, $\tilde x$ belongs to the right-hand side of (3.4). This establishes (3.4).

The above preparations allow us to show that the optimal value function $v(.)$ is Lipschitz on the chosen neighborhood U of $\bar a$, hence completing the proof of the theorem. Let $\tilde a, a\in U$ be given arbitrarily. Select a point $\tilde x = (\tilde x^1,\dots,\tilde x^k)$ from $\mathbf{S}(\tilde a)$. For each $j\in J$, since $\tilde x^j\in\operatorname{co}\widetilde{A}$ (see [7, Theorem 3.7]), by (3.4) we can find some $x^j\in\operatorname{co} A$ and $v^j\in\bar B_{\mathbb{R}^n}$ such that

$$\tilde x^j = x^j + \big(\|\tilde a^1 - a^1\| + \cdots + \|\tilde a^m - a^m\|\big)\,v^j. \tag{3.5}$$

By the Cauchy–Schwarz inequality, one has

$$\begin{aligned}\|\tilde a^1 - a^1\| + \cdots + \|\tilde a^m - a^m\| &\leq \sqrt{m}\,\Big(\|\tilde a^1 - a^1\|^2 + \cdots + \|\tilde a^m - a^m\|^2\Big)^{1/2}\\ &= \sqrt{m}\,\|\tilde a - a\|.\end{aligned}$$

Therefore, (3.5) yields

$$\big\|\tilde x^j - x^j\big\| \leq \sqrt{m}\,\|\tilde a - a\| \quad (\forall j\in J). \tag{3.6}$$

Setting $x = (x^1,\dots,x^k)$, from (3.6) we get

$$\|\tilde x - x\|^2 = \|\tilde x^1 - x^1\|^2 + \ldots + \|\tilde x^k - x^k\|^2 \leq km\|\tilde a - a\|^2. \tag{3.7}$$

Since $x = (x^1, \ldots, x^k) \in (\mathrm{co}\, A)^k$, we see that x is a feasible point of the constrained optimization problem

$$\min \Big\{ f(u,a) \mid u \in (\mathrm{co}\, A)^k \Big\}.$$

Hence, from (3.3) it follows that $v(a) \le f(x,a)$. So, using Claim 3 and (3.7) we have

$$\begin{array}{rcl} v(\tilde a) - v(a) = f(\tilde x, \tilde a) - v(a) & \ge & f(\tilde x, \tilde a) - f(x,a) \\ & \ge & -\ell_0 \|(\tilde x, \tilde a) - (x,a)\| \\ & = & -\ell_0 \big(\|\tilde x - x\|^2 + \|\tilde a - a\|^2\big)^{1/2} \\ & \ge & -\ell_0 \left(km\, \|\tilde a - a\|^2 + \|\tilde a - a\|^2\right)^{1/2} \\ & = & -\ell_0 \sqrt{km+1}\, \|\tilde a - a\|, \end{array}$$

where $\ell_0 = \sqrt{2}\left(\sum_{i\in I} s_i\right)$. It follows that $v(a) - v(\tilde a) \le \ell_0 \sqrt{km+1}\, \|\tilde a - a\|$. By symmetry, one has $v(\tilde a) - v(a) \le \ell_0 \sqrt{km+1}\, \|a - \tilde a\|$. Therefore,

$$|v(\tilde a) - v(a)| \le \ell_0 \sqrt{km+1}\, \|\tilde a - a\| \quad \forall\, \tilde a, a \in U.$$

Since one has $\ell_0 \sqrt{km+1} = \sqrt{2(km+1)}\left(\sum_{i\in I} s_i\right)$, this completes the proof of the theorem. □

Slightly modifying the proof of Theorem 3.1, we can obtain the next result on the global Lipschitz continuity of the optimal value function of the single-source Weber problem when the data points are subject to perturbations.

Theorem 3.2. *The optimal value function $v(.)$ of the single-source Weber problem (2.2) is Lipschitz on $\mathbb{R}^{nm}$ with the Lipschitz constant $\sqrt{2(m+1)}\Big(\sum_{i\in I} s_i\Big)$, i.e.,*

$$|v(\tilde a) - v(a)| \le \sqrt{2(m+1)}\Big(\sum_{i\in I} s_i\Big)\, \|\tilde a - a\|$$

for all $\tilde a, a \in \mathbb{R}^{nm}$. In particular, the function $v(.)$ is continuous on $\mathbb{R}^{nm}$.

