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STABILITY AND DATA DEPENDENCY OF A NEW FIXED POINT TECHNIQUE WITH APPLICATION TO POLYNOMIOGRAPHS

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Abstract. The current study focuses on the introduction and convergence analysis of new two-step fixed point approximation method, based on a single parameter. The method is used to approximate fixed points of a contractive type mapping in the sense of Imoru and Olantinwo [7]. The algorithm offers a wider interval of convergence compared to the classical fixed point procedure. By analyzing the stability and data dependency of the new technique, we investigate the rate at which the new scheme converges compared to the Picard [16], Mann [13], Ishikawa [8], and Noor [14] iterative procedures. We exhibit analytically and graphically that the new algorithm outperforms in terms of speed and time. We also present that the new technique is of at least linear order of convergence, and depending on the choice of parameter, it may increase to second order. In addition, we utilize numerical examples, tables and graphs in MATLAB 2023(a) software to compare the proposed scheme with certain other fixed point processes, including absolute error, relative error, elapsed time, number of iterations, and order of convergence. Finally, we apply our algorithm to solve various complex polynomials and build polynomiographs. These polynomiographs enhance the utility of polynomiograph software, which uses standard Picard iteration to create visually pleasing patterns. The acquired results are novel and widen the scope of earlier findings in the literature.

1. Introduction

The field of fixed point theory primarily encompasses two overarching topics. There are two approaches to address the operator equation $\mathtt{L}(\mathtt{u}) = 0$, where $\mathtt{L}$ is a self mapping equipped with specific properties depending upon the problem at hand:

1) To identify the conditions under which a fixed point exists,
2) To determine these fixed points either by analytical means or by employing suitable algorithms.

It is worth noting that the analytical technique often fails to provide a solution for numerous situations, which urges one to resort to the transformation of the operator equation into a fixed point equation and thereafter utilize the most appropriate algorithm. The Banach Fixed Point Theorem [2] proposes the use of the elementary Picard [16] iteration method given by

$$\mathtt{u}_{\mathtt{n+1}} = \mathtt{L}\mathtt{u}_{\mathtt{n}}, n \in \mathbb{N} \cup \{0\}, \tag{1.1}$$

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when dealing with contraction maps. Due to the lack of convergence of Picard iterates for the class of nonexpansive mappings, alternative iterative procedures with distinct steps and parameters are employed, recognizing Krasnoselskii's [17] and Mann's work [13], for instance.

The iterative approximation of fixed points is essential for solving a wide range of issues faced in numerous study fields. As the application domains of iterative algorithms have expanded, considerable iterative algorithms have been fabricated for the mappings endowed with set properties. These algorithms have been further analyzed for their qualitative characteristics, including convergence, rate of convergence, stability, and data dependency(see, for example [3–5, 8, 11, 14, 16, 18, 19] and references therein).

The classical fixed point approach converges for $|\mathtt{L}'(\mathtt{u})| < 1$. Then, an obvious question arises.

Does there exist any iterative technique that has a wider interval of convergence?

Answering to this, V. Kanwar et al. [10] recently presented a novel iterative approach that relies on geometrical construction and involves only one parameter, defined by

$$\mathtt{u}_{\mathtt{n}+1} = \frac{m\mathtt{u}_{\mathtt{n}} + \mathtt{L}\mathtt{u}_{\mathtt{n}}}{m+1}, n \in \mathbb{N} \cup \{0\}, m \in \mathbb{R}^+. \tag{1.2}$$

It is proven that the aforementioned approach converges for a larger interval, namely $-2m-1 < L'(\mathtt{u}) < 1$. This improvement of the technique (1.2) surpasses fixed point iteration method in use.
Inspired by the potency of algorithm (1.2) and the lack of efficient two-step iterative schemes relying on a single parameter, we introduce a novel approach that combines iterative method (1.2) with the widely recognized Picard iteration [16].
The root finding property of fixed point procedures encourages researchers to solve polynomial equations of varied degrees. The study of finding roots in polynomials has been a significant focus since 3000 B.C. In 2005, Kalantari's [9] advanced work on polynomiography brought new life to the issue of finding roots of polynomials. This work showcased the aesthetic connection between mathematical sciences and the world of art and design. Kalantari [9] defined Polynomiography as the "art and science of visualization in an approximation of zeros of complex polynomials via fractal and non-fractal images produced using the mathematical convergence properties of iterative functions". The resulting image is referred to as a "polynomiograph". Polynomiographs are distinct from (Mandelbrot's) fractals, as the forms of the latter cannot be effectively controlled and are solely determined by the iteration function's coefficient. Nevertheless, the shapes of a polynomiograph may be manipulated and designed in a more foreseeable manner by employing varied iterations on different complex polynomials which are applicable for several purposes, such as creating textures, designing carpets, and producing tapestries.

The researchers utilized Picard [16], Mann [13], and Ishikawa [8] iterative schemes to create polynomiographs. Inspired by the artistic patterns achieved through these schemes, we propose our own scheme to generate polynomiographs with various shapes and designs.

The paper is structured in the following manner:

(i) Section 3 introduces a fixed point procedure with a single parameter.
(ii) Section 4 examines the qualitative behavior of the technique, including convergence analysis, stability, and data dependency for contractive type mappings defined by Imoru and Olantinwo [7].
(iii) Section 5 presents a numerical example and discusses the efficiency, time complexity, and order of convergence of our new technique to demonstrate the superiority of our approach compared to other algorithms in the literature using MATLAB 2023(a) software.
(iv) The approximation procedure is employed to generate polynomiographs for different complex polynomials in Section 6.

2. Requisite definitions and results

This section recalls few definitions and results of the literature required in the sequel.

In 2003, Berinde [3] presented a class of mappings that are distinct from contraction mappings. Later, Osilike [15] extended the condition on contractive type mappings to establish stability results for certain fixed point procedures.

Imoru and Olantinwo [7] further expanded upon the definition of Osilike [15] and utilized it to demonstrate the stability of the Picard and Mann iterative methods.

Definition 2.1 ([7])**.** Assume that X_b is a Banach space. A self-mapping $\mathtt{L}$ on a subset A of Banach space X_b is contractive if there is constant $\delta \in [0,1)$ and a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ which is

(a) continuous on $\mathbb{R}^+$,
(b) monotonic increasing in its domain,
(c) $\phi(0) = 0$,

such that

$$\|\mathtt{Lu} - \mathtt{Lv}\| \leq \phi(\|\mathtt{u} - \mathtt{Lu}\|) + \delta\|\mathtt{u} - \mathtt{v}\|, \text{ for all } \mathtt{u}, \mathtt{v} \in \text{A}. \tag{2.1}$$

In his doctoral dissertation, A. M. Harder [6] discussed the stability of an iterative process in the following manner.

Definition 2.2 ([6])**.** An iterative procedure $\{\mathtt{u_n}\}$ may be expressed as some function of $\mathtt{L}$ and $\mathtt{u_n}$, i.e., $\mathtt{u_{n+1}} = f(\mathtt{L}, \mathtt{u_n})$ converging to fixed point ν is said to be stable with respect to $\mathtt{L}$ if for an arbitrary sequence τ_n, the value of $\lim_{n\to\infty} \mathring{\delta}_n = \|\tau_{n+1} - f(\mathtt{L}, \tau_n)\|$ equals zero if and only if $\lim_{n\to\infty} \tau_n = \nu$.

The stability of the suggested iterative technique relies on the following lemma.

Lemma 2.3 ([3])**.** *For any sequence* $\{\mathtt{u_n}\}$ *of positive numbers satisfying*

$$\mathtt{u_{n+1}} \leq \mathfrak{t}\mathtt{u_n} + \mathfrak{s}_n, n \in \mathbb{N} \cup \{0\}$$

one has $\lim_{n\to\infty} \mathtt{u_n} = 0$ *where* $\mathfrak{t} \in [0,1] \subset \mathbb{R}$*, and* $\{\mathfrak{s}_n\}$ *is a sequence of positive numbers whenever* $\lim_{n\to\infty} \mathfrak{s}_n = 0$.

The identification of fixed points may be a difficult undertaking, particularly when the behavior of an operator is unknown. This motivates to investigate the data dependency of fixed points. Data dependence in an iterative scheme refers to the existence of an approximation method that allows one to determine the fixed point of an unknown operator based on the fixed point of a known operator.
To examine the data dependence of fixed points in our study, let us review the following definition.

