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PARALLEL INERTIAL PROXIMAL ALGORITHM WITH APPLICATIONS TO IMAGE RECOVERY PROBLEMS

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Abstract. We study variational inequality problems defined over the intersection of fixed point sets in a real Hilbert space. By using inertial extrapolation, we propose a parallel proximal algorithm for solving the problems. By incorporating several ideas of proximal projection and parallel methods, we present a strongly convergent method that can be easily implemented, for example for solving image restoration problems via proximal operators. Primary numerical experiments illustrate and compare the performances of the proposed algorithm.

1. Introduction

Let $\mathbb{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, norm $\| \cdot \|$, respectively. Let $C \subset \mathbb{H}$ be a nonempty, closed and convex set. For a sequence $\{x^k\}$, we denote by $x^k \to \bar{x}$ (resp. $x^k \rightharpoonup \bar{x}$) the strong (resp. weak) convergence to $\bar{x}$. Denote $\mathbb{N}, \mathbb{R}^s$ the set of natural numbers (including zero) and the s Euclidean space, respectively.

We start by focusing on the following problem, first introduced by Yamada [21]. This problem is applied for various applications such as image reconstruction, computerized tomography, data compression and many more, see for example, [2–4, 6, 7, 14, 16, 18, 22].

Problem 1. Let $i \in I := \{1, 2, ..., n\}$, and for each $i \in I$, consider the self-mappings $S_i : \mathbb{H} \to \mathbb{H}$. Denote the fixed point set of S_i by $Fix(S_i) := \{x \in \mathbb{H} : S_i(x) = x\}$. Given this data, we consider the variational inequality problem, shortly $VI(\Omega, F)$, of finding a point $x^* \in \Omega := \cap_{i \in I} Fix(S_i)$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega.$$

Moreover, Problem 1 has several important special cases as seen below.

1. Convex Minimization Problem. Let C be a nonempty, closed and convex subset of $\mathbb{H}$, $S_i = 0 (i \in I)$ and $\varphi : C \to \mathbb{R}$ be subdifferentiable and convex. We consider the optimization problem (OP):

$$\min\{\varphi(x) : x \in C\}$$

Setting $F(x) := \nabla \varphi(x)$, it is easy to see that the problem (OP) becomes a case of $VI(\Omega, F)$.

2. Classical Variational Inequality Problem. Let C be a nonempty, closed and

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convex subset of $\mathbb{H}$ and $Fix(S_i) = C$ for all $i \in I$. Problem $VI(\Omega, F)$ is equivalent to the variational inequality problem:

$$\text{Find } x^* \in C \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C.$$

3. Common Fixed Point Problem. Let $F(x) = 0$. The following problem is called the common fixed point problem (CFPP):

$$\text{Find } x^* \in Fix(S_i), \quad \forall i \in I.$$

We can easily see that the problem (CFPP) becomes a case of $VI(\Omega, F)$.

Clearly Problem 1 is a generalization of the classical variational inequality problem in which the feasible set Ω is not given explicitly, and thus, standard fixed point/projection methods cannot be applied directly. In [21], Yamada introduced the following cyclic procedure:

$$x^{k+1} = S_{[k+1]}(x^k) - \lambda_{k+1}\mu F(S_{[k+1]}(x^k)), \quad k \geq 0, \tag{1.1}$$

where $\{\lambda_k\}_{k\in\mathbb{N}}$ and μ are parameters fulfilling certain conditions and $S_{[k]} := S_{k \bmod n}$ for $k \in \mathbb{N}$ with the mod function. Inspired by (1.1), many iterative methods for solving $\text{VI}(\Omega, F)$ have been developed, see for example [8, 20, 23].

As seen above, a differentiable convex minimization problem is a special case of Problem 1, but in many applications the objective function is the sum of two functions, not necessarily differentiable and thus direct translation to Problem 1 is not obvious. Due to this reason we recall the next problem 2 and its relationship to Problem 1.

Problem 2. Let $f_1 : \mathbb{R}^s \to \mathbb{R}$ be a convex and differentiable function with a $L-$Lipchitz continuous gradient ∇f_1 and $f_2 : \mathbb{R}^s \to \mathbb{R}$ a proper lower semi-continuous and convex function. With this data the convex minimization problem is formulated as

$$\min\{f_1(x) + f_2(x) : x \in \mathbb{R}^s\}.$$

Clearly when f_2 is differentiable, one can see that Problem 2 is equivalent to Problem $VI(C, F)$ with $F := \nabla f_1 + \nabla f_2$ and $C := \mathbb{R}^s$.

Combettes and Wajs [7] showed that a solution of Problem 2 is characterized by the fixed point of the proximity operator:

$$S_\lambda(x) = x - prox_{\mathbb{R}^s}^{\lambda f_2}(Id - \lambda\nabla f_1)(x),$$

where Id is the identity mapping and *prox* is the proximity operator that is defined in the next section.

Returning to the general Problem 2, the *Forward-Backward Splitting Algorithm* (shortly FBSA) of Lions and Mercier [13] is a classical method for solving Problem 2. Choose arbitrary starting point $x^0 \in \mathbb{R}^s$ and $\lambda_k \in (0, 2/L)$. The algorithm generates a sequence $\{x^k\}$ according the following rule:

$$x^{k+1} = S_{\lambda_k}(x^k), \quad \forall k \geq 0. \tag{1.2}$$

For solving Problem 2, Beck and Teboulle [5] introduced the following *Fast Iterative Shrinkage-Thresholding Algorithm* (shortly FISTA):

$$\begin{cases} x^0 = y^0 \in \mathbb{R}^s, t_0 = 1, \\ y^k = S_{\frac{1}{L}}(x^k), \\ t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2}, \quad \theta_k = \frac{t_k - 1}{t_{k+1}}, \\ x^{k+1} = y^k + \theta_k(y^k - y^{k-1}), \quad k \geq 0. \end{cases} \tag{1.3}$$

Problem 2 received a lot of attention and many iterative methods are introduced for solving it, for example, iterative denoising method of Figueiredo and Nowak [10], fixed point continuation algorithm of Hale et al. [11], improved FISTA of Liang et al. [14], *Fast Viscosity Forward-Backward Algorithm* (FVFBA) [12], viscosity approximation method [19].

Motivated and inspired by the above algorithms, as well as the proximal projection method [4, 9], the parallel techniques [3] and the inertial proximal approach [6], for solving Problems 1 and 2, the purpose of this paper is two-fold. *First*, we present a new strong convergence algorithm for solving Problem 1 in a real Hilbert spaces. One of the main theoretical and practical advantage of our proposed method, compared with other related algorithms, is that there is no need to compute/approximate any metric projection onto the VI's feasible set Ω, especially since such task might be computationally expansive due to the structure of Ω. *Second,* by using the properties of the proximal operator we apply our proposed method to the image restoration problem (4.1) reformulated as Problem 2.

The outline of the paper is as follows. In Section 2 we recall some useful definitions, notations and results. The new algorithm and its analysis are presented in Section 3 and then in Section 4, numerical testings are presented including application to image recovery problem.

2. Preliminaries

As before, $\mathbb{H}$ is a real Hilbert space and $C \subseteq \mathbb{H}$ nonempty, closed and convex set.

Definition 2.1. Given a mapping $T : \mathbb{H} \to \mathbb{H}$.