Proof. Note that the single-source Weber problem (2.2) is the same as the convex optimization problem (3.2) in [7]. Hence, the arguments given in the proof of Theorem 3.7 in that paper assure that, if $x \in \mathbf{S}(a)$ with $a = \big(a^1, \ldots, a^m\big) \in \mathbb{R}^{nm}$ and $\mathbf{S}(a)$ standing for the solution set of (2.2), then $x \in \mathrm{co}\, A$ with $A := \{a^1, \ldots, a^m\}$. So,

$$\mathbf{S}(a) \subset \mathrm{co}\, A \quad \forall a \in \mathbb{R}^{nm}. \tag{3.8}$$

For problem (2.2), we have

$$v(a) = \min \Big\{ f(x,a) = \sum_{i\in I} s_i \|a^i - x\| \mid x \in \mathbb{R}^n \Big\}. \tag{3.9}$$

Clearly, combining (3.9) with (3.8) yields

$$v(a) = \min \{ f(x,a) \mid x \in \mathrm{co}\, A \} \quad \forall a \in \mathbb{R}^{nm}. \tag{3.10}$$

Using (3.10) instead of (3.3) and repeating the arguments of the preceding proof, we obtain the desired result. □

4. Upper Semicontinuity of the Map S(.)

The subject of our study in this section is the *global solution map* $\mathbf{S} : \mathbb{R}^{nm} \rightrightarrows \mathbb{R}^{nk}$ of (2.1), which assigns every data point $a = \big(a^1, \ldots, a^m\big) \in \mathbb{R}^{nm}$ to the global solution set $\mathbf{S}(a)$.

Theorem 4.1. *The global solution map* $\mathbf{S}$ *of* (2.1) *is upper semicontinuous at every point* $\bar{a} = \big(\bar{a}^1, \ldots, \bar{a}^m\big) \in \mathbb{R}^{nm}$ *satisfying the condition* $\bar{a}^i \neq \bar{a}^p$ *for any* $i, p \in I$ *with* $i \neq p$.

Proof. Let $\bar{a} = \big(\bar{a}^1, \ldots, \bar{a}^m\big) \in \mathbb{R}^{nm}$ be such that $\bar{a}^i \neq \bar{a}^p$ for any $i, p \in I$ with $i \neq p$. Choose $\rho > 0$ to be sufficiently small such that for every $a = \big(a^1, \ldots, a^m\big) \in \prod_{i\in I} \bar{B}(\bar{a}^i, \rho)$, one has $a^i \neq a^p$ for any $i, p \in I$ with $i \neq p$.

Suppose on the contrary that $\mathbf{S}$ is not upper semicontinuous at $\bar{a}$. Then one can find an open set V in $\mathbb{R}^{nk}$ such that

$$\begin{cases} \mathbf{S}(\bar{a}) \subset V \text{ and for any neighborhood } U \text{ of } \bar{a} \text{ in } \mathbb{R}^{nm} \\ \text{there exists a point } a \in U \text{ satisfying } \mathbf{S}(a) \setminus V \neq \emptyset. \end{cases} \tag{4.1}$$

Set $Q = \big[\mathrm{co}\,\big(\bigcup_{i\in I} \bar{B}(\bar{a}^i, \rho)\big)\big]^k$, then observe by [20, Corollary 1.9] (see also [15, Corollary 5.7]) and the Tychonoff Theorem (see, e.g., [9, Theorem 13, Chap. 5] and [24, Theorem 17.8]) that Q is a compact set in $\mathbb{R}^{nk}$.

Claim 5. *For every* $a \in \prod_{i\in I} \bar{B}(\bar{a}^i, \rho)$, *it holds that*

$$\mathbf{S}(a) \subset Q. \tag{4.2}$$

Indeed, for any $a \in \prod_{i\in I} \bar{B}(\bar{a}^i, \rho)$, since $a^i \neq a^p$ for any $i, p \in I$ such that $i \neq p$, by [7, Theorem 3.7] we get $\mathbf{S}(a) \subset (\mathrm{co}A)^k$ with $A := \{a^1, \ldots, a^m\}$. So, to obtain (4.2), it suffices to prove that

$$(\mathrm{co}A)^k \subset Q. \tag{4.3}$$

Let $x = (x^1, \ldots, x^k) \in (\mathrm{co}\,A)^k$ be given arbitrarily. For each $j \in J$, we choose $\lambda_1^j \geq 0, \ldots, \lambda_m^j \geq 0$ such that $\sum_{i\in I} \lambda_i^j = 1$ and $x^j = \sum_{i\in I} \lambda_i^j a^i$. As $a^i \in \bar{B}(\bar{a}^i, \rho)$ for all $i \in I$, we have $x^j \in \mathrm{co}\,\big(\bigcup_{i\in I} \bar{B}(\bar{a}^i, \rho)\big)$ for every $j \in J$. Therefore,

$$x = (x^1, \ldots, x^m) \in \left[\mathrm{co}\left(\bigcup_{i\in I} \bar{B}(\bar{a}^i, \rho)\right)\right]^k = Q.$$

This justifies (4.3).