Definition 2.4 (see [3])**.** For two self mappings $\mathtt{L}, \mathtt{L}^\star$ defined on A, $\mathtt{L}^\star$ is said to be an approximate operator of $\mathtt{L}$, if $\|\mathtt{L}^\star - \mathtt{L}\| < \epsilon$, for all $\mathtt{u} \in \mathrm{A}$.

We establish the superiority of the recently presented approach by comparing it to the well-known iterative strategies are shown in Table 1.

Sr.No.	Name of the Iterative Scheme	Iterative Scheme
1.	Ishikawa [8]	$d_n = (1-\mathfrak{b}_n)f_n + \mathfrak{b}_n L f_n$ $f_{n+1} = (1-\mathfrak{a}_n)f_n + \mathfrak{a}_n L d_n$
2.	Noor [14]	$q_n = (1-\mathfrak{c}_n)e_n + \mathfrak{c}_n L e_n$ $w_n = (1-\mathfrak{b}_n)e_n + \mathfrak{b}_n L q_n$ $e_{n+1} = (1-\mathfrak{a}_n)e_n + \mathfrak{a}_n L w_n$
3.	Mann [13]	$g_{n+1} = (1-\mathfrak{a}_n)g_n + \mathfrak{a}_n L g_n$

Table 1. Table listing different iterative procedures used for comparison

where $\{\mathfrak{a}_n\}, \{\mathfrak{b}_n\}$ and $\{\mathfrak{c}_n\}$ are real sequences in $(0,1), n \in \mathbb{N} \cup \{0\}$.
The following lemma will be needed in the sequel.

Lemma 2.5 ([1])**.** *Let $\{\mathfrak{p}_n\}$ be a sequence of positive reals and there exists $M \in \mathbb{Z}^+$, such that for all $n \geq N$, $\{\mathfrak{p}_n\}$ satisfies the following inequality:*

$$\mathfrak{p}_n \leq (1-\mathfrak{q}_n)\mathfrak{p}_n + \mathfrak{r}_n,$$

where $\mathfrak{q}_n \in (0,1)$ $\forall n \in \mathbb{Z}^+$ such that $\sum_{n=0}^{\infty} \mathfrak{q}_n = \infty$ and $\mathfrak{r}_n \geq 0$ is a bounded sequence. Then

$$0 \leq \limsup_{n\to\infty} \mathfrak{p}_n \leq \limsup_{n\to\infty} \mathfrak{r}_n.$$

3. Proposed fixed point estimation technique

The present section suggests a new two-step fixed point approximation technique.

Algorithm 1 Fixed Point Iterative Algorithm

Step 1. Choose $\mathtt{u}_0 \in \mathrm{A}$ *as an initial guess for the approximation.*
Step 2. Compute

$$\mathtt{v_n} = \frac{m\mathtt{u_n} + \mathtt{Lu_n}}{m+1}$$

$$\mathtt{u_{n+1}} = \mathtt{Lv_n},\ n \in \mathbb{N} \cup \{0\}, \text{where } m \in \mathbb{R}^+. \tag{3.1}$$

Remark 3.1.
- For $m = 1$, the method becomes Picard- Krasnoselskii scheme (see [17]).
- For $m = \frac{1-\lambda_n}{\lambda_n}$, where sequence $\{\lambda_n\} \subset [0, 1]$, the algorithm reduces to Picard-Mann [12] or Normal-S [18] iterative scheme.

The wider interval of convergence of iterative scheme (1.2) proved in [10], makes our proposed scheme (3.1) different from other vital techniques existing in the literature.

4. Convergence analysis

This section studies the qualitative features of proposed scheme (3.1) including convergence behavior, stability analysis and data dependency.

Theorem 4.1. *Suppose X is a Banach space and $\mathrm{A}(\neq \phi)$ is a closed convex subset of X_b. Let $\mathtt{L} : \mathrm{A} \to \mathrm{A}$ be a mapping such that*
(1) $\mathtt{L}$ *satisfies contractive type mapping* (2.1)*,*
(2) $\mathtt{L}$ *has a fixed point* ν*.*

Then the iteration process given by (3.1) converges strongly to a unique fixed point of $\mathtt{L}$ *for some* $\mathtt{u}_0 \in \mathrm{A}, m \in \mathbb{R}^+$*.*

Proof. We shall prove that $\lim_{n\to\infty} \mathtt{u_n} = \nu$. Using Definition 2.1 and iteration process (3.1),

$$\begin{aligned} \|\mathtt{v_n} - \nu\| &= \Big\|\frac{m\mathtt{u_n} + \mathtt{Lu_n}}{m+1} - \nu\Big\| \\ &= \frac{1}{m+1}\|m\mathtt{u_n} + \mathtt{Lu_n} - (m+1)\nu\| \\ &\leq \frac{m}{m+1}\|\mathtt{u_n} - \nu\| + \frac{1}{m+1}\|\mathtt{Lu_n} - \mathtt{L}\nu)\| \\ &\leq \frac{m}{m+1}\|\mathtt{u_n} - \nu\| + \frac{1}{m+1}\|\big[\phi(\|\nu - \mathtt{L}\nu\|) + \delta\|\nu - \mathtt{u_n}\|\big] \\ &\leq \Big(\frac{m+\delta}{m+1}\Big)\|\mathtt{u_n} - \nu\|, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \|\mathtt{u_{n+1}} - \nu\| &= \|\mathtt{Lv_n} - \nu\| \\ &= \|\mathtt{L}\nu - \mathtt{Lv_n}\| \\ &\leq \phi(\|\nu - \mathtt{L}\nu\|) + \delta\|\nu - \mathtt{v_n}\| \\ &= \delta\|\mathtt{v_n} - \nu\|. \end{aligned} \tag{4.2}$$

Owing to (4.1), the inequality (4.2) becomes,

$$\|\mathtt{u_{n+1}} - \nu\| \leq \delta\big(\frac{m+\delta}{m+1}\big)\|\mathtt{u_n} - \nu\|.$$

Repetition of the above process yields

$$\|\mathtt{u_n} - \nu\| \leq \delta\Big(\frac{m+\delta}{m+1}\Big)\|\mathtt{u_{n-1}} - \nu\|$$

$$
\begin{aligned}
\|\mathtt{u_{n-1}} - \nu\| &\leq \delta\Big(\frac{m+\delta}{m+1}\Big)\|\mathtt{u_{n-2}} - \nu\| \\
&\cdot \\
&\cdot \\
&\cdot \\
\|\mathtt{u_1} - \nu\| &\leq \delta\Big(\frac{m+\delta}{m+1}\Big)\|\mathtt{u_0} - \nu\|
\end{aligned}
$$

Combining all the above inequalities, one gets

$$
\|\mathtt{u_{n+1}} - \nu\| \leq \delta^{n+1}\Big(\frac{m+\delta}{m+1}\Big)^{n+1}\|\mathtt{u_0} - \nu\|. \tag{4.3}
$$

Because $\delta \in (0,1)$, so $m+\delta < m+1$, which gives $\frac{m+\delta}{m+1} < 1$ and hence $\left(\frac{m+\delta}{m+1}\right)^{n+1} \to 0$ as $n \to \infty$.
Consequently, with respect to inequality (4.3), the sequence $\{\mathtt{u_n}\}$ converges strongly to ν.

Uniqueness of fixed point

Take $\nu, \mu \in Fix(\mathtt{L})$ with $\nu \neq \mu$ satisfying $\mathtt{L}\nu \neq \mathtt{L}\mu$, using 2.1, we have

$$
\begin{aligned}
\|\nu - \mu\| &= \|\mathtt{L}\nu - \mathtt{L}\mu\| \\
&\leq \phi(\|\nu - \mathtt{L}\nu\|) + \delta\|\nu - \mu\| \\
&= \delta\|\nu - \mu\| \\
&< \|\nu - \mu\|,
\end{aligned}
$$

which is absurd. Henceforth, the fixed point is unique. □

The subsequent part of this section reflects $\mathtt{L}$ - stability of the fixed point approximation procedure defined by (3.1).