(1) T is called *β-strongly monotone* on C with constant $\beta > 0$, if

$$\langle T(x) - T(y), x - y\rangle \geq \beta\|y - x\|^2, \quad \forall x, y \in C.$$

(2) T is called *L-Lipschitz continuous* on C with constant $L > 0$, if

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

It is called *nonexpansive* if it is 1-Lipschitz.

(3) T is called *quasi-nonexpansive* on $\mathbb{H}$, if

$$\|T(x) - \hat{x}\| \leq \|x - \hat{x}\|, \quad \forall (x, \hat{x}) \in \mathbb{H} \times \mathrm{Fix}(T).$$

(4) T is called *firmly nonexpansive*, that is, for all $x, y \in H$,

$$\langle T(x) - T(y), x - y\rangle \geq \|T(x) - T(y)\|^2.$$

(5) T is called *$\tau-$strictly pseudocontractive* on $\mathbb{H}$, where $\tau \in [0, 1)$, if

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + \tau\|(x - y) - [T(x) - T(y)]\|^2, \quad \forall x, y \in \mathbb{H}.$$

(6) T is called $\beta-$*demicontractive* on $\mathbb{H}$ where $\beta \in [0,1)$, if

$$\|T(x)-\hat{x}\|^2 \leq \|x-\hat{x}\|^2 + \beta\|x-T(x)\|^2, \quad \forall (x,\hat{x}) \in \mathbb{H} \times \text{Fix}(T).$$

(7) T is called *demiclosed* at zero, if $\{x^k\}$ weakly converges to $\bar{x}$ and $\{(I-T)(x^k)\}$ strongly converges to 0, then $\bar{x} \in \text{Fix}(T)$.

Let $f : \mathbb{H} \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The function f is called *proper* if its effective domain $D(f) := \{x \in \mathbb{H} : f(x) < +\infty\} \neq \emptyset$. f is *lower semicontinuous* at $x_0 \in D(f)$ if $f(x_0) \leq \liminf_{x \to x_0} f(x)$. It is called lower semicontinuous if it is lower semicontinuous at every $x_0 \in D(f)$. The *subdifferential* ∂f of a proper convex function f at $x \in \mathbb{H}$ is defined by

$$\partial f(x) := \{z \in \mathbb{H} : f(x) + \langle z, y-x \rangle \leq f(y), \ \forall y \in \mathbb{H}\}.$$

The following lemmas are useful for our algorithm's analysis.

Lemma 2.2. *Let C be a nonempty, closed and convex subset of $\mathbb{H}$, $f : C \to \mathbb{R}$ be proper, convex and lower semicontinuous function. The proximity operator of f with $\gamma \in \mathbb{R}$ is defined as*

$$prox_C^f(x) := argmin_{y \in C} \left\{ f(y) + \frac{1}{2\gamma}\|x-y\|^2 \right\}.$$

Now for any $x, y \in C$, the following are equivalent.

(a) $u = prox_C^f(x)$;
(b) $x - u \in \partial f(u)$;
(c) $\langle x-u, y-u \rangle \leq f(y) - f(u), \ \forall y \in C$.

Further properties of the proximal operator of f on C are collected in the next lemma.

Lemma 2.3. *Let C be a nonempty, closed and convex subset of $\mathbb{H}$, $f : C \to \mathbb{R}$ be proper, convex and lower semicontinuous function. Then, we have*

(i) *The operator $prox_C^f$ is firmly nonexpansive and thus clearly nonexpansive;*
(ii) *For any $x \in \mathbb{H}$ and $y \in C$ the following inequality holds.*

$$\|prox_C^f(x) - y\|^2 \leq \|x-y\|^2 - \|x - prox_C^f(x)\|^2 - 2\left[f\left(prox_C^f(x)\right) - f(y)\right].$$

Lemma 2.4 ([19, Lemma 2.5])**.** *Let $\{s_k\}$ be a sequence of nonnegative real numbers and $\{p_k\}$ a sequence of real numbers. Let $\{\alpha_k\}$ be a sequence of real numbers in $(0,1)$ such that $\sum_{k=1}^{\infty} \alpha_k = \infty$. Assume that*

$$s_{k+1} \leq (1-\alpha_k)s_k + \alpha_k p_k, \quad k \in \mathbb{N}.$$

If $\limsup_{i\to\infty} p_{k_i} \leq 0$ then $\lim_{k\to\infty} s_k = 0$.

Lemma 2.5 ([15, Remark 4.2])**.** *Let $S : \mathbb{H} \to \mathbb{H}$ be a β-demicontractive mapping with $Fix(S) \neq \emptyset$ and set $S_\omega = (1-\omega)Id + \omega S$ for $\omega \in (0,1]$. Then S_ω is quasi-nonexpansive if $\omega \in [0, 1-\beta]$ and*

$$\|S_\omega(x) - \bar{x}\|^2 \leq \|x-\bar{x}\|^2 - \omega(1-\beta-\omega)\|S(x)-x\|^2, \quad \forall \bar{x} \in \text{Fix}(S), x \in \mathbb{H}.$$

Lemma 2.6 ([15, Remark 4.4]). *Let $\{a_k\}$ be a sequence of nonnegative real numbers. Suppose that for any integer m, there exists an integer p such that $p \geq m$ and $a_p \leq a_{p+1}$. Let k_0 be an integer such that $a_{k_0} \leq a_{k_0+1}$ and define, for all integer $k \geq k_0$,*

$$\tau(k) = \max\{i \in \mathbb{N} : k_0 \leq i \leq k, a_i \leq a_{i+1}\}.$$

Then, $0 \leq a_k \leq a_{\tau(k)+1}$ for all $k \geq k_0$. Furthermore, the sequence $\{\tau(k)\}_{k \geq k_0}$ is nondecreasing and tends to $+\infty$ as $k \to \infty$.

3. Main result

In this section, we introduce a new iteration algorithm for approximating a solution of Problem 1 and prove its strong convergence. The algorithm uses a parallel technique and combines the inertial iteration method with an explicit self adaptive stepsize rule.

The parameters setup for the algorithm is as follows.

$$\begin{cases} a \in (0,1), \quad \{\lambda_k\} \subset [\bar{a}, \hat{a}] \subset \left(0, \frac{2\beta}{L^2}\right), \quad \sqrt{1 - 2\lambda_k\beta + \lambda_k^2 L^2} < 1 - a, \\ \zeta_k \in (0,1), \quad \sum_{k=1}^{\infty} \zeta_k = +\infty, \quad \lim_{k\to\infty} \zeta_k = 0, \\ 0 \leq \tau_k \leq \zeta_k^2, \quad \mu_k > 0, \\ \gamma_{k,i} \in (\bar{b}, \hat{b}) \subset (0, 1 - \max\{\beta_i : i \in I\}), \quad \forall i \in I, \end{cases} \tag{3.1}$$

The Parallel Inertial Proximal Algorithm (PIPA) is presented next.