Choose $\bar{\ell} \in \mathbb{N}$ to be sufficiently large such that $\bar{\ell}^{-1} < \rho$. For every $\ell \in \mathbb{N}$ with $\ell \geq \bar{\ell}$, let us set $U = \prod_{i\in I} B\big(\bar{a}^i, \frac{1}{\ell}\big)$ and apply (4.1) to find a point $a_{[\ell]} \in \prod_{i\in I} B\big(\bar{a}^i, \frac{1}{\ell}\big)$ and a point

$$x_{[\ell]} \in \mathbf{S}(a_{[\ell]}) \setminus V. \tag{4.4}$$

Since

$$a_{[\ell]} \in \prod_{i\in I} B\big(\bar{a}^i, \ell^{-1}\big) \subset \prod_{i\in I} \bar{B}(\bar{a}^i, \rho)$$

for all $\ell \geq \bar{\ell}$, by (4.2) we have $x_{[\ell]} \in Q$ for all $\ell \geq \bar{\ell}$. By the compactness of Q, there exists a subsequence $\{x_{[\ell']}\}$ of $\{x_{[\ell]}\}_{\ell \geq \bar{\ell}}$ such that $\{x_{[\ell']}\}$ converges to a point $\bar{x} \in Q$.

Claim 6. *One has*

$$\bar{x} \in \mathbf{S}(\bar{a}). \tag{4.5}$$

To obtain (4.5), we deduce from (4.4) that $f\big(x_{[\ell']}, a_{[\ell']}\big) = v\big(a_{[\ell']}\big)$. Passing this equality to the limit as $\ell' \to \infty$ and using Theorem 3.1 give $f(\bar{x}, \bar{a}) = v(\bar{a})$. Thus $\bar{x} \in \mathbf{S}(\bar{a})$.

Since $\{x_{[\ell']}\} \subset \mathbb{R}^{nk} \setminus V$, by the openness of V, we must have $\bar{x} \in \mathbb{R}^{nk} \setminus V$. This contradicts (4.5) and the inclusion in the first line of (4.1).

The proof is complete. □

In connection with Theorem 4.1, the following natural question arises: *Whether the global solution map* $\mathbf{S}$ *of* (2.1) *can possess some lower semicontinuity or inner semicontinuity properties, or not?* The next example shows that, even if $\bar{a}^i \neq \bar{a}^p$ for any $i, p \in I$ with $i \neq p$, there may exist some $\bar{x} \in \mathbf{S}(\bar{a})$ such that $\mathbf{S}$ is not inner semicontinuous at $(\bar{a}, \bar{x})$ with $\bar{a} := \big(\bar{a}^1, \ldots, \bar{a}^m\big) \in \mathbb{R}^{nm}$. Hence, according to Definition 2.5 and the subsequent observation, the solution map in question is not lower semicontinuous at $\bar{a}$.

Example 4.2. Let $n = 2, m = 3, k = 2$, $s_1 = s_2 = s_3 = 1$, and $\bar{a}^1 = (0,0)$, $\bar{a}^2 = (1,0)$, $\bar{a}^3 = (0,1)$, and $\bar{a} = \big(\bar{a}^1, \bar{a}^2, \bar{a}^3\big)$. By [7, Example 3.9] we know that

$$\mathbf{S}(\bar{a}) = \Big(\big[\bar{a}^1, \bar{a}^2\big] \times \{\bar{a}^3\}\Big) \cup \Big(\{\bar{a}^2\} \times \big[\bar{a}^1, \bar{a}^3\big]\Big)$$