Theorem 4.2. *Suppose X is a Banach space and* $\mathrm{A}(\neq \phi)$ *is a closed convex subset of X_b. Let $\mathtt{L} : \mathrm{A} \to \mathrm{A}$ be a mapping such that*

(1) $\mathtt{L}$ *satisfies contractive type mapping* (2.1),
(2) $\mathtt{L}$ *has a fixed point ν.*

Then, the iteration process given by (3.1) is $\mathtt{L}$ *- stable.*

Proof. Let us denote the sequence $\{\mathtt{u_n}\}$ defined by (3.1) as some function of $\mathtt{L}$ and $\mathtt{u_n}$, i.e., $\mathtt{u_{n+1}} = f(\mathtt{L}, \mathtt{u_n})$ converging to fixed point ν. Consider an arbitrary sequence $\{\tau_n\}$ in A and define $\mathring{o}_n = \|\tau_{n+1} - f(\mathtt{L}, \tau_n)\|$.

By Harder's definition of stability [6], the iterative procedure (3.1) will be stable with respect to $\mathtt{L}$, if $\lim_{n\to\infty} \mathring{o} = 0$ if and only if $\lim_{n\to\infty} \tau_n = \nu$.

$$
\begin{aligned}
\|\tau_{n+1} - \nu\| &= \|\tau_{n+1} - f(\mathtt{L}, \tau_n) + f(\mathtt{L}, \tau_n) - \nu\| \\
&\leq \|\tau_{n+1} - f(\mathtt{L}, \tau_n)\| + \|f(\mathtt{L}, \tau_n) - \nu\| \\
&= \mathring{o}_n + \|f(\mathtt{L}, \tau_n) - \nu\| \\
&= \mathring{o}_n + \Big\|\mathtt{L}\nu - \mathtt{L}\Big(\frac{m\tau_n + \mathtt{L}\tau_n}{m+1}\Big)\Big\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mathring{o}_n + \phi(\|\nu - \mathtt{L}\nu\|) + \delta\Big\|\nu - \Big(\frac{m\tau_n + \mathtt{L}\tau_n}{m+1}\Big)\Big\| \\
&= \mathring{o}_n + \delta\Big\|\frac{(m+1)\nu \ - (m\tau_n + \mathtt{L}\tau_n)}{m+1}\Big\| \\
&\leq \mathring{o}_n + \delta\Big[\frac{m}{m+1}\|\tau_n - \nu\| + \frac{1}{m+1}\|\mathtt{L}\nu - \mathtt{L}\tau_n\|\Big] \\
&\leq \mathring{o}_n + \frac{m\delta}{m+1}\|\tau_n - \nu\| + \frac{\delta}{m+1}\Big[\phi(\nu - \mathtt{L}\nu) + \delta\|\nu - \tau_n\|\Big] \\
&\leq \mathring{o}_n + \frac{m\delta}{m+1}\|\tau_n - \nu\| + \frac{\delta^2}{m+1}\|\nu - \tau_n\| \\
&= \mathring{o}_n + \frac{\delta(m+\delta)}{m+1}\|\tau_n - \nu\|. \qquad (4.4)
\end{aligned}
$$

Since $\delta < 1$, $\frac{m+\delta}{m+1} < 1$ and $\lim_{n\to\infty} \mathring{o}_n = 0$. Using Lemma 2.3, $\lim_{n\to\infty} \tau_n = \nu$. Conversely, assume that $\lim_{n\to\infty} \tau_n = \nu$. We shall claim that $\lim_{n\to\infty} \mathring{o} = 0$. Consider

$$
\begin{aligned}
\mathring{o} &= \|\tau_{n+1} - f(\mathtt{L}, \tau_n)\| \\
&= \|\tau_{n+1} - \nu + \nu - f(\mathtt{L}, \tau_n)\| \\
&\leq \|\tau_{n+1} - \nu\| + \|f(\mathtt{L}, \tau_n) - \nu\|. \qquad (4.5)
\end{aligned}
$$

Proceeding as the previous arguments, the inequality (4.5) reduces to

$$
\mathring{o} \leq \|\tau_{n+1} - \nu\| + \frac{\delta(m+\delta)}{m+1}\|\tau_n - \nu\|.
$$

By assumption, it holds that $\lim_{n\to\infty} \mathring{o} = 0$ and hence the result. □

Next, we attempt to estimate the fixed point of mapping $\mathtt{L}$ by taking into account an approximate mapping $\mathtt{L}^\star$ with known fixed point.

Theorem 4.3. *Suppose X is a Banach space and $\mathrm{A}(\neq \phi)$ is a closed convex subset of X_b. Let $\mathtt{L} : \mathrm{A} \to \mathrm{A}$ be a mapping such that*

(1) *$\mathtt{L}$ satisfies contractive type mapping (2.1),*
(2) *$\mathtt{L}$ has a fixed point ν.*

With maximum permissible error ϵ, assume $\mathtt{L}^\star$ is an approximate mapping of contractive type mapping $\mathtt{L}$ with $\mathtt{L}^\star(\nu^\star) = \nu^\star$. Consider the iteration process given by (3.1) for $\mathtt{L}$ and the iteration process $\{\mathtt{u}_\mathtt{n}^\star\}$ generated by

$$
\begin{aligned}
\mathtt{v}_\mathtt{n}^\star &= \frac{m\mathtt{u}_\mathtt{n}^\star + \mathtt{L}^\star\mathtt{u}_\mathtt{n}^\star}{m+1}, \\
\mathtt{u}_{\mathtt{n}+1}^\star &= \mathtt{L}^\star\mathtt{v}_\mathtt{n}^\star \; n \in \mathbb{N}, \qquad (4.6)
\end{aligned}
$$

for the approximate mapping $\mathtt{L}^\star$ such that $\lim_{n\to\infty} \mathtt{u}_\mathtt{n}^\star = \nu^\star$. Then $\|\nu - \nu^\star\| < \frac{2\epsilon}{(1-\delta)}$.

Proof. Owing to (2.1), (3.1), (4.6) and by definition of an approximate mapping 2.4, we have

$$
\begin{aligned}
\|\mathtt{u}_{\mathtt{n}+1} - \mathtt{u}_{\mathtt{n}+1}^\star\| &= \|\mathtt{L}\mathtt{v}_\mathtt{n} - \mathtt{L}^\star\mathtt{v}_\mathtt{n}^\star\| \\
&\leq \|\mathtt{L}\mathtt{v}_\mathtt{n} - \mathtt{L}\mathtt{v}_\mathtt{n}^\star\| + \|\mathtt{L}\mathtt{v}_\mathtt{n}^\star - \mathtt{L}^\star\mathtt{v}_\mathtt{n}^\star\| \\
&< \phi(\|\mathtt{v}_\mathtt{n} - \mathtt{L}\mathtt{v}_\mathtt{n}\|) + \delta\|\mathtt{v}_\mathtt{n} - \mathtt{v}_\mathtt{n}^\star\| + \epsilon. \qquad (4.7)
\end{aligned}
$$

Also,

$$\begin{aligned}
\|\mathtt{v_n} - \mathtt{v_n^\star}\| &= \Big\|\frac{m\mathtt{u_n} + \mathtt{Lu_n}}{m+1} - \frac{m\mathtt{u_n^\star} + \mathtt{L^\star u_n^\star}}{m+1}\Big\| \\
&\leq \frac{m}{m+1}\|\mathtt{u_n} - \mathtt{u_n^\star}\| + \frac{1}{m+1}\|\mathtt{Lu_n} - \mathtt{L^\star u_n^\star}\| \\
&\leq \frac{m}{m+1}\|\mathtt{u_n} - \mathtt{u_n^\star}\| + \frac{1}{m+1}\Big[\|\mathtt{Lu_n} - \mathtt{Lu_n^\star}\| + \|\mathtt{Lu_n^\star} - \mathtt{L^\star u_n^\star}\|\Big] \\
&\leq \frac{m}{m+1}\|\mathtt{u_n} - \mathtt{u_n^\star}\| + \frac{1}{m+1}\Big[\phi(\|\mathtt{u_n} - \mathtt{Lu_n}\|) + \delta\|\mathtt{u_n} - \mathtt{u_n^\star}\|\Big] + \frac{\epsilon}{m+1} \\
&= \frac{m+\delta}{m+1}\|\mathtt{u_n} - \mathtt{u_n^\star}\| + \frac{1}{m+1}\phi(\|\mathtt{u_n} - \mathtt{Lu_n}\|) + \frac{\epsilon}{m+1}. \qquad (4.8)
\end{aligned}$$