Algorithm 3.1. *Choose starting points $x^0, x^1 \in \mathbb{H}$.*

Step 1. *Given the iterates x^{k-1} and x^k, compute*

$$w^k = x^k + \alpha_k(x^k - x^{k-1}), \tag{3.2}$$

where

$$\alpha_k = \begin{cases} \min\left\{\frac{\tau_k}{\|x^k - x^{k-1}\|}, \mu_k\right\}, & if \quad \|x^k - x^{k-1}\| \neq 0, \\ \mu_k & otherwise. \end{cases} \tag{3.3}$$

Step 2. *Take $u_i^k = (1-\gamma_{k,i})w^k + \gamma_{k,i}S_i(w^k)$. Set $t^k := u_{i_0}^k$, where $i_0 \in \operatorname{argmax}\{\|u_i^k - w^k\| : i \in I\}$.*

Step 3. *Compute $x^{k+1} = (1-\zeta_k)t^k + \zeta_k\left[t^k - \lambda_k F(t^k)\right]$. Let $k := k+1$ and go to Step 1.*

Note that, computing w^k is used by inertial technique and t^k is by parallel technique. Then, the iteration point x^{k+1} is based on the Mann iteration method and the hybrid steepest-descent method. We call a point x^k generated by Algorithm 3.1 $\epsilon-$solution of Problem 1 if $\|x^{k+1} - x^k\| \leq \epsilon$.

For the convergence of the algorithm we assume the following.

Condition 3.2. The mapping $F : \mathbb{H} \to \mathbb{H}$ is $\beta-$strongly monotone and $L-$Lipschitz continuous.

Condition 3.3. For all $i \in I$ the mappings $S_i : \mathbb{H} \to \mathbb{H}$ are β_i-demicontractive and demiclosed at zero and the set $\Omega := \cap_{i\in I} Fix(S_i)$ is nonempty.

Theorem 3.4. *Assume that Conditions 3.2 and 3.3 hold and consider the parameters setup (3.1). Then, the sequence $\{x^k\}$ generated by Algorithm 3.1 converges strongly to a unique solution x^* of Problem 1.*

Proof. Since F is $\beta-$strongly monotone and $L-$Lipschitz continuous, we have

$$\begin{aligned}\|[t^k - \lambda_k F(t^k)] - [x^* - \lambda_k F(x^*)]\|^2 =&\|t^k - x^*\|^2 - 2\lambda_k \langle F(t^k) - F(x^*), t^k - x^*\rangle \\ &+ \lambda_k^2 \|F(t^k) - F(x^*)\|^2 \\ \leq&(1 - 2\lambda_k\beta + \lambda_k^2 L^2)\|t^k - x^*\|^2.\end{aligned}$$

Consequently

$$\|[t^k - \lambda_k F(t^k)] - [x^* - \lambda_k F(x^*)]\| \leq \delta_k \|t^k - x^*\|, \tag{3.4}$$

where $\delta_k := \sqrt{1 - 2\lambda_k\beta + \lambda_k^2 L^2}$. Since (3.2) and (3.3), for every $x \in \mathbb{H}$ we have

$$\begin{aligned}\|w^k - x\| =&\|x^k - \alpha_k(x^k - x^{k-1}) - x\| \\ \leq&\|x^k - x\| + \alpha_k\|x^k - x^{k-1}\| \\ \leq&\|x^k - x\| + \tau_k. \end{aligned} \tag{3.5}$$

For each $\bar{x} \in \Omega$, it follows from Step 2 and Lemma 2.5 that

$$\begin{aligned}\|t^k - \bar{x}\|^2 =&\|u_{i_0}^k - \bar{x}\|^2 \\ =&\left\|(1 - \gamma_{k,i_0})w^k + \gamma_{k,i_0} S_{i_0}(w^k) - \bar{x}\right\|^2 \\ \leq&\|w^k - \bar{x}\|^2 - \gamma_{k,i_0}(1 - \gamma_{k,i_0} - \beta_{i_0})\|S_{i_0}(w^k) - w^k\|^2. \end{aligned} \tag{3.6}$$

Combining Step 3, (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned}\|x^{k+1} - x^*\| =& \left\|(1 - \zeta_k)t^k + \zeta_k \left[t^k - \lambda_k F(t^k)\right] - x^*\right\| \\ \leq&(1 - \zeta_k)\|t^k - x^*\| + \zeta_k \left\|[t^k - \lambda_k F(t^k)] - x^*]\right\| \\ \leq&(1 - \zeta_k)\|t^k - x^*\| + \zeta_k \left\|[t^k - \lambda_k F(t^k)] - [x^* - \lambda_k F(x^*)]\right\| \\ &+ \zeta_k\lambda_k\|F(x^*)\| \\ \leq&[1 - \zeta_k(1 - \delta_k)]\|t^k - x^*\| + \zeta_k\lambda_k\|F(x^*)\| \\ \leq&[1 - \zeta_k(1 - \delta_k)]\sqrt{\|w^k - x^*\|^2 - \gamma_{k,i_0}(1 - \gamma_{k,i_0} - \beta_{i_0})\|S_{i_0}(w^k) - w^k\|^2} \\ &+ \zeta_k\lambda_k\|F(x^*)\| \\ \leq&[1 - \zeta_k(1 - \delta_k)]\|w^k - x^*\| + \zeta_k\lambda_k\|F(x^*)\| \\ \leq&[1 - \zeta_k(1 - \delta_k)](\|x^k - x^*\| + \tau_k) + \zeta_k\lambda_k\|F(x^*)\| \\ =&[1 - \zeta_k(1 - \delta_k)]\|x^k - x^*\| + \tau_k[1 - \zeta_k(1 - \delta_k)] + \zeta_k\lambda_k\|F(x^*)\|.\end{aligned}$$

Then, using the conditions $a \in (0,1), \delta_k := \sqrt{1-2\lambda_k\beta+\lambda_k^2L^2} < 1-a$ of (3.1) we get

$$\begin{aligned}\|x^{k+1}-x^*\| \leq& [1-\zeta_k(1-\delta_k)]\|x^k-x^*\| + \tau_k[1-\zeta_k(1-\delta_k)] + \zeta_k\lambda_k\|F(x^*)\| \\ \leq& (1-a\zeta_k)\|x^k-x^*\| + \tau_k(1-a\zeta_k) + \zeta_k\lambda_k\|F(x^*)\| \\ \leq& (1-a\zeta_k)\|x^k-x^*\| + \zeta_k\left[1-a\zeta_k+\lambda_k\|F(x^*)\|\right] \\ \leq& (1-a\zeta_k)\|x^k-x^*\| + a\zeta_k\frac{1-a\zeta_k+\lambda_k\|F(x^*)}{a} \\ \leq& (1-a\zeta_k)\|x^k-x^*\| + a\zeta_k M \\ \leq& \max\{\|x^k-x^*\|, M\} \\ \leq& \cdots \\ \leq& \max\{\|x^0-x^*\|, M\},\end{aligned}$$

where $M = \sup\left\{\frac{1-a\zeta_k+\lambda_k\|F(x^*)}{a} : k=1,2,...\right\} < +\infty$ is deduced from the parameters setup (3.1). Thus, the sequence $\{x^k\}$ is bounded. Since the condition $\lim_{k\to\infty}\tau_k = 0$, (3.5) and (3.6), we deduce that both sequences $\{w^k\}$ and $\{t^k\}$ are also bounded. Applying the inequality