(up to permutations of the coordinates of every $\bar{x} = (\bar{x}^1, \bar{x}^2) \in \mathbf{S}(\bar{a})$; see Remark 2.4)) and $v(\bar{a}) = 1$. Choose $\bar{x} = (\bar{x}^1, \bar{a}^3) \in \mathbf{S}(\bar{a})$ with $\bar{x}^1 := (\frac{1}{2}, 0)$. Let $a^1 = \bar{a}^1$, $a^2 = \bar{a}^2$, $a^3 = a^3(\varepsilon)$, and $a(\varepsilon) = (a^1, a^2, a^3(\varepsilon))$, where $a^3(\varepsilon) := (1-\varepsilon)\bar{a}^3 + \varepsilon\bar{a}^3$ with $\varepsilon \in (0, \frac{1}{2})$. Then, using Proposition 3.5(a) and Theorem 3.8 in [7], we can show that

$$\mathbf{S}(a(\varepsilon)) = \{\bar{a}^2\} \times \big[\bar{a}^1, \bar{a}^3(\varepsilon)\big] \quad \forall \varepsilon \in \Big(0, \frac{1}{2}\Big)$$

(up to permutations). Hence, for the neighborhood $V := B(\bar{x}^1, \frac{1}{4}) \times B(\bar{a}^3, \frac{1}{4})$ of $\bar{x} \in \mathbb{R}^4$, one has $\mathbf{S}(a(\varepsilon)) \cap V = \emptyset$ for all $\varepsilon \in \big(0, \frac{1}{2}\big)$. Since $\lim_{\varepsilon \to 0^+} a(\varepsilon) = \bar{a}$, there does not exist any neighborhood U of $\bar{a}$ such that $\mathbf{S}(a) \cap V \neq \emptyset$ for every $a \in U$. By Definition 2.5(d), $\mathbf{S}$ is not inner semicontinuous at $(\bar{a}, \bar{x})$. This implies that $\mathbf{S}$ is not lower semicontinuous at $\bar{a}$.

The above method for proving Theorem 4.1 does not work if the assumption $\bar{a}^i \neq \bar{a}^p$ for any $i, p \in I$ with $i \neq p$ is violated. The reason is that the global solution $\mathbf{S}(\bar{a})$ can be unbounded then. To justify this remark, let us consider the following example.

Example 4.3. Choose $n = 2,\ m = 3,\ k = 3$, and $s_1 = s_2 = s_3 = 1$. Let $\bar{a}^1 = \bar{a}^2$ and $\bar{a}^3 \neq \bar{a}^1$ be the given demand points. Put $\bar{a} = \big(\bar{a}^1, \bar{a}^2, \bar{a}^3\big)$. Note that

$$\mathbf{S}(\bar{a}) = \big\{\bar{x} = \big(\bar{x}^1, \bar{x}^2, \bar{x}^3\big) \mid \bar{x}^1 = \bar{a}^1,\ \bar{x}^2 = \bar{a}^3,\ \bar{x}^3 \in \mathbb{R}^2\big\}$$

(up to permutations of the coordinates of $\bar{x}$) and $v(\bar{a}) = 0$. Clearly, $\mathbf{S}(\bar{a})$ is unbounded.

For the single-source Weber problem, we now obtain an analogue of Theorem 4.1 without the requirement $\bar{a}^i \neq \bar{a}^p$ for any $i, p \in I$ with $i \neq p$.

Theorem 4.4. *The global solution map* $\mathbf{S}$ *of* (2.2) *is upper semicontinuous at every point* $\bar{a} = \big(\bar{a}^1, \ldots, \bar{a}^m\big) \in \mathbb{R}^{nm}$.

Proof. To obtain a contradiction, suppose that $\mathbf{S}$ is not upper semicontinuous at $\bar{a}$. Then one can find an open set V in $\mathbb{R}^n$ such that (4.1) holds.

Set $Q_1 = \mathrm{co}\big(\bigcup_{i\in I} \bar{B}(\bar{a}^i, 1)\big)$ and observe by [20, Corollary 1.9] (see also [15, Corollary 5.7]) that Q_1 is a compact set in $\mathbb{R}^n$.

Claim 7. *For every* $a \in \prod_{i\in I} \bar{B}(\bar{a}^i, 1)$, *it holds that*

$$\mathbf{S}(a) \subset Q_1. \tag{4.6}$$

Indeed, the inclusion (3.8), where $A := \{a^1, \ldots, a^m\}$, is valid for any

$$a = \big(a^1, \ldots, a^m\big) \in \prod_{i\in I} \bar{B}(\bar{a}^i, 1)$$

(see the proof of Theorem 3.2). Since $\mathrm{co}\, A \subset \mathrm{co}\big(\bigcup_{i\in I} \bar{B}(\bar{a}^i, 1)\big)$, the inclusion (4.6) follows from (3.8).