Keeping in view $\frac{m+\delta}{m+1} < 1$, $\frac{\delta}{m+1} < 1$, substituting (4.8) in (4.7), one gets

$$\begin{aligned}
&\|\mathtt{u_{n+1}} - \mathtt{u_{n+1}^\star}\| \\
&\leq \phi(\|\mathtt{v}_n - \mathtt{Lv}_n\|) + \delta\Big[\frac{m+\delta}{m+1}\|\mathtt{u_n} - \mathtt{u_n^\star}\| + \frac{1}{m+1}\phi(\|\mathtt{u_n} - \mathtt{Lu_n}\|) + \frac{\epsilon}{m+1}\Big] + \epsilon \qquad (4.9) \\
&\leq \delta\|\mathtt{u_n} - \mathtt{u_n^\star}\| + \phi(\|\mathtt{v}_n - \mathtt{Lv}_n\|) + \frac{\delta}{m+1}\phi(\|\mathtt{u_n} - \mathtt{Lu_n}\|) + 2\epsilon.
\end{aligned}$$

Writing $\mathfrak{p}_n = \|\mathtt{u}_n - \mathtt{u}_n^*\|$, $\mathfrak{q}_n = (1-\delta) \in (0,1)$, $\mathfrak{r}_n = \frac{\phi(\|\mathtt{v}_n - \mathtt{Lv}_n\|) + \frac{\delta}{m+1}\phi(\|\mathtt{u_n} - \mathtt{Lu_n}\|) + 2\epsilon}{(1-\delta)}$, (4.9) takes the form

$$\mathfrak{p}_{n+1} \leq (1-\mathfrak{q}_n)\mathfrak{p}_n + \mathfrak{r}_n.$$

Therefore, using Lemma 2.5, we get

$$0 \leq \limsup_{n\to\infty} \|\mathtt{u_n} - \mathtt{u_n^\star}\| \leq \limsup_{n\to\infty} \frac{\phi(\|\mathtt{v}_n - \mathtt{Lv}_n\|) + \frac{\delta}{m+1}\phi(\|\mathtt{u_n} - \mathtt{Lu_n}\|) + 2\epsilon}{(1-\delta)} = \frac{2\epsilon}{(1-\delta)}.$$

In view of Theorem 4.1 and above hypothesis, one gets

$$\|\nu - \nu^*\| < \frac{2\epsilon}{(1-\delta)}, \text{for all n} \in \mathbb{N}.$$

This completes the proof. □

Remark 4.4. As the value of $m \to \infty$, an improved upper limit can be obtained. Taking $m \to \infty$ in (4.9), we have

$$\|\mathtt{u_{n+1}} - \mathtt{u_{n+1}^\star}\| \leq \phi(\|\mathtt{v}_n - \mathtt{Lv}_n\|) + \delta\|\mathtt{u_n} - \mathtt{u_n^\star}\| + \epsilon. \qquad (4.10)$$

Writing $\mathfrak{p}_n = \|\mathtt{u}_n - \mathtt{u}_n^*\|$, $\mathfrak{q}_n = (1-\delta) \in (0,1)$, $\mathfrak{r}_n = \frac{\phi(\|\mathtt{v}_n - \mathtt{Lv}_n\|) + \epsilon}{(1-\delta)}$, (4.10) becomes

$$\mathfrak{p}_{n+1} \leq (1-\mathfrak{q}_n)\mathfrak{p}_n + \mathfrak{r}_n.$$

Therefore, using Lemma 2.5, we get

$$0 \leq \limsup_{n\to\infty} \|\mathtt{u_n} - \mathtt{u_n^\star}\| \leq \limsup_{n\to\infty} \frac{\phi(\|\mathtt{v}_n - \mathtt{Lv}_n\|) + \epsilon}{(1-\delta)} = \frac{\epsilon}{(1-\delta)}.$$

In view of Theorem 4.1 and above hypothesis, one gets

$$\|\nu - \nu^*\| < \frac{\epsilon}{(1-\delta)}, \text{for all n} \in \mathbb{N}.$$

5. Discussion on efficiency, time complexity and order of convergence for algorithm (3.1) with numerical example

In this part, we will analyze the rate of convergence of the algorithms listed in the Table 1 and demonstrate that the new fixed point approach is efficient among them in terms of both speed and time.

Theorem 5.1. *Suppose X_b is a Banach space and $\mathrm{A}(\neq \phi)$ is a closed convex subset of X_b. Let $\mathtt{L} : \mathrm{A} \to \mathrm{A}$ be a mapping such that*

(1) $\mathtt{L}$ *satisfies contractive type mapping* (2.1)*,*
(2) $\mathtt{L}$ *has a fixed point* ν*.*

Assume that all the iterative techniques in Table 1 along with algorithm (3.1) converge to same fixed point ν. Then for $\mathtt{u_0} \in \mathrm{A}, m > 0$, the new fixed point technique (3.1) converges faster than the ones listed in Table 1 for the class of mappings agreeing Definition 2.1.

Proof. Using inequality (4.3) of Theorem 4.1, we observe,

$$\|\mathtt{u_{n+1}} - \nu\| \leq \delta^{n+1}\Big(\frac{m+\delta}{m+1}\Big)^{n+1}\|\mathtt{u_0} - \nu\| m \in \mathbb{R}^+, n \in \mathbb{N}\cup\{0\}. \tag{5.1}$$

In the first part, we shall prove that for the class of mapping under consideration, the new fixed point scheme provided by (3.1) converges faster than the Ishikawa scheme [8].
By (2.1), we have

$$\begin{aligned}
\|d_n - \nu\| &= \|(1-\mathfrak{b}_n)f_n + \mathfrak{b}_n\mathtt{L}f_n - \nu\| \\
&\leq (1-\mathfrak{b}_n)\|f_n - \nu\| + \mathfrak{b}_n\|\mathtt{L}\nu - \mathtt{L}f_n\| \\
&\leq (1-\mathfrak{b}_n)\|f_n - \nu\| + \mathfrak{b}_n\Big[\phi(\|\nu - \mathtt{L}\nu\|) + \delta\|f_n - \nu\|\Big] \\
&= \Big[1 - \mathfrak{b}_n(1-\delta)\Big]\|f_n - \nu\|. \qquad (5.2)
\end{aligned}$$

Therefore, we can write

$$\begin{aligned}
\|f_{n+1} - \nu\| &= \|(1-\mathfrak{a}_n)f_n + \mathfrak{a}_n\mathtt{L}d_n - \nu\| \\
&= \|(1-\mathfrak{a}_n)f_n + \mathfrak{a}_n\mathtt{L}d_n - (1-\mathfrak{a}_n+\mathfrak{a}_n)\nu\| \\
&\geq (1-\mathfrak{a}_n)\|f_n - \nu\| - \mathfrak{a}_n\|\mathtt{L}\nu - \mathtt{L}d_n\| \\
&\geq (1-\mathfrak{a}_n)\|f_n - \nu\| - \mathfrak{a}_n\Big[\phi(\|\nu - \mathtt{L}\nu\|) + \delta\|d_n - \nu\|\Big] \\
&= (1-\mathfrak{a}_n)\|f_n - \nu\| - \delta\mathfrak{a}_n\|d_n - \nu\| \\
&\geq (1-\mathfrak{a}_n)\|f_n - \nu\| - \delta\mathfrak{a}_n\Big(1 - \mathfrak{b}_n(1-\delta)\Big)\|f_n - \nu\| \\
&= \Big[1 - \mathfrak{a}_n\Big(1 + \delta(1 - \mathfrak{b}_n(1-\delta))\Big)\Big]\|f_n - \nu\|.
\end{aligned}$$

Repeating the above argument for $n = 0, 1, 2, 3...$, we obtain

$$\begin{aligned} \|f_{n+1} - \nu\| &\geq \prod_{\mathfrak{s}=0}^{n} \Big[1 - \mathfrak{a}_s\Big(1 + \delta(1 - \mathfrak{b}_s(1-\delta)\Big)\Big] \|f_0 - \nu\| \\ &\geq \prod_{\mathfrak{s}=0}^{n} \Big[1 - \mathfrak{a}_s\Big(1 + \delta\Big)\Big] \|f_0 - \nu\|. \end{aligned} \tag{5.3}$$