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,y \in \mathbb{H},$$

for $x := [t^k - \lambda_k F(t^k)] - [x^* - \lambda_k F(x^*)]$ and $y := -\lambda_k F(x^*)$, we obtain

$$\begin{aligned}\|x^{k+1}-x^*\|^2 =& \left\|(1-\zeta_k)(t^k-x^*) + \zeta_k\left[t^k-\lambda_kF(t^k)-x^*\right]\right\|^2 \\ \leq& (1-\zeta_k)\|t^k-x^*\|^2 + \zeta_k\left\|t^k-\lambda_kF(t^k)-x^*\right\|^2 \\ =& (1-\zeta_k)\|t^k \\ & -x^*\|^2 + \zeta_k\left\|[t^k-\lambda_kF(t^k)-(x^*-\lambda_kF(x^*))] - \lambda_kF(x^*)\right\|^2 \\ \leq& (1-\zeta_k)\|t^k-x^*\|^2 + \zeta_k\left\|[t^k-\lambda_kF(t^k)] - [x^*-\lambda_kF(x^*)]\right\|^2 \\ & - 2\zeta_k\lambda_k\langle F(x^*), x^{k+1}-x^*\rangle \\ \leq& [1-\zeta_k(1-\delta_k^2)]\|t^k-x^*\|^2 - 2\zeta_k\lambda_k\langle F(x^*), x^{k+1}-x^*\rangle \qquad (3.7)\end{aligned}$$

Combining (3.5), (3.6) and (3.7), we obtain

$$\begin{aligned}
\|x^{k+1}-x^*\|^2 \le&[1-\zeta_k(1-\delta_k^2)]\|t^k-x^*\|^2-2\zeta_k\lambda_k\langle F(x^*),x^{k+1}-x^*\rangle\\
\le&[1-\zeta_k(1-\delta_k^2)]\Big[\|w^k-\bar{x}\|^2-\gamma_{k,i_0}(1-\gamma_{k,i_0}-\beta_{i_0})\|S_{i_0}(w^k)-w^k\|^2\Big]\\
&-2\zeta_k\lambda_k\langle F(x^*),x^{k+1}-x^*\rangle\\
\le&[1-\zeta_k(2\lambda_k\beta-\lambda_k^2L^2)]\Big[\Big(\|x^k-x^*\|+\alpha_k\|x^k-x^{k-1}\|\Big)^2\\
&-\gamma_{k,i_0}(1-\gamma_{k,i_0}-\beta_{i_0})\|S_{i_0}(w^k)-w^k\|^2\Big]-2\zeta_k\lambda_k\langle F(x^*),x^{k+1}-x^*\rangle\\
\le&[1-\zeta_k(2\lambda_k\beta-\lambda_k^2L^2)]\Big[\Big(\|x^k-x^*\|+\tau_k\Big)^2\\
&-\gamma_{k,i_0}(1-\gamma_{k,i_0}-\beta_{i_0})\|S_{i_0}w^k-w^k\|^2\Big]-2\zeta_k\lambda_k\langle F(x^*),x^{k+1}-x^*\rangle\\
\le&[1-\zeta_k(2\lambda_k\beta-\lambda_k^2L^2)]\Big[\Big(\|x^k-x^*\|+\zeta_k^2\Big)^2\\
&-\gamma_{k,i_0}(1-\gamma_{k,i_0}-\beta_{i_0})\|S_{i_0}(w^k)-w^k\|^2\Big]-2\zeta_k\lambda_k\langle F(x^*),x^{k+1}-x^*\rangle\\
=&[1-\zeta_k(2\lambda_k\beta-\lambda_k^2L^2)]\Big[\Big(\|x^k-x^*\|^2+2\zeta_k^2\|x^k-x^*\|+\zeta_k^4\Big)\\
&-\gamma_{k,i_0}(1-\gamma_{k,i_0}-\beta_{i_0})\|S_{i_0}(w^k)-w^k\|^2\Big]-2\zeta_k\lambda_k\langle F(x^*),x^{k+1}-x^*\rangle\\
\le&[1-\zeta_k(2\lambda_k\beta-\lambda_k^2L^2)]\|x^k-x^*\|^2+\zeta_k(2\lambda_k\beta-\lambda_k^2L^2)\Gamma_k\\
&-\gamma_{k,i_0}(1-\gamma_{k,i_0}-\beta_{i_0})[1-\zeta_k(2\lambda_k\beta-\lambda_k^2L^2)]\|S_{i_0}(w^k)-w^k\|^2, \qquad (3.8)
\end{aligned}$$

where

$$\Gamma_k:=\frac{\zeta_k[1-\zeta_k(2\lambda_k\beta-\lambda_k^2L^2)]\left(2\|x^k-x^*\|+\zeta_k^2\right)}{\lambda_k(2\beta-\lambda_kL^2)}-\frac{2\langle F(x^*),x^{k+1}-x^*\rangle}{2\beta-\lambda_kL^2}.$$

Since $\{x^k\}$ is bounded and (3.1), we have $\Gamma:=\sup\{\Gamma_k:k=1,2,...\}<+\infty$. Consequently

$$\begin{aligned}
\|x^{k+1}-x^*\|^2\le&[1-\zeta_k(2\lambda_k\beta-\lambda_k^2L^2)]\|x^k-x^*\|^2+\zeta_k(2\lambda_k\beta-\lambda_k^2L^2)\Gamma\\
&-\gamma_{k,i_0}(1-\gamma_{k,i_0}-\beta_{i_0})[1-\zeta_k(2\lambda_k\beta-\lambda_k^2L^2)]\|S_{i_0}(w^k)-w^k\|^2. \qquad (3.9)
\end{aligned}$$

Set $a_k:=\|x^k-x^*\|^2$. Let us consider two following cases.

Case 1. There exists k_0 such that $a_{k+1}\le a_k$ for all $k\ge k_0$. Then, $\lim_{k\to\infty}a_k=A<+\infty$. Passing the limit (3.9) as $k\to\infty$, using $\lim_{k\to\infty}\zeta_k=0$ and the conditions (3.1), we obtain $\lim_{k\to\infty}\|S_{i_0}(w^k)-w^k\|=0$. From the condition $\gamma_{k,i}\in(\bar{b},\hat{b})\subset(0,1-\min\{\beta_i:i\in I\})$ for all $k\in\mathbb{N},i\in I$, it follows that

$$0\le\bar{b}\|w^k-S_i(w^k)\|\le\gamma_{k,i}\|w^k-S_i(w^k)\|\le\hat{b}\|w^k-S_{i_0}(w^k)\|\to0.$$

This implies $\lim_{k\to\infty}\|w^k-S_i(w^k)\|=0$ for all $i\in I$. Let $\{x^{k_j}\}$ be a subsequence of $\{x^k\}$ such that

$$\liminf_{k\to\infty}\langle F(x^*),x^{k+1}-x^*\rangle=\lim_{j\to\infty}\langle F(x^*),x^{k_j+1}-x^*\rangle.$$

By the boundedness of $\{x^{k_j+1}\}$, there exists subsequence $\{x^{k_{j_h}}\}$ such that $x^{k_{j_h}+1} \rightharpoonup \bar{x}$ as $h \to \infty$. Since S_i is demiclosed at zero for each $i \in I$, we deduce $\bar{x} \in \cap_{i\in I} Fix(S_i)$. This leads to

$$\liminf_{k\to\infty} \langle F(x^*), x^{k+1} - x^*\rangle = \lim_{h\to\infty} \langle F(x^*), x^{k_{j_h}+1} - x^*\rangle = \langle F(x^*), \bar{x} - x^*\rangle \geq 0.$$

The last inequality is deduced from that x^* is a solution of Problem 1. due to the setup (3.1), (3.8) yields

$$\limsup_{k\to\infty} \Gamma_k \leq 0.$$

By Lemma 2.4, we can conclude that $x^k \to x^*$ as $k \to \infty$.