For every $\ell \in \mathbb{N}$, setting $U = \prod_{i\in I} B\big(\bar{a}^i, \frac{1}{\ell}\big)$ and applying (4.1), we can get a point $a_{[\ell]} \in \prod_{i\in I} B\big(\bar{a}^i, \frac{1}{\ell}\big)$ and a point $x_{[\ell]} \in \mathbf{S}(a_{[\ell]}) \setminus V$. As

$$a_{[\ell]} \in \prod_{i\in I} B\big(\bar{a}^i, \ell^{-1}\big) \subset \prod_{i\in I} \bar{B}(\bar{a}^i, 1),$$

by (4.6) we have $x_{[\ell]} \in Q_1$ for all ℓ. By the compactness of Q_1, there is a subsequence $\big\{x_{[\ell']}\big\}$ of $\big\{x_{[\ell]}\big\}$ that converges to a point $\bar{x} \in Q_1$.

Since $x_{[\ell']} \in \mathbf{S}(a_{[\ell']})$, one has $f\big(x_{[k']}, a_{[k']}\big) = v\big(a_{[k']}\big)$. Letting $\ell' \to \infty$ and using Theorem 3.2, from the last equality we obtain $f(\bar{x}, \bar{a}) = v(\bar{a})$. Hence $\bar{x} \in \mathbf{S}(\bar{a})$.

Since $\big\{x_{[\ell']}\big\} \subset \mathbb{R}^n \setminus V$, by the openness of V, we must have $\bar{x} \in \mathbb{R}^n \setminus V$. This contradicts the property $\bar{x} \in \mathbf{S}(\bar{a})$ and the inclusion in the first line of (4.1).

The proof is complete. □

Let us show by an example that the global solution map $\mathbf{S}$ of (2.2) may not be inner semicontinuous at a point in its graph. Hence, it cannot be lower semicontinuous at a point in its effective domain.

Example 4.5. Consider the single-source Weber problem (2.2) with $n = 2$, $m = 4$, $s_1 = s_2 = s_3 = s_4 = 1$, and demand points $\bar{a}^1, \ldots, \bar{a}^4$ having zero ordinates such that $\bar{a}^i = (1 - \tau_i)\bar{a}^3 + \tau_i \bar{a}^4$, $\tau_i \in (0, 1)$ for $i = 1, 2$, and $\tau_1 < \tau_2$. Note that the chosen demand points are pairwise distinct. Set $\bar{a} = \big(\bar{a}^1, \ldots, \bar{a}^4\big)$ and note that $\mathbf{S}(\bar{a}) = [\bar{a}^1, \bar{a}^2]$. For every $\varepsilon \in (0, \frac{1}{2})$, put

$$a^1(\varepsilon) = \bar{a}^1 + (0, \varepsilon),\ a^2(\varepsilon) = \bar{a}^2 - (0, \varepsilon),\ a^3(\varepsilon) = \bar{a}^3,\ a^4(\varepsilon) = \bar{a}^4,$$

and $a(\varepsilon) = \big(a^1(\varepsilon), a^2(\varepsilon), a^3(\varepsilon), a^4(\varepsilon)\big)$. Since there exist at least three points in the set $\{a^1(\varepsilon), a^2(\varepsilon), a^3(\varepsilon), a^4(\varepsilon)\}$ which are not colinear (i.e., there is no straight line containing all the three chosen points), by the proof of [12, Lemma 1.1] (see also [15, Proposition 8.2]) we know that the objective function of (2.2) with $a(\varepsilon)$ taking the role of a is strictly convex. Thus, by [7, Theorem 3.3] (see also [15, Proposition 8.1]), this single-source Weber problem has a unique solution, which is denoted $\bar{u}(\varepsilon)$. By the well-known necessary and optimality condition in convex programming, $\bar{u}(\varepsilon)$ is the unique solution of the inclusion $0 \in \partial_x f(\cdot, a(\varepsilon))(\bar{u}(\varepsilon))$ with $\partial_x f(\cdot, a(\varepsilon))(\bar{u}(\varepsilon))$ denoting the subdiferential of the convex function $f(\cdot, a(\varepsilon))$ at $\bar{u}(\varepsilon)$. As noted in the proof of [7, Theorem 3.7], if $\bar{u}(\varepsilon) \notin \{a^1(\varepsilon), a^2(\varepsilon), a^3(\varepsilon), a^4(\varepsilon)\}$, then the inclusion $0 \in \partial_x f(\cdot, a(\varepsilon))(\bar{u}(\varepsilon))$ can be rewritten equivalently as

$$\sum_{i=1}^{4} \|\bar{u}(\varepsilon) - a^i(\varepsilon)\|^{-1} \|\bar{u}(\varepsilon) - a^i(\varepsilon)\| = 0. \tag{4.7}$$