In view of (5.1) and (5.3), we can write,

$$\frac{\|\mathtt{u}_{\mathtt{n+1}} - \nu\|}{\|f_{n+1} - \nu\|} \leq \frac{\delta^{n+1}\Big(\frac{m+\delta}{m+1}\Big)^{n+1} \|\mathtt{u}_0 - \nu\|}{\prod_{\mathfrak{s}=0}^{\mathfrak{s}=n} \Big[1 - \mathfrak{a}_s\Big(1+\delta\Big)\Big] \|f_0 - \nu\|} = \mathcal{V}_n \text{ (say)}. \tag{5.4}$$

Therefore,

$$\begin{aligned} \frac{\mathcal{V}_{n+1}}{\mathcal{V}_n} &= \frac{\delta^{n+2}\Big(\frac{m+\delta}{m+1}\Big)^{n+2} \prod_{\mathfrak{s}=0}^{n} \Big[1 - \mathfrak{a}_s\Big(1+\delta\Big)\Big]}{\prod_{\mathfrak{s}=0}^{n+1} \Big[1 - \mathfrak{a}_s\Big(1+\delta\Big)\Big] \delta^{n+1}\Big(\frac{m+\delta}{m+1}\Big)^{n+1}} \\ &= \frac{\delta(\frac{m+\delta}{m+1})}{\Big(1 - \mathfrak{a}_{n+1}\Big(1+\delta\Big)\Big)}. \end{aligned}$$

Taking the limit $n \to \infty$, we get

$$\begin{aligned} \lim_{n\to\infty} \frac{\mathcal{V}_{n+1}}{\mathcal{V}_n} &= \lim_{n\to\infty} \frac{\delta(\frac{m+\delta}{m+1})}{\Big(1 - \mathfrak{a}_{n+1}\Big(1+\delta\Big)\Big)} \\ &= \lim_{n\to\infty} \delta\Big(\frac{m+\delta}{m+1}\Big). \end{aligned}$$

Owing to $\delta < 1, \frac{m+\delta}{m+1} < 1$, and by ratio test, $\sum \mathcal{V}_n$ is convergent which implies $\lim_{n\to\infty} \mathcal{V}_n = 0$. Henceforth, new fixed point procedure $\{\mathtt{u}_\mathtt{n}\}$ converges faster than the Ishikawa iterative procedure $\{f_n\}$ for the said class of mapping.

We will now assess the rate of convergence of new algorithm (3.1) with Mann iteration [13].

In view of Definition 2.1, for $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} \|g_{n+1} - \nu\| &= \|(1-\mathfrak{a}_n)g_n + \mathfrak{a}_n \mathrm{L} g_n - \nu\| \\ &\geq (1-\mathfrak{a}_n)\|g_n - \nu\| - \mathfrak{a}_n\|\mathrm{L}\nu - \mathrm{L}g_n\| \\ &\geq (1-\mathfrak{a}_n)\|g_n - \nu\| - \mathfrak{a}_n\Big[\phi(\|\nu - \mathrm{L}\nu\|) + \delta\|g_n - \nu\|\Big] \\ &= (1-\mathfrak{a}_n)\|g_n - \nu\| - \delta\mathfrak{a}_n\|g_n - \nu\| \\ &= \Big(1 - \mathfrak{a}_n\Big(1+\delta)\Big)\|g_n - \nu\| \\ &\geq \prod_{\mathfrak{s}=0}^{n} \Big[1 - \mathfrak{a}_s\Big(1+\delta\Big)\Big] \|g_0 - \nu\|, n = 0, 1, 2, 3, ... \end{aligned} \tag{5.5}$$

On account of (5.1) and (5.5), we can write,

$$\frac{\|\mathtt{u_{n+1}} - \nu\|}{\|g_{n+1} - \nu\|} \le \frac{\delta^{n+1}\Big(\frac{m+\delta}{m+1}\Big)^{n+1}\|\mathtt{u_0} - \nu\|}{\prod_{\mathfrak{s}=0}^{\mathfrak{s}=n}\Big[1 - \mathfrak{a}_s\Big(1+\delta\Big)\Big]\|g_0 - \nu\|} = \mathcal{M}_n(say). \tag{5.6}$$

Therefore,

$$\begin{aligned}
\frac{\mathcal{M}_{n+1}}{\mathcal{M}_n} &= \frac{\delta^{n+2}\Big(\frac{m+\delta}{m+1}\Big)^{n+2}\prod_{\mathfrak{s}=0}^{n}\Big[1 - \mathfrak{a}_s\Big(1+\delta\Big)\Big]}{\prod_{\mathfrak{s}=0}^{n+1}\Big[1 - \mathfrak{a}_s\Big(1+\delta\Big)\Big]\delta^{n+1}\Big(\frac{m+\delta}{m+1}\Big)^{n+1}} \\
&= \frac{\delta(\frac{m+\delta}{m+1})}{\Big(1 - \mathfrak{a}_{n+1}\Big(1+\delta\Big)\Big)}.
\end{aligned}$$

Taking the limit $n \to \infty$, we get

$$\begin{aligned}
\lim_{n\to\infty}\frac{\mathcal{M}_{n+1}}{\mathcal{M}_n} &= \lim_{n\to\infty}\frac{\delta(\frac{m+\delta}{m+1})}{\Big(1 - \mathfrak{a}_{n+1}\Big(1+\delta\Big)\Big)} \\
&= \lim_{n\to\infty}\delta\Big(\frac{m+\delta}{m+1}\Big).
\end{aligned}$$

Because $\delta < 1, \frac{m+\delta}{m+1} < 1$, and by ratio test, $\sum \mathcal{M}_n$ is convergent which implies $\lim_{n\to\infty}\mathcal{M}_n = 0$. Henceforth, new fixed point procedure $\{\mathtt{u_n}\}$ converges faster than the Mann iterative procedure $\{f_n\}$ for Definition 2.1 type mapping.

We will now compare the rate of convergence of algorithm (3.1) and Noor iteration [14].

For $n \in \mathbb{N}\cup\{0\}$,

$$\begin{aligned}
\|q_n - \nu\| &= \|(1-\mathfrak{c}_n)e_n + \mathfrak{c}_n\mathrm{L}e_n - \nu\| \\
&\le (1-\mathfrak{c}_n)\|e_n - \nu\| + \mathfrak{c}_n\|\mathrm{L}e_n - \mathrm{L}\nu\| \\
&\le (1-\mathfrak{c}_n)\|e_n - \nu\| + \mathfrak{c}_n\Big[\phi(\|\nu - \mathrm{L}\nu\|) + \delta\|e_n - \nu\|\Big] \\
&= \Big[1 - \mathfrak{c}_n(1-\delta)\Big]\|e_n - \nu\|. \qquad (5.7)
\end{aligned}$$

Also

$$\begin{aligned}
\|w_n - \nu\| &= \|(1-\mathfrak{b}_n)e_n + \mathfrak{b}_n\mathrm{L}q_n - \nu\| \\
&\le (1-\mathfrak{b}_n)\|e_n - \nu\| + \mathfrak{b}_n\|\mathrm{L}q_n - \mathrm{L}\nu\| \\
&\le (1-\mathfrak{b}_n)\|e_n - \nu\| + \mathfrak{b}_n\Big[\phi(\|\nu - \mathrm{L}\nu\|) + \delta\|q_n - \nu\|\Big] \\
&\le (1-\mathfrak{b}_n)\|e_n - \nu\| + \delta\mathfrak{b}_n\Big[1 - \mathfrak{c}_n(1-\delta)\Big]\|e_n - \nu\| \\
&= \Big[1 - \mathfrak{b}_n\Big(1 - \delta(1 - \mathfrak{c}_n(1-\delta))\Big)\Big]\|e_n - \nu\|. \qquad (5.8)
\end{aligned}$$

