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AN EFFECTIVE FIXED POINT APPROACH BASED ON GREEN'S FUNCTION FOR SOLVING BVPS

BARAKAH ALMARRI, MOHAMMAD SAIF, MOHAMMAD AKRAM*, AND IZHAR UDDIN

Abstract. In this paper, we study the convergence behavior of the Abbas et al. iterative scheme for Zamfirescu mapping in an arbitrary Banach space. Also, we construct a numerical example to ensure a better rate of convergence. Further, we apply this method to find the approximate solution of nonlinear boundary value problems (BVPs). To demonstrate the validity, applicability, and high efficiency of the proposed iterative method, an illustrative numerical example is provided. The results of this paper generalize and extend the corresponding results in the literature.

1. Introduction

Fixed point theory provides essential tools for solving various types of nonlinear problems arising in modern mathematics. It is used to prove the existence and uniqueness of solutions for differential equations, including BVPs. The Banach contraction principle [6] is a fundamental result in metric fixed point theory, which states that a contraction mapping defined on a complete metric space has a unique fixed point. The important feature of the Banach contraction principle is that it gives existence and uniqueness, which can be approximated by Picard's iteration process [17]. It has been observed that the Banach contraction principle does not hold for nonexpansive mappings, and Picard's iteration process need not converge for nonexpansive mappings in a complete metric space. Therefore, Mann [15], Halpern [10], and Ishikawa [11] are basic iterative methods that have been studied to approximate fixed points of nonexpansive mappings. Following this, several authors studied various iteration processes to approximate the fixed points of different classes of nonlinear mappings, like Agarwal [3], Khan [13], Thakur et al. [20] and many others. A mapping $\mathcal{P} : \mathcal{K} \to \mathcal{K}$, where $\mathcal{K}$ is a nonempty closed convex subset of a Banach space $\mathcal{X}$. Agarwal et al. [3] introduced the S-iterative process as follows:

$$\begin{cases} c_0 \in \mathcal{K}, \\ d_n = (1-\beta_n)c_n + \beta_n \mathcal{P} c_n \\ c_{n+1} = (1-\alpha_n)\mathcal{P} c_n + \alpha_n \mathcal{P} d_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.1}$$

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*Corresponding author.

Khan [13] introduced the Picard-Mann hybrid iterative process as follows:

$$\begin{cases} c_0 \in \mathcal{K}, \\ d_n = (1-\alpha_n)c_n + \alpha_n \mathcal{P}c_n \\ c_{n+1} = \mathcal{P}d_n \quad n \in \mathbb{N}, \end{cases} \tag{1.2}$$

Recently, Abbas et al. [1] introduced a three-step iteration process as follows:

$$\begin{cases} c_0 \in \mathcal{K}, \\ e_n = \mathcal{P}c_n, \\ d_n = \mathcal{P}e_n, \\ c_{n+1} = (1-\alpha_n)d_n + \alpha_n \mathcal{P}d_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $(0,1)$.
The purpose of this paper is to show a strong convergence result for the Zamfirescu operator using the iterative method (1.3) in arbitrary Banach spaces. We also show the rate of convergence with the help of an example. Further, we find the approximate solution of a second order boundary value problem and show that the newly constructed algorithm is highly efficient and more exact than the results in the literature [4, 5, 14, 18, 21].

2. Preliminaries

In this section, we collect some definitions from Berinde [7] that will be used in our main result.

Definition 2.1. Let $\{\rho_n\}$ and $\{\sigma_n\}$ be the sequences in positive real numbers, $\rho_n \to \rho$ and $\sigma_n \to \sigma$. Suppose that

$$l = \lim_{n\to\infty} \frac{|\rho_n - \rho|}{|\sigma_n - \sigma|}$$

Then, we say that

(1) $\{\rho_n\}$ converges faster than $\{\sigma_n\}$ if $l = 0$.
(2) $\{\sigma_n\}$ and $\{\rho_n\}$ have the same convergence rate if $0 < l < \infty$.