Case 2. There does not exist k_1 such that $a_{k+1} \leq a_k$ for all $k \geq k_1$. So, there exists an integer k_0 such that $a_{k_0} \leq a_{k_0+1}$. By Lemma 2.6, Maingé introduced a subsequence $\{a_{\tau(k)}\}$ of $\{a_k\}$ which is defined as

$$\tau(k) = \max\{i \in \mathbb{N} : k_0 \leq i \leq k, a_i \leq a_{i+1}\}, \quad \forall k \geq k_0.$$

Then, he showed that

$$\tau(k) \nearrow +\infty, 0 \leq a_k \leq a_{\tau(k)+1}, a_{\tau(k)} \leq a_{\tau(k)+1}, \quad \forall k \geq k_0. \tag{3.10}$$

Since $\{a_{\tau(k)}\}$ is decreasing and bounded, there exists the limit $\lim_{k\to\infty} a_{\tau(k)} = M < +\infty$. By the boundedness of $\{x^{\tau(k)}\}$, so there exists a subsequence which weakly converges to $\bar{x}$. Without loss of generality, we may assume that $x^{\tau(k)} \rightharpoonup \bar{x}$. From (3.9) and (3.10), it follows that

$$\begin{aligned} a_{\tau(k)} \leq a_{\tau(k)+1} &= \|x^{\tau(k)+1} - x^*\|^2 \\ &\leq [1 - \zeta_{\tau(k)}(2\lambda_{\tau(k)}\beta - \lambda_{\tau(k)}^2 L^2)]a_{\tau(k)} + \zeta_{\tau(k)}(2\lambda_{\tau(k)}\beta - \lambda_{\tau(k)}^2 L^2)\Gamma \\ &- \gamma_{\tau(k),i_0}(1 - \gamma_{\tau(k),i_0} - \beta_{i_0})[1 - \zeta_{\tau(k)}(2\lambda_{\tau(k)}\beta - \lambda_{\tau(k)}^2 L^2)]\|S_{i_0}(w^{\tau(k)}) - w^{\tau(k)}\|^2. \end{aligned}$$

Taking the limit as $k \to \infty$ and using $\lim_{k\to\infty} a_{\tau(k)} = M$, we deduce

$$\lim_{k\to\infty} \|S_{i_0}(w^{\tau(k)}) - w^{\tau(k)}\| = 0,$$

and hence

$$\lim_{k\to\infty} \|S_i(w^{\tau(k)}) - w^{\tau(k)}\| = 0, \quad \forall i \in I.$$

Since S_i is demiclosed, we get $\bar{x} \in \Omega$. By a similar way as the above case, we also have

$$\limsup_{k\to\infty} \Gamma_{\tau(k)} \leq 0. \tag{3.11}$$

From (3.8), it yields

$$\begin{aligned} a_{\tau(k)} \leq& a_{\tau(k)+1} \\ =& \|x^{\tau(k)+1} - x^*\|^2 \\ \leq& [1 - \zeta_{\tau(k)}(2\lambda_{\tau(k)}\beta - \lambda_{\tau(k)}^2 L^2)]a_{\tau(k)} + \zeta_{\tau(k)}(2\lambda_{\tau(k)}\beta - \lambda_{\tau(k)}^2 L^2)\Gamma_{\tau(k)}. \end{aligned}$$

Consequently

$$0 \leq a_{\tau(k)} \leq \Gamma_{\tau(k)}, \quad \forall k \geq k_0.$$

Combining this and (3.11), we deduce

$$\limsup_{k\to\infty} a_{\tau(k)} = 0,$$

and so $\lim_{k\to\infty} a_{\tau(k)+1} = 0$. By (3.10), $0 \leq a_k \leq a_{\tau(k)+1} \to 0$ as $k \to \infty$. Which completes the proof. $\square$

4. Numerical experiments

We start with a numerical example in which we compare our proposed Algorithm 3.1 with the parallel projection algorithm (PPA) introduced by Anh and Hong [1, Scheme (3.1)] and the hybrid steepest descent algorithm ($HSDA$) suggested by Yamada [21, Scheme (23)] where $T := S_n S_{n-1} \cdots S_2 S_1$.

Example 4.1. We use a linear mapping $F : \mathbb{R}^m \to \mathbb{R}^m$ defined in the form $F(x) = Qx + q$ in [4] with $q \in \mathbb{R}^m$ and $Q = BB^T + D + E$ with B is a $m \times m$ matrix with their entries being generated in $(0, 2)$, D is a $m \times m$ skew-symmetric matrix with their entries being generated in $[-11, 11]$, E is a $m \times m$ diagonal matrix, whose diagonal entries are positive in $(0, 2)$. So, Q is positive semidefinite. It is clear that F is $L-$Lipschitz continuous and $\beta-$strongly monotone with $L := \max\{t : t \in eig(Q)\}$ and $\beta := \min\{t : t \in eig(Q)\}$, where the set $eig(G)$ represents all eigenvalues of Q. Next, we consider the feasible set C and mappings S_1, S_2, S_3 given as follows:

$$\begin{aligned}
&C = \left\{x \in \mathbb{R}^m : 0 \leq x, e^\top x \leq g\right\}, e \in \mathbb{R}^m, g \in \mathbb{R},\\
&S_1(x) = x \quad \forall x \in C,\\
&S_2(x) = Pr_A(x), A = C \cap \left\{x = (x_1, x_2, ..., x_m)^\top \in \mathbb{R}^m : x_i \leq 3 \quad \forall i = 1, 2, ..., m\right\},\\
&S_3(x) = (\sin^2 x_1, 1 + x_2, x_3, ..., x_m)^\top,
\end{aligned}$$

where Pr_A is the metric projection onto A. Then, for each $i \in I$, the mapping $S_i : C \to C$ is nonexpansive.

Test 1. Consider in $\mathbb{R}^3$. The matrices B, D, E and the vectors q, e and real number g are chosen:

$$B = \begin{bmatrix} 1.5 & 1 & 0 \\ 1 & 1.3 & 1.25 \\ 0 & 1 & 0.5 \end{bmatrix}, D = \begin{bmatrix} 0 & 1 & 0.5 \\ -1 & 1 & -0.5 \\ -0.5 & 0.5 & -0.8 \end{bmatrix}, E = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 12 \end{bmatrix},$$

$$q = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}, e = \begin{bmatrix} 3 \\ -5 \\ 10 \end{bmatrix}, g = 7.$$

It is easy to evaluate that

$$eig(Q) = \{5.5817, 9.5633, 13.8076\},$$

and hence

$$L := \max\{t : t \in eig(Q)\} = 13.8076 \text{ and } \beta := \min\{t : t \in eig(Q)\} = 5.5817.$$

The parameters satisfying (3.1) are set as follows:

$$\begin{cases} a = 0.001 \in (0,1), \lambda_k = 0.001 + \frac{10^{-3}}{3k+1} \in \left(0, \frac{2\beta}{L^2}\right) = (0, 0.0586), \\ \zeta_k = \frac{1}{15k+1} \in (0,1), \mu_k = \frac{k}{10k+1} > 0, \tau_k = \frac{1}{(20k+7)^2}, \gamma_{k,i} = \frac{1}{40k}. \end{cases}$$

We take $x^0 = (1,2,0)^\top, x^1 = (1,1,1)^\top$ and the tolerance $\epsilon = 10^{-3}$. The numerical results are showed in Figure 1 and Table 1.