Using (5.7) and (5.8), we get

$$\begin{aligned}
\|e_{n+1} - \nu\| &= \|(1-\mathfrak{a}_n)e_n + \mathfrak{a}_n\mathrm{L}w_n - \nu\| \\
&\ge (1-\mathfrak{a}_n)\|e_n - \nu\| - \mathfrak{a}_n\|\mathrm{L}\nu - \mathrm{L}w_n\|
\end{aligned}$$

$$
\begin{aligned}
&\geq (1-\mathfrak{a}_n)\|e_n-\nu\| - \delta\mathfrak{a}_n\|w_n-\nu\| \\
&\geq (1-\mathfrak{a}_n)\|e_n-\nu\| - \delta\mathfrak{a}_n\Big[1-\mathfrak{b}_n\Big(1-\delta(1-\mathfrak{c}_n(1-\delta))\Big)\Big]\|e_n-\nu\| \\
&\geq \Big[1-\mathfrak{a}_n\Big(1+\delta(1-\mathfrak{b}_n(1-\delta(1-\mathfrak{c}_n(1-\delta))))\Big)\Big]\|e_n-\nu\| \\
&\geq \Big(1-(1+\delta)\mathfrak{a}_n\Big)\|e_n-\nu\|.
\end{aligned}
$$

It implies that

$$
\|e_{n+1}-\nu\| \geq \prod_{\mathfrak{s}=0}^{\mathfrak{s}=n}(1-(1+\delta)\mathfrak{a}_\mathfrak{s})\|e_0-\nu\|, n=0,1,2,3,... \tag{5.9}
$$

In view of (5.1) and (5.9), we have

$$
\frac{\|\mathtt{u_{n+1}}-\nu\|}{\|e_{n+1}-\nu\|} \leq \frac{\delta^{n+1}(\frac{m+\delta}{m+1})^{n+1}}{\prod_{\mathfrak{s}=0}^{\mathfrak{s}=n}(1-(1+\delta)\mathfrak{a}_\mathfrak{s})} = \mathcal{E}_n(\text{say}). \tag{5.10}
$$

Therefore,

$$
\lim_{n\to\infty}\frac{\mathcal{E}_{n+1}}{\mathcal{E}_n} = \lim_{n\to\infty}\frac{\delta(\frac{m+\delta}{m+1})}{1-(1+\delta)\mathfrak{a}_{\mathtt{n+1}}} = \delta\Big(\frac{m+\delta}{m+1}\Big) < 1.
$$

By ratio test, we conclude that new iterative procedure (3.1) converges faster than Noor iteration for the class of contractive type mappings defined by (2.1). ☐

Next, we will discuss the order of convergence for a new iterative procedure (3.1).

Theorem 5.2. *Suppose X_b is a Banach space and $\mathrm{A}(\neq \phi)$ is a closed convex subset of X_b. Let $\mathtt{L}:\mathrm{A}\to\mathrm{A}$ be a mapping such that*

(1) $\mathtt{L}$ *satisfies contractive type mapping* (2.1)*,*
(2) $\mathtt{L}$ *has a fixed point ν.*

Then the order of convergence of new iterative algorithm given by (3.1) is

- *at least linear.*
- *second order in case $m=-\mathtt{L}'(\nu)$.*

Proof. Applying Taylor's series expansion on $\mathtt{L}$ about ν and denoting $(\mathtt{u_n}-\nu)$ by $\epsilon_{\mathtt{u_n}}$ we have,

$$
\begin{aligned}
\mathtt{Lu_n} &= \mathtt{L}(\nu)+\mathtt{L}'(\nu)(\mathtt{u_n}-\nu)+\frac{\mathtt{L}''(\nu)}{2!}(\mathtt{u_n}-\nu)^2+O(\mathtt{u_n}-\nu)^3 \\
&= \nu+\mathtt{L}'(\nu)\epsilon_{\mathtt{u_n}}+\frac{\mathtt{L}''(\nu)}{2!}\epsilon_{\mathtt{u_n}}^2+O(\epsilon_{\mathtt{u_n}}^3).
\end{aligned}
$$

Therefore, using the above expansion, we can write

$$
\begin{aligned}
\epsilon_{\mathtt{v_n}} &= \mathtt{v_n}-\nu \\
&= \frac{m\mathtt{u_n}+\mathtt{Lu_n}}{m+1}-\nu \\
&= \frac{m\mathtt{u_n}+\nu+\mathtt{L}'(\nu)\epsilon_{\mathtt{u_n}}+\frac{\mathtt{L}''(\nu)}{2!}\epsilon_{\mathtt{u_n}}^2+O(\epsilon_{\mathtt{u_n}}^3)}{m+1}-\nu
\end{aligned}
$$

$$= \frac{m\mathtt{u_n} + \mathtt{L}'(\nu)\epsilon_{\mathtt{u_n}} + \frac{\mathtt{L}''(\nu)}{2!}\epsilon_{\mathtt{u_n}}^2 + O(\epsilon_{\mathtt{u_n}}^3) - m\nu}{m+1}$$
$$= \frac{m + \mathtt{L}'(\nu)}{m+1}\epsilon_{\mathtt{u_n}} + \frac{\mathtt{L}''(\nu)}{2!(m+1)}\epsilon_{\mathtt{u_n}}^2 + O(\epsilon_{\mathtt{u_n}}^3).$$

Proceeding in a similar pattern and employing the above expansion, we obtain

$$\begin{aligned}
\epsilon_{\mathtt{u_{n+1}}} &= \mathtt{u_{n+1}} - \nu = \mathtt{Lv_n} - \nu \\
&= \mathtt{L}'(\nu)\epsilon_{\mathtt{v_n}} + \frac{\mathtt{L}''(\nu)}{2!}\epsilon_{\mathtt{v_n}}^2 + O(\epsilon_{\mathtt{v_n}}^3) \\
&= \mathtt{L}'(\nu)\Big[\Big(\frac{m + \mathtt{L}'(\nu)}{m+1}\Big)\epsilon_{\mathtt{u_n}} + \frac{\mathtt{L}''(\nu)}{2!(m+1)}\epsilon_{\mathtt{u_n}}^2 + O(\epsilon_{\mathtt{u_n}}^3)\Big] \\
&\quad + \frac{\mathtt{L}''(\nu)}{2!}\Big[\Big(\frac{m + \mathtt{L}'(\nu)}{m+1}\Big)\epsilon_{\mathtt{u_n}} + \frac{\mathtt{L}''(\nu)}{2!(m+1)}\epsilon_{\mathtt{u_n}}^2 + O(\epsilon_{\mathtt{u_n}}^3)\Big]^2 + O(\epsilon_{\mathtt{u_n}}^3) \\
&= \Big(\frac{m + \mathtt{L}'(\nu)}{m+1}\Big)\mathtt{L}'(\nu)\epsilon_{\mathtt{u_n}} + \frac{\mathtt{L}''(\nu)\epsilon_{\mathtt{u_n}}^2}{2!(m+1)^2}\Big[(m+1)\mathtt{L}'(\nu) + (m + \mathtt{L}'(\nu))^2\Big] + O(\epsilon_{\mathtt{u_n}}^3).
\end{aligned}$$

Also, if $m = -\mathtt{L}'(\nu)$, then above expression gives

$$\epsilon_{\mathtt{u_{n+1}}} = \frac{\mathtt{L}''(\nu)\epsilon_{\mathtt{u_n}}^2}{2!(m+1)^2}\Big[(m+1)\mathtt{L}'(\nu) + (m + \mathtt{L}'(\nu))^2\Big] + O(\epsilon_{\mathtt{u_n}}^3).$$

It indicates that the aforementioned iterative procedure is at least linearly convergent, however depending upon the choice of m, the order of convergence becomes 2. □

Now, we'll illustrate to substantiate our assertions.