Put $\bar{u} = \frac{1}{2}(\bar{a}^1 + \bar{a}^2)$ and observe that $\bar{u} \notin \{a^1(\varepsilon), a^2(\varepsilon), a^3(\varepsilon), a^4(\varepsilon)\}$. It is easy to check that

$$\|\bar{u} - a^1(\varepsilon)\|^{-1} \|\bar{u} - a^1(\varepsilon)\| + \|\bar{u} - a^2(\varepsilon)\|^{-1} \|\bar{u} - a^2(\varepsilon)\| = 0$$

and

$$\|\bar{u} - a^3(\varepsilon)\|^{-1} \|\bar{u} - a^3(\varepsilon)\| + \|\bar{u} - a^4(\varepsilon)\|^{-1} \|\bar{u} - a^4(\varepsilon)\| = 0.$$

Hence, $\bar{u}(\varepsilon) := \bar{u}$ is the unique solution of (4.7). Therefore, $\mathbf{S}(a(\varepsilon)) = \{\bar{u}\}$ for every $\varepsilon \in (0, \frac{1}{2})$. Choose $\bar{x} = \bar{a}^1$ and put $V = B(\bar{x}, \rho)$, where $\rho \in \left(0, \frac{1}{2}\|a^2 - \bar{a}^1\|\right)$ can be taken arbitrarily. Observe that $\bar{x} \in \mathbf{S}(\bar{a})$ and $\mathbf{S}(a(\varepsilon)) \cap V = \emptyset$ for all $\varepsilon \in (0, \frac{1}{2})$. Since $\lim_{\varepsilon \to 0^+} a(\varepsilon) = \bar{a}$, one cannot find any neighborhood U of $\bar{a}$ such that $\mathbf{S}(a) \cap V \neq \emptyset$ for every $a \in U$. By Definition 2.5(d), $\mathbf{S}$ is not inner semicontinuous at $(\bar{a}, \bar{x})$. As a consequence, $\mathbf{S}$ is not lower semicontinuous at $\bar{a}$.

5. Inner semicontinuity of the map $\mathbf{S_1}(.)$

In this section, we are interested to know what properties the map $\mathbf{S_1} : \mathbb{R}^{nm} \rightrightarrows \mathbb{R}^{nk}$, which puts every data point $a = \left(a^1, \ldots, a^m\right) \in \mathbb{R}^{nm}$ of (2.1) in correspondence with the local solution set $\mathbf{S_1}(a)$, may have.

Theorem 5.1. *Suppose that* $(\bar{a}, \bar{x}) \in \operatorname{gph} \mathbf{S_1}$, *where* $\bar{a} = \left(\bar{a}^1, \ldots, \bar{a}^m\right) \in \mathbb{R}^{nm}$ *is such that* $\bar{a}^i \neq \bar{a}^p$ *for any* $i, p \in I$ *with* $i \neq p$. *Let* $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^k)$, $\bar{A} = \{\bar{a}^1, \ldots, \bar{a}^m\}$, $J_1 = \{j' \in J \mid \bar{A}[\bar{x}^{j'}] \neq \emptyset\}$ *with* $\bar{A}[\bar{x}^{j'}]$ *being the attraction set of* $\bar{x}^{j'}$ *with respect to the data set* $\bar{A}$, *and the following conditions be satisfied:*

(i) $\bar{x}^{j_1} \neq \bar{x}^{j_2}$ *whenever* $j_1, j_2 \in J$ *and* $j_1 \neq j_2$;
(ii) $\bar{x}^j \notin \bar{A}$ *for all* $j \in J$;
(iii) *For each* $j_1 \in J_1$, *the data points from* $\bar{A}[\bar{x}^{j_1}]$ *are not colinear, i.e., they do not belong to the same line.*

Then, the local solution map $\mathbf{S_1}$ *of* (2.1) *is inner semicontinuous at* $(\bar{a}, \bar{x})$.