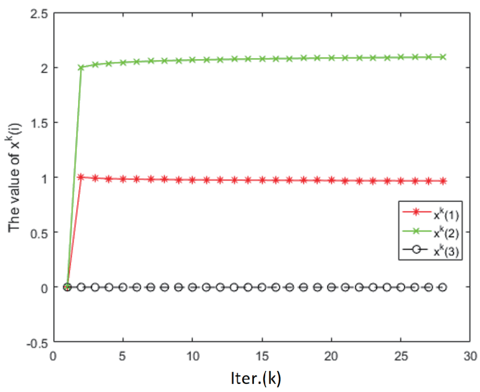

FIGURE 1. Performance of PIPA with Test 1 setting. The approximate solution is $x^{28} = (0.9371, 2.0069, 0.0031)$.

Problems	λ_k	ζ_k	τ_k	μ_k	$\gamma_{k,i}$	No. Iter.	CPU times
1	0.0015	$\frac{1}{10k+1}$	$\frac{1}{(30k+7)^2}$	$\frac{k}{20k+1}$	$\frac{1}{50k}$	20	0.7344
2	$0.001 + \frac{10^{-3}}{3k+1}$	$\frac{1}{15k+1}$	$\frac{1}{(20k+7)^2}$	$\frac{k}{10k+1}$	$\frac{1}{40k}$	27	0.8438
3	$0.04 + \frac{10^{-3}}{3k+1}$	$\frac{1}{15k+1}$	$\frac{1}{(20k+7)^2}$	$\frac{k}{10k+1}$	$\frac{1}{40k}$	57	1.5469
4	$0.04 + \frac{10^{-3}}{3k+1}$	$\frac{1}{55k+1}$	$\frac{1}{(20k+7)^2}$	$\frac{k}{10k+1}$	$\frac{1}{40k}$	24	0.6719
5	$0.04 + \frac{10^{-2}}{13k+6}$	$\frac{1}{55k+1}$	$\frac{1}{(20k+7)^2}$	$\frac{k}{10k+1}$	$\frac{1}{110k}$	18	0.5001
6	$0.03 + \frac{10^{-3}}{3k+100}$	$\frac{1}{5k+1}$	$\frac{1}{(6k+7)^2}$	$\frac{k}{30k+1}$	$\frac{1}{11k+32}$	101	3,0002
7	$0.0013 + \frac{10^{-5}}{3k+9}$	$\frac{1}{15k+11}$	$\frac{1}{(20k+37)^2}$	$\frac{k}{3k+37}$	$\frac{1}{49k+32}$	21	0.5469

TABLE 1. Test 1 with different parameters.

Test 2. Consider in $\mathbb{R}^5$. Compare the $(PIPA)$ with the (PPA) and the $(HSDA)$. The stopping criterion of the algorithms is $\|x^{k+1} - x^k\| \leq \epsilon$. Let $e = (1,2,-3,4,-5)^\top, g = 6$, all entries B, D, E and vector q be randomly generated by using the commands in Matlab $B = 2 * rand(5,5); D = skewdec(5,1); E = 3 * diag(1:5)$ giving $E = (e_{ij})_{5\times 5}$ where $e_{ij} = 0$ for all $i \neq j$ and $e_{ii} = 3i$ for all $i \in \{1,...,5\}$; $q = rand(5,1)$. The termination criterion is $\|x^{k+1} - x^k\| \leq \epsilon$. Data of the algorithms are given as follows:

(a) (PPA): $\alpha_{k,i} := 0.001 + \frac{1}{k+100}$ for all $i \in I, \epsilon_k = 0, \tau_k = 0, \gamma_k = \frac{1}{7k+10}$, for all $k \in \mathbb{N}$, the starting point $x^0 = (0,0,0,1,1)^\top$. The tolerance: $\epsilon = 10^{-3}$.

(b) $(HSDA)$: $\mu = 1.65\frac{\beta}{L^2} \in (0, \frac{2\beta}{L^2})$ where $\beta = \min\{m : m \in eig(Q)\}$ and $L = \max\{k : k \in eig(Q)\}$. Parameters $\lambda_k := \frac{1}{\sqrt{3k+5}}$ (k=1,2,...) satisfy the conditions

$$\lim_{k\to\infty} \lambda_k = 0, \sum_{k=1}^{\infty} \lambda_k = +\infty, \lim_{k\to\infty} \frac{\lambda_k - \lambda_{k+1}}{\lambda_{k+1}} = 0.$$

The starting point: $x^0 = (0,0,0,1,1)^\top$. The tolerance: $\epsilon = 10^{-2}$.

(c) $(PIPA)$: $\lambda_k = \frac{\beta}{L^2} - \frac{1}{10^4(3k+1)}, \zeta_k = \frac{1}{15k+1}, \mu_k = \frac{k}{10k+1}, \tau_k = \frac{1}{(20k+7)^2}, \gamma_{k,i} = \frac{1}{40k+1}$. Starting points: $x^0 = (1,0,0,0,0)^\top, x^1 = (0,0,1,1,0)^\top$.

The numerical results are showed in Figure 2 and Table 2.

	No. Iter.			CPU times		
Problems	(PPA)	$(HSDA)$	$(PIPA)$	(PPA)	$(HSDA)$	$(PIPA)$
1	114	5918	32	8.9219	154.6094	1.1563
2	174	983	28	14.4688	28.8125	1.0313
3	85	2317	27	6.6563	56.1719	0.9375
4	86	8573	29	6.7344	219.3906	1.1563
5	101	3885	31	10.1250	102.0938	1.0625
6	124	3425	29	11.1719	88.9375	1.0313
7	130	2095	33	11.0938	97.1034	1.2188
8	138	4095	30	11.7344	173.0448	1.2188
9	83	6033	32	7.0313	195.36500	1.4063
10	126	5031	28	9.9688	168.4924	1.0938

TABLE 2. Comparison results.

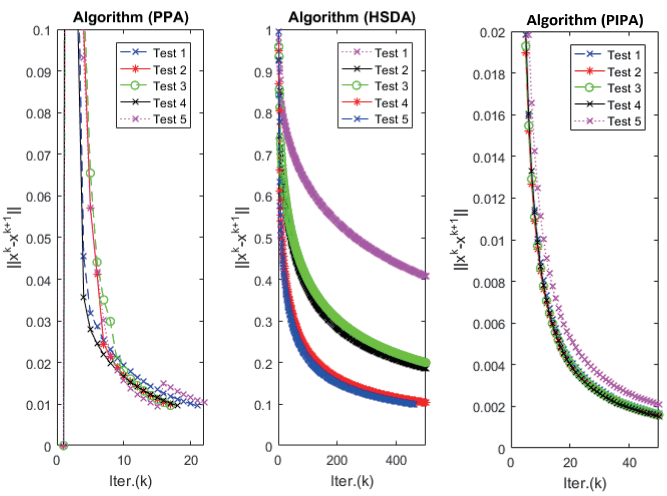

FIGURE 2. Comparisons of successive iteration difference ($\|x^k - x^{k+1}\|$) for 5 random tests (problems).