Example 5.3. Suppose $X_b = \mathbb{R}, \mathrm{A} = [0, 14]$, define $\mathtt{L} : \mathrm{A} \to \mathrm{A}$ by

$$\mathtt{L(u)} = \begin{cases} \frac{\mathtt{u}}{7}, & \mathtt{u} \in [0, 7] \\ \frac{\mathtt{u}}{14}, & \mathtt{u} \in [7, 14]. \end{cases}$$

Here, the fixed point of mapping $\mathtt{L}$ is zero.
Further, define $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(\mathtt{u}) = \frac{\mathtt{u}}{4}$ for all $\mathtt{u} \in [0, \infty)$. It is evident that ϕ fulfils all the requirements specified in Definition 2.1.
Also, for $\mathtt{u} \in [0, 7]$, $\phi(\|\mathtt{u} - \mathtt{Lu}\|) = \phi(\|\mathtt{u} - \frac{\mathtt{u}}{7}\|) = \frac{3\mathtt{u}}{14}$ and $\mathtt{u} \in [7, 14]$, $\phi(\|\mathtt{u} - \mathtt{Lu}\|) = \phi(\|\mathtt{u} - \frac{\mathtt{u}}{14}\|) = \frac{13\mathtt{u}}{56}$
We only need to demonstrate that $\mathtt{L}$ is a contractive type mapping.
Case 1 When $\mathtt{u}, \mathtt{v} \in [0, 7]$,

$$\begin{aligned}
\|\mathtt{Lu} - \mathtt{Lv}\| &= \Big\|\frac{\mathtt{u}}{7} - \frac{\mathtt{v}}{7}\Big\| \\
&= \frac{1}{7}\|\mathtt{u} - \mathtt{v}\| \\
&\le \frac{3\mathtt{u}}{14} + \frac{1}{7}\|\mathtt{u} - \mathtt{v}\| \\
&= \phi(\|\mathtt{u} - \mathtt{Lu}\|) + \delta\|\mathtt{Lu} - \mathtt{Lv}\| \text{ for } \delta \in \Big[\frac{1}{7}, 1\Big).
\end{aligned}$$

Case 2 When $\mathtt{u}, \mathtt{v} \in [7, 14]$,

$$\begin{aligned} \|\mathtt{Lu} - \mathtt{Lv}\| &= \left\| \frac{\mathtt{u}}{14} - \frac{\mathtt{v}}{14} \right\| \\ &= \frac{1}{14}\|\mathtt{u} - \mathtt{v}\| \\ &\leq \frac{13\mathtt{u}}{56} + \frac{1}{7}\|\mathtt{u} - \mathtt{v}\| \\ &= \phi(\|\mathtt{u} - \mathtt{Lu}\|) + \delta\|\mathtt{Lu} - \mathtt{Lv}\|, \text{ for } \delta \in \left[\frac{1}{7}, 1\right). \end{aligned}$$

Case 3 When one of the $\mathtt{u}, \mathtt{v} \in [0, 7]$ while other in $[7, 14]$, say, $\mathtt{u} \in [0, 7]$ and $\mathtt{v} \in [7, 14]$,

$$\begin{aligned} \|\mathtt{Lu} - \mathtt{Lv}\| &= \left\| \frac{\mathtt{u}}{7} - \frac{\mathtt{v}}{14} \right\| \\ &= \| \frac{\mathtt{u}}{14} + \frac{\mathtt{u}}{14} - \frac{\mathtt{v}}{14} \| \\ &\leq \frac{\mathtt{u}}{14} + \frac{1}{14}\|\mathtt{u} - \mathtt{v}\| \\ &\leq \frac{3\mathtt{u}}{14} + \frac{1}{7}\|\mathtt{u} - \mathtt{v}\| \\ &= \phi(\|\mathtt{u} - \mathtt{Lu}\|) + \delta\|\mathtt{Lu} - \mathtt{Lv}\|, \text{ for } \delta \in \left[\frac{1}{7}, 1\right). \end{aligned}$$

Combining, we obtain, $\|\mathtt{Lu} - \mathtt{Lv}\| \leq \phi(\|\mathtt{u} - \mathtt{Lu}\|) + \delta\|\mathtt{Lu} - \mathtt{Lv}\|$ for $\delta \in \left[\frac{1}{7}, 1\right)$, for all $\mathtt{u}, \mathtt{v} \in [0, 14]$.

Hence, $\mathtt{L}$ is a contractive type mapping in the sense of Definition 2.1, with a fixed point at 0. Using MATLAB 2023(a) software, we have demonstrated that our proposed iterative scheme exhibits faster convergence compared to other iterative algorithms. This is achieved by selecting control sequences with specific values: $\mathfrak{a}_n = \frac{n}{n+2}, \mathfrak{b}_n = \frac{3}{(n+1)^2}, \mathfrak{c}_n = \frac{14n}{n^5+8}, m = 0.1$ and prescribed tolerance as 0.0001. We start the iteration with an initial guess value of $\mathtt{u} = 10$ (refer to Figure 1 and Table 2 for further details).

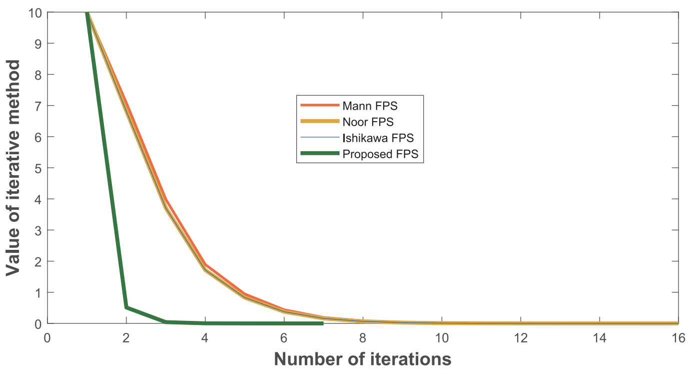


FIGURE 1. **Graphical representation of rate of convergence of various iterative schemes under study.**

Iteration.No.	Proposed FPS	Ishikawa FPS	Noor FPS	Mann FPS
1.	0.5114	6.8099	6.8186	7.0833
2.	0.0407	3.7064	3.7072	3.9844
3.	0.0032	1.7149	1.7143	1.8926
4.	0.0002	0.8317	0.8311	0.9463
5.	0.0000	0.3769	0.3765	0.4393
6.	0.0000	0.1616	0.1615	0.1922
7.	0.0000	0.0662	0.0662	0.0801
8.	0.0000	0.0261	0.0261	0.0320
9.	0.0000	0.0100	0.0100	0.0124
10.	0.0000	0.0037	0.0037	0.0046
11.	0.0000	0.0013	0.0013	0.0017
12.	0.0000	0.0005	0.0005	0.0006
13.	0.0000	0.0002	0.0002	0.0002
14.	0.0000	0.0001	0.0001	0.0001
15.	0.0000	0.0000	0.0000	0.0000

Table 2. Tabular comparison of various iterative methods in the context of Example 5.3.

In addition, we establish the contrast for absolute and relative error, time complexity, number of iterations and order of convergence of different iterative algorithms (see Table 3).

	Proposed FPS	Ishikawa FPS	Noor FPS	Mann FPS
Absolute Error	1.8845e-05	3.7414e-05	3.7376e-05	4.8225e-05
Relative Error	0.09205	0.6637	0.6637	0.6618
Order of Convergence	1	1.0175	1.0175	1.0185
No.of Iterations	5	15	15	15
Time involved(in seconds)	0.978224	1.062799	1.03538	1.070094

Table 3. Comparison in terms of error and time in the context of Example 5.3.

6. Application to polynomiograph

Polynomiography unveils the aesthetic allure and artistic essence inherent in the process of solving polynomial equations. It has transformed the arduous and complex work of root finding problem of polynomials into a captivating and aesthetically pleasing form of artistic expression, design, invention, creativity, scientific exploration, and educational engagement. To explore the theoretical foundation and artistic implementation of polynomiography, see [9]. Polynomiography uses iteration methods to approximate polynomial roots, such as Newton's and Halley's methods. We employ the suggested iteration strategy instead of Picard iteration to modify the Newton technique to generate polynomiographs.