Proof. By our assumptions, $\bar{a}^i \neq \bar{a}^p$ for any $i, p \in I$ with $i \neq p$. Then, there exists $\rho > 0$ such that for each $a = \left(a^1, \ldots, a^m\right) \in \prod_{i \in I} \bar{B}(\bar{a}^i, \rho)$ one has $a^i \neq a^p$ for any $i, p \in I$ with $i \neq p$.

To prove that $\mathbf{S_1}$ is inner semicontinuous at $(\bar{a}, \bar{x})$, given any $\varepsilon > 0$, we need to show the existence of a neighborhood U of $\bar{a}$ in $\mathbb{R}^{nm}$ having the following property:

$$\mathbf{S_1}(a) \cap V \neq \emptyset \quad (\forall a \in U) \tag{5.1}$$

with $V := \prod_{j\in J} B(\bar{x}^j, \varepsilon)$.

First, observe by the assumption (i) that $\bar{x}$ is a nontrivial local solution of (2.1), where $\bar{a}$ plays the role of a, in the meaning of [8, Definition 3.3]. Since $\bar{x}^j \notin \bar{A}$ for all $j \in J$ thanks to the assumption (ii), by [8, Theorem 3.5] we obtain the following three properties:

(a) *For every $i \in I$, the index set*

$$J_i(\bar{x}) := \{j \in J \mid \bar{a}^i \in \bar{A}[\bar{x}^j]\}$$

is a singleton.

(b) *For every $j \in J$, if $\bar{A}[\bar{x}^j]$ is nonempty, then $\bar{x}^j$ is a solution of the single-source Weber problem defined by the data set $\bar{A}[\bar{x}^j]$.*

(c) *For every $j \in J$, if $\bar{A}[\bar{x}^j]$ empty, then one has*

$$\bar{x}^j \notin \bar{\mathcal{A}}[\bar{x}], \tag{5.2}$$

where $\bar{\mathcal{A}}[\bar{x}]$ is the union of closed balls $\bar{B}(\bar{a}^p, \|\bar{a}^p - \bar{x}^q\|)$ with $p \in I$ and $q \in J$ satisfying $p \in I(q)$, where $I(q) := \{i \in I \mid \bar{a}^i \in \bar{A}[\bar{x}^q]\}$.

By (a), for every $i \in I$ there is a unique index $j(i) \in J$ such that $J_i(\bar{x}) = \{j(i)\}$. Moreover, by (a) we have

$$\|\bar{a}^i - \bar{x}^{j(i)}\| < \|\bar{a}^i - \bar{x}^{j'}\| \quad (\forall j' \in J \setminus \{j(i)\}). \tag{5.3}$$

Since all the inequalities in (5.3) are strict, there exists a constant $\delta_1 \in (0, \varepsilon)$ such that for any $a \in \prod_{i\in I} B(\bar{a}^i, \delta_1)$ and for any $x \in \prod_{j\in J} B(\bar{x}^j, \delta_1)$ the next property holds:

$$\|a^i - x^{j(i)}\| < \|a^i - x^{j'}\| \quad (\forall j' \in J \setminus \{j(i)\}). \tag{5.4}$$

Furthermore, the property (b) implies that for every $j_1 \in J_1$, the facility $\bar{x}^{j_1}$ is a solution of the single-source Weber problem defined by the data set $\bar{A}[\bar{x}^{j_1}]$. As the latter set is not colinear, this single-source Weber problem has a unique solution (see, e.g., [16, Proposition 8.2]).

Recall that the sets $I(q)$ for $q \in J$ have been defined in (c). Since

$$J_1 = \{j' \in J \mid \bar{A}[\bar{x}^{j'}] \neq \emptyset\},$$

by the above property (a) we can infer that the nonempty sets $I(j_1)$, $j_1 \in J_1$, are pairwise disjoint and I is the union of these sets. By the assumption (iii), for each $j_1 \in J_1$, the data points $\{\bar{a}^i \mid i \in I(j_1)\}$ are not colinear. So, one can find a constant $\delta_2 \in (0, \delta_1)$ such that, for every $a \in \prod_{i\in I} B(\bar{a}^i, \delta_2)$, the data points $\{a^i \mid i \in I(j_1)\}$ are not colinear for all $j_1 \in J_1$. Hence, for every $a \in \prod_{i\in I} B(\bar{a}^i, \delta_2)$, the single-source Weber problem with the data set $\{a^i \mid i \in I(j_1)\}$ has a unique solution, which is denoted by $x^{j_1}(a)$. By Theorem 4.4 we know that, for each $j_1 \in J_1$, the single-valued mapping

$$x^{j_1}(\cdot) : \prod_{i\in I} B(\bar{a}^i, \delta_2) \to \mathbb{R}^n, \ \ a \mapsto x^{j_1}(a),$$

is continuous.