4. **Image recovery.** Problem 2 is a useful mathematical model for many inverse problems, one of them is signal/image restoration. In this problem, the goal is to recover an original signal/image $x \in \mathbb{R}^s$ from a noisy observed signal/image $y \in \mathbb{R}^m$. Let $\mathbb{B}$ be the (linear) blurring operator (i.e., a $m \times s$ matrix) and ϵ be a sample of zero-mean white Gaussian noise with variance σ^2. It means that $p(s) = \mathbb{N}(s|0, \sigma^2 Id)$, where $N(g|\mu, \Sigma)$ denotes a multivariate Gaussian density with mean μ and covariance Σ, evaluated at g. The image restoration problem is formulated as:

$$y = \mathbb{B}x + \epsilon. \tag{4.1}$$

Examples of observation mechanisms which are adequately approximated by (4.1) are optical or motion blur, tomographic projections, electronic noise, photoelectric noise, and more. One classical approach for handling (4.1) is the following *Least Absolute Shrinkage and Selection Operator*, mainly known as Lasso [16]:

$$\min\left\{\frac{1}{2}\|y - \mathbb{B}x\|^2 + \lambda\|x\|_1 : x = (x_1, x_2, ..., x_s)^\top \in \mathbb{R}^s\right\}, \tag{4.2}$$

where $\|x\|_1 = \sum_{k=1}^{s} |x_k|$. Under standard assumption, (4.2) is a special case of Problem 2.

In this section we consider a special case of Problem 2 which is the image restoration problem (4.1). We choose

$$f_1(x) := \frac{1}{2}\|\mathbb{B}x - y\|^2, \quad f_2(x) = \lambda\|x\|_1.$$

One can easily verify that $f_1 : \mathbb{R}^s \to \mathbb{R}$ is convex and differentiable and its gradient of f_1 is $\nabla f_1(x) = \langle \mathbb{B}, \mathbb{B}x - y\rangle$ which is $\mathbb{L} := \|\mathbb{B}\|^2$-Lipschitz continuous. Moreover, $f_2 : \mathbb{R}^s \to \mathbb{R}$ is proper, lower semicontinuous and convex.

Now, choose $\epsilon_i \in (0, 2/\mathbb{L})$ and mappings $S_i : \mathbb{R}^s \to \mathbb{R}^s$ be defined by $S_i(x) = prox_{\mathbb{R}^s}^{\epsilon_i f_2}(Id - \epsilon_i \nabla f_1)(x)$ for all $x \in \mathbb{R}^s, i \in I$. Via the fixed point characterization of the proximity operator and by choosing $F = 0$ in Problem 1, we treat the image restoration problem (4.1) as a common fixed point problem with this nonexpansive mappings S_i (also $0-$demicontractive on $\mathbb{R}^s$) and then apply Algorithm 3.1 for solving Problem 2. We call this algorithm the *Modified Parallel Inertial Proximal Algorithm* (MPIPA) and its strong convergence follows directly from Theorem 3.4.

All programming is implemented in Matlab R2016a running on a PC with Intel(R) Core(TM) i9-9900KS CPU @ 4.00GHz 32.0 GB Ram. The proximal operators evaluation is computed via the Matlab optimization toolbox (fmincon). The stopping criterion for all tested algorithms is $\|x^{k+1} - x^k\| \leq \epsilon$.

Test 1. We test the convergence of the MPIPA for the image restoration problem by means of Peak Signal-to-Noise Ratio (PSNR) in decibel (dB) in [17] and Structural Similarity Index Metric (SSIM) [22]. The parameters $\alpha_k, \tau_k, \mu_k, \gamma_{k,i}$ and ζ_k are different. The blurring operator is chosen as $\mathbb{B} := fspecial('gaussian', [256\ \ 256], 4)$. Let $F(x) := 0.7x, \epsilon_1 = 0.1, \epsilon_2 = 0.3$ and $\epsilon_3 = 0.7$. Then, we have $\beta = L = 0.7$. *Case 1.2.* Consider Gaussian blur of filter size with standard deviation $\sigma = 4$ and noise 10^{-4}. The values of PSNR (dB) and SSIM for the " bird " image corrupted by Gaussian blur are PSNR = 20.532 dB and SSIM = 0.6803. For all $k = 1, 2, \ldots,$

the parameters of the IPPA are chosen as follows:

$$\mu_k = \frac{k}{100k+1}, \quad \tau_k = \frac{1}{(k+1)^2}, \quad \gamma_{k,i} = \frac{1}{50k}, \quad \zeta_k = \frac{1}{10k+1}, \quad \lambda_k = 0,01.$$

The numerical results are showed in Figure 3 and Table 3.

Case	μ_k	τ_k	$\gamma_{k,i}$	ζ_k	λ_k	PSNR	SSIM	CPU time/s
1	$\frac{k}{100k+1}$	$\frac{1}{(k+1)^2}$	$\frac{1}{50k}$	$\frac{1}{10k+1}$	0.01	28.1825	0.9016	3677.0425
2	0.01	$\frac{1}{(3k+2)^2}$	$\frac{1}{50k}$	$\frac{1}{10k+1}$	0.01	31.3307	0.9615	3402.5581
3	$\frac{k}{100k+1}$	$\frac{1}{(k+2)^2}$	$\frac{1}{50k}$	$\frac{1}{10k+1}$	10^{-3}	28.9304	0.9083	3677.7022
4	0.8	$\frac{1}{(k+1)^2}$	$\frac{1}{100k}$	$\frac{1}{10k+1}$	0.01	26.9910	0.7048	3633.9307
5	$\frac{k}{10k+300}$	$\frac{1}{(2k+1)^2}$	$\frac{1}{100k}$	$\frac{1}{100k+1}$	10^{-3}	32.3016	0.9204	3509.5591

TABLE 3. Results for the algorithm MPIPA with different parameters and the tolerance error $\epsilon = 10^{-2}$.

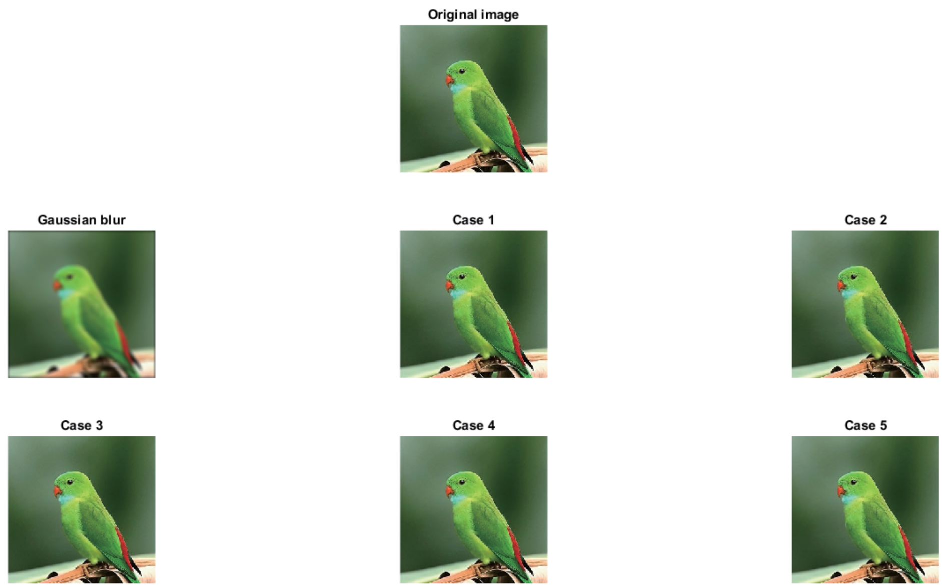

FIGURE 3. Restoration results of Algorithm MPIPA with different parameters.