The Newton technique for determining the roots is given by

$$\mathtt{z}_{n+1} = \mathtt{z}_n - \frac{\mathtt{q}(\mathtt{z_n})}{\mathtt{q}'(\mathtt{z_n})}, n \in \mathbb{N} \cup \{0\}, \tag{6.1}$$

where $\mathtt{z}_0$ represents the initial point for the approximation of root of complex polynomial $\mathtt{q}(\mathtt{z})$. Modifying (6.1) using the suggested algorithm (3.1) , we obtain

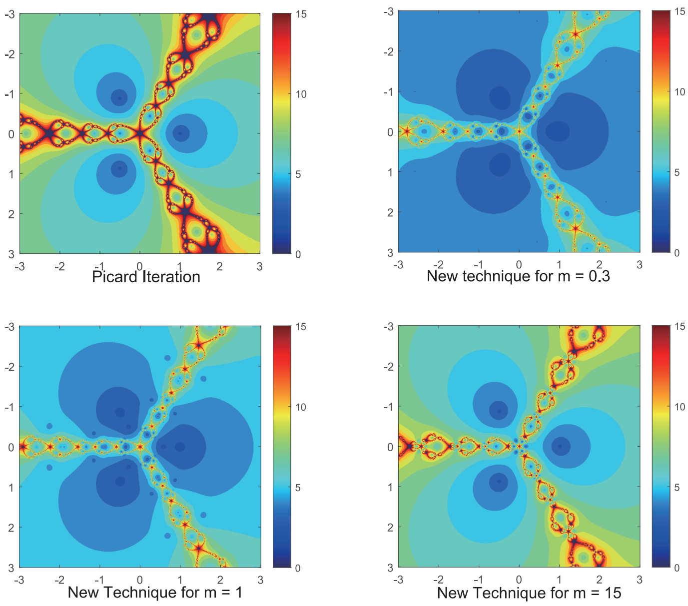
$$\begin{aligned} \mathtt{z}_{\mathtt{n}+1} &= \mathtt{v}_n - \frac{\mathtt{q}(\mathtt{v}_\mathtt{n})}{\mathtt{q}'(\mathtt{v}_\mathtt{n})} \\ \mathtt{v}_\mathtt{n} &= \frac{m\mathtt{z}_\mathtt{n} + (\mathtt{z}_n - \frac{\mathtt{q}(\mathtt{z}_\mathtt{n})}{\mathtt{q}'(\mathtt{z}_\mathtt{n})})}{m+1}. \end{aligned} \tag{6.2}$$

The sequence $\{\mathtt{z}_n\}$ is referred to as the orbit of the point $\mathtt{z}_0$ converging to a root $\mathtt{z}^\#$ of $\mathtt{q}$, then we say that $\mathtt{z}_0$ is attracted to $\mathtt{z}^\#$. The set of all such initial points for which the sequence $\{\mathtt{z}_n\}$ converges to root $\mathtt{z}^\#$ is termed as basin of attraction of $\mathtt{z}^\#$. Therefore, in view of Fundamdntal theorem of Algebra, the degree of polynomial determines the number of basins of attraction.

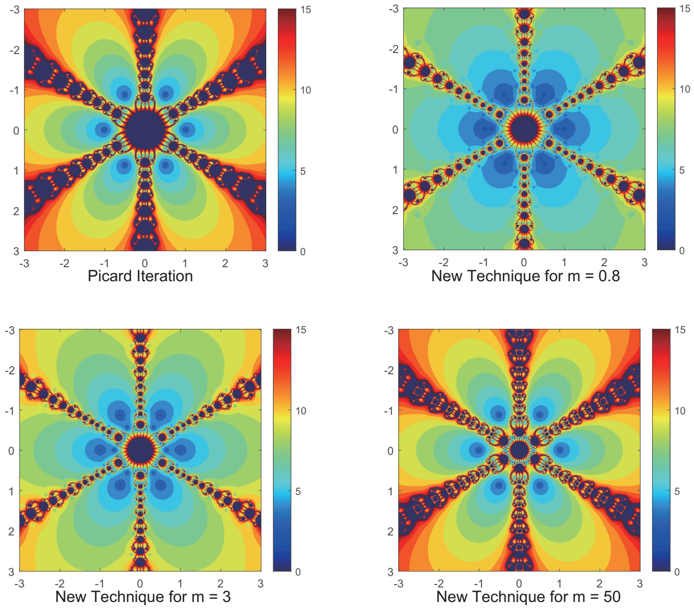
We apply (6.2) over complex Banach space with initial guess $\mathtt{u}_0 = (\mathtt{x}_0, \mathtt{y}_0)$ to generate polynomiographs for different complex polynomials. The various colors of an image are determined by the number of iterations performed to get a root with a specified precision of $\epsilon = 0.00001$. By varying the number of iterations and parameter m, one may generate an endless number of aesthetically pleasing polynomiographs. However, for the sake of this study, we have set the maximum number of iterations to be fixed at 15 with escape criterion $|\mathtt{z}_{\mathtt{n}+1} - \mathtt{z}_\mathtt{n}| < \epsilon$.

Below are the polynomiographs of various complex polynomial equations $\mathtt{q}(\mathtt{z}) = 0$ and few special polynomials. Resolution for each of the images is 800 by 800 pixels. It can be easily observed that the basins of attraction obtained by the suggested technique are entirely different from the orbits of standard Picard iteration.

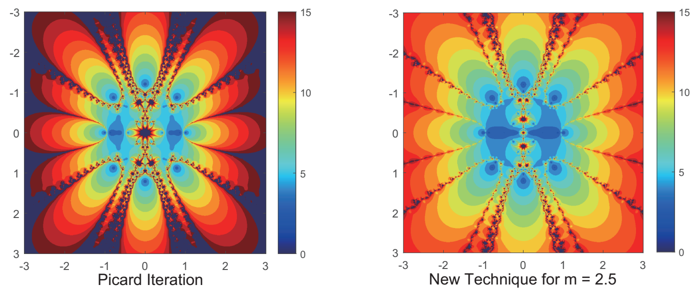
(1) *Polynomiograph of* $\mathtt{z}^3 - 1$



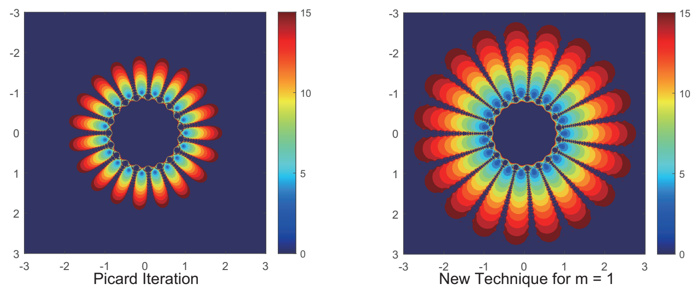
(2) *Polynomiograph of* $z^6 - 1$



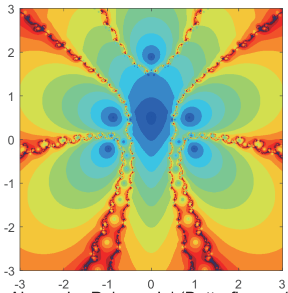
(3) *Polynomiograph of* $z^{10} - 3z^2 + 2$



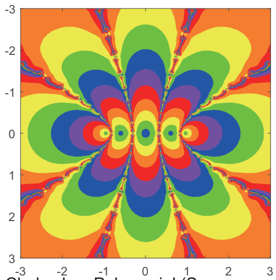
(4) *Polynomiograph of* $z^{19} - 1$



(5) *Polynomiograph of Special Polynomials*



Alexander Polynomial (Butterfly, m=10)



Chebyshev Polynomial (Sparrow, m=10)

7. Conclusion

The article presents a suggestion for Picard-Kanwar hybrid two-step fixed point procedure [10], based on a single parameter and has a wider interval of convergence, which sets it apart from conventional fixed point procedure. We demonstrated that our novel technique exhibits strong convergence towards the invariant points of contractive type mappings, as defined by Imoru and Olantinwo [7]. After discussing the data dependency of fixed points, we proceeded to illustrate the stability of our technique. We have demonstrated by analytical, numerical, and graphical methods that the new algorithm has faster rate of convergence compared to the previous important algorithms through MATLAB 2023(a) software. We have additionally proven that the novel technique exhibits at least linear convergence, which can potentially reach second order depending on the choice of parameter m. The efficiency and superiority of new fixed point technique is also established by taking into account, the elapsed time, absolute error, relative error, number of iterations, and order of convergence using a relevant example. Finally, we showcased visually appealing aesthetic patterns called Polynomiographs, which were generated by utilizing our technique to determine the roots of various complex polynomials.

Few Open Problems:

(1) Is it possible to define an iterative technique with indeed larger interval of convergence?
(2) What will be the convergence behavior of the new technique when dealing with non-unique fixed points?
(3) Is it possible to discuss the convergence of the suggested technique for a different class of mappings, such as nonexpansive, quasi nonexpansive or generalized nonexpansive mappings?

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