Put $J_2 = J \setminus J_1$ and observe by the property (c) that (5.2) holds for all $j \in J_2$. Since $\bar{\mathcal{A}}[\bar{x}]$ is a nonempty compact set, there exists $\rho > 0$ such that $B(\bar{x}^j, \rho) \cap \bar{\mathcal{A}}[\bar{x}] = \emptyset$ for all $j \in J_2$.

Let $x^j(a) = \bar{x}^j$ for all $a \in \prod_{i\in I} B(\bar{a}^i, \delta_2)$ and $j \in J_2$. Put $x(a) = (x^1(a), \dots, x^k(a))$, where the functions $x^j(.)$ with $j \in J_1$ have been defined above. By the continuity of the function

$$x(\cdot) : \prod_{i\in I} B(\bar{a}^i, \delta_2) \to \mathbb{R}^{nk}, \ \ a \mapsto x(a),$$

we can find $\delta_3 \in (0, \delta_2)$ such that $x(a) \in \prod_{j\in J} B(\bar{x}^j, \delta_1)$ for every $a \in \prod_{i\in I} B(\bar{a}^i, \delta_3)$. Therefore, from (5.4) we can deduce for each $a \in \prod_{i\in I} B(\bar{a}^i, \delta_3)$ that

$$\|a^i - x^{j(i)}(a)\| < \|a^i - x^{j'}(a)\| \quad (\forall j' \in J \setminus \{j(i)\}). \tag{5.5}$$

Thus, for each $a \in \prod_{i\in I} B(\bar{a}^i, \delta_3)$, the property (5.5) implies that the attraction set $A[x^j(a)]$ of $x^j(a)$ with respect to the data set $A := \{a^1, \dots, a^m\}$ is nonempty for every $j \in J_1$, while $A[x^j(a)]$ is empty for every $j \in J_2$. Moreover, the facilities system $\{x^1(a), \dots, x^k(a)\}$ has the following properties:

(a') *For every $i \in I$, the index set $J_i(x(a)) := \big\{j \in J \mid a^i \in A[x^j(a)]\big\}$ is a singleton.*

(b') *For every $j \in J_1$, $x^j(a)$ is a solution of the single-source Weber problem defined by the data set $A[x^j(a)]$.*

(c') *For every $j \in J_2$, one has $x^j(a) \notin \mathcal{A}[x(a)]$, where $\mathcal{A}[x(a)]$ is the union of closed balls $\bar{B}(a^p, \|a^p - x^q\|)$ with $p \in I$ and $q \in J$ satisfying $p \in I(q)$, where*

$$I(q) := \big\{i \in I \mid a^i \in A[x^q(a)]\big\}.$$

Therefore, applying the sufficient conditions for nontrivial local solutions of the multi-source Weber problem given in [8, Theorem 3.7] shows that $x(a) \in \mathbf{S_1}(a)$ for each $a \in \prod_{i\in I} B(\bar{a}^i, \delta_3)$. Since $\delta_1 \in (0, \varepsilon)$ and $x(a) \in \prod_{j\in J} B(\bar{x}^j, \delta_1)$ for every $a \in \prod_{i\in I} B(\bar{a}^i, \delta_3)$, choosing $U = \prod_{i\in I} B(\bar{a}^i, \delta_3)$, we get (5.1).

The proof is complete. □

6. Conclusions

Stability analysis of the multi-source Weber problem has been considered in detail for the first time in this paper. Allowing the data set to change, we have obtained sufficient conditions for the locally Lipschitz continuity of the optimal value function, the upper semicontinuity of the global solution map, as well as the inner semicontinuity of the local solution map.

For further investigations, the following questions seem to be interesting.

(Q1) *Whether the assumption $\bar{a}^i \neq \bar{a}^p$ for any $i, p \in I$ with $i \neq p$ in Theorems 3.1, 4.1 and 5.1 is redundant, i.e., there exist other proofs of the results not relying on the condition?*

(Q2) *Is the assumption (ii) in Theorem 5.1 essential for the validity of the theorem?*

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