Test 2. In this experiment we compare the performances of the MPIPA with the FBSA (1.2), the FISTA (1.3) and the Fast Viscosity Forward-Backward Algorithm in [12] (shortly FVFBA) by means of PSNR and SSIM. *First*, we take a color image of the Halong bay. Then, the blurring operator is chosen as $\mathbb{B} := fspecial('gaussian', [288\ \ 288], 13)$, i.e., size of the original image is 288×288 with standard deviation $\sigma = 13$. Parameters in each algorithm are chosen as follows:

(i) In the FBSA: $\lambda_k := \frac{1}{50k+1}$ for all $k = 1, 2, \cdots;$
(ii) In the FISTA: $y^0 = x^0, \quad t_0 = 1.5;$

(iii) In the FVFBA: $\tau_k = \frac{10^{15}}{k^2}, \quad \mu_k = \frac{1}{100k+1}, \quad \gamma_k = \frac{1}{50(k+1)}, \quad \beta_k = \frac{0.99k}{k+1};$

(iv) In the MPIPA: The parameters are chosen as in Test 1.

The comparative results are shown in Figures 4-6.

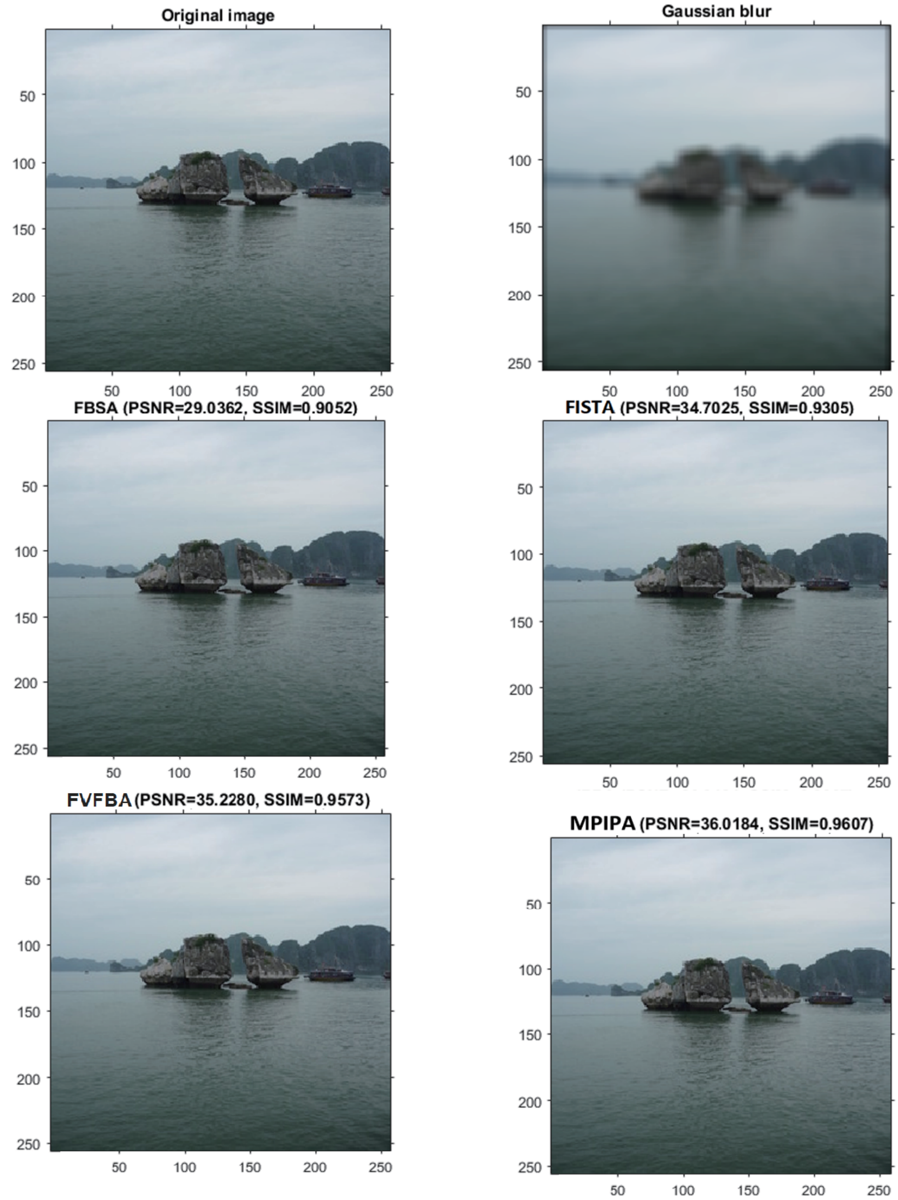

FIGURE 4. Restoration results using the different algorithms.

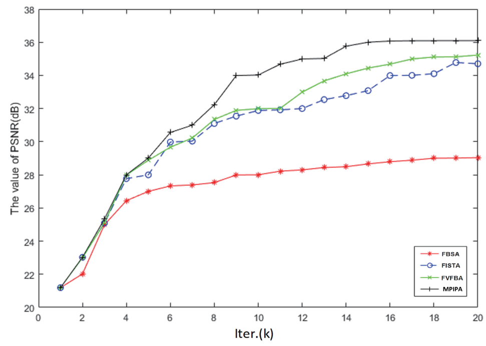

FIGURE 5. The PSNR comparison.

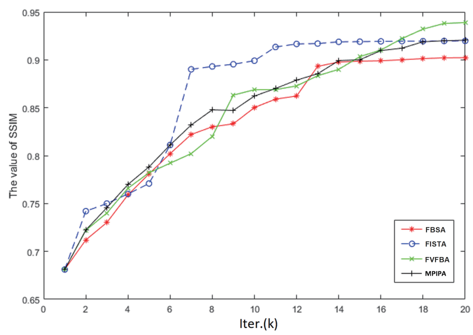

FIGURE 6. The SSIM comparison.

Test 3. In this experiment we use the same data as in Test 2 and compare the previous three algorithms with the PSNR and SSIM quality measurement. We take the "flower" image and consider Motion blur $\mathbb{B} = fspecial('motion', len, \theta)$ specifying with Motion length 20 pixels ($len = 20$) and Motion orientation $45^o(\theta = 45)$. The results are showed in Figure 7.

FIGURE 7. Restoration results using the different algorithms.

In the above experiments we tried to illustrate the performances of our scheme besides its theoretical advantages, one of which is that no projections are computed/approximated. We also decided to present results and comparison for the image restoration problem since we think that the reader can have an idea on the practical usage of our scheme.

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