Definition 2.2. Let $\{\theta_n\}$ and $\{\omega_n\}$ be two fixed point iterative methods converging to same fixed point p^*. Let $\{\rho_n\}$ and $\{\sigma_n\}$ be sequences in positive real numbers converging to 0, such that

$$\|\theta_n - p^*\| \leq \rho_n,$$
$$\|\omega_n - p^*\| \leq \sigma_n,$$

for all $n \geq 1$. If $\lim_{n\to\infty} \frac{\rho_n}{\sigma_n} = 0$, then $\{\theta_n\}$ converges faster than $\{\omega_n\}$.
As a background to our exposition, we now mention some contractive mappings.
A mapping $\mathcal{P} : \mathcal{K} \to \mathcal{K}$ is said to be a:

(z_1) contraction if there exists a constant $\delta \in [0,1)$ such that

$$\|\mathcal{P}x - \mathcal{P}y\| \leq \delta\|x - y\|, \quad \forall\ x, y \in \mathcal{P}. \tag{2.1}$$

(z_2) Kannan map [12] if there exists a constant $b \in (0, \frac{1}{2}))$ such that

$$\|\mathcal{P}x - \mathcal{P}y\| \leq b(\|x - \mathcal{P}x\| + \|y - \mathcal{P}y\|), \quad \forall\ x, y \in \mathcal{P}. \tag{2.2}$$

(z_3) Chatterjea map [8] if there exists a constant $c \in (0, \frac{1}{2})$ such that

$$\|\mathcal{P}x - \mathcal{P}y\| \leq b(\|x - \mathcal{P}y\| + \|y - \mathcal{P}x\|), \quad \forall\ x, y \in \mathcal{P}. \tag{2.3}$$

In 1972, Zamfirescu [22] defined a very interesting fixed point theorem by combining contraction, Kannan [12], and Chatterjea [8] mappings.

Definition 2.3. An operator $\mathcal{P} : \mathcal{K} \to \mathcal{K}$ is said to be a Zamfirescu operator if it satisfies at least one of the conditions $(z_1), (z_2)$ and (z_3).

3. Convergence analysis with numerical example

In this section, we prove a convergence result for the Zamfirescu operator using the iteration process (1.3) in an arbitrary Banach space.

Theorem 3.1. *Let $\mathcal{P} : \mathcal{K} \to \mathcal{K}$, where $\mathcal{K}$ be a non-empty closed convex subset of a arbitrary Banach space $\mathcal{X}$ and $\mathcal{P}$ be a Zamfirescu operator. If $\{\alpha_n\}$ is sequence in $(0, 1)$ and $\{c_n\}$ be the sequence defined by (1.3) and $c_0 \in \mathcal{K}$. Then, $\{c_n\}$ converges strongly to a fixed point of $\mathcal{P}$.*

Proof. We know that $\mathcal{P}$ has a unique fixed point in $\mathcal{X}$. Let $\{c_n\}$ be the sequence defined by (1.3) and $c_0 \in \mathcal{K}$ arbitrary and $x, y \in \mathcal{K}$. At least one of conditions from $(z_1), (z_2), (z_3)$ is satisfied. If (z_2) holds, then

$$\begin{aligned} \|\mathcal{P}x - \mathcal{P}y\| &\leq b[\|x - \mathcal{P}x\| + \|y - \mathcal{P}y\|], \\ &\leq b\big[\|x - \mathcal{P}x\| + \|y - x\| + \|x - \mathcal{P}x\| + \|\mathcal{P}x - \mathcal{P}y\|\big]. \end{aligned}$$

So,

$$(1 - b)\|\mathcal{P}x - \mathcal{P}y\| \leq b\|x - y\| + 2b\|x - \mathcal{P}x\|.$$

Since $0 \leq b < \frac{1}{2}$, we get

$$\|\mathcal{P}x - \mathcal{P}y\| \leq \frac{b}{(1 - b)}\|x - y\| + \frac{b}{(1 - b)}\|x - \mathcal{P}x\|. \tag{3.1}$$

Similarly if (z_3) holds, we get

$$\|\mathcal{P}x - \mathcal{P}y\| \leq \frac{c}{(1 - c)}\|x - y\| + \frac{c}{(1 - c)}\|x - \mathcal{P}x\|. \tag{3.2}$$

Let

$$\delta = max\Big\{a, \frac{b}{(1 - b)}, \frac{c}{(1 - c)}\Big\} \tag{3.3}$$

Then, we have $0 \leq \delta < 1$ and from (3.1), (3.2) and (3.3) we get the following inequality

$$\|\mathcal{P}x - \mathcal{P}y\| \leq \delta\|x - y\| + 2\delta\|x - \mathcal{P}x\| \quad \forall x, y \in \mathcal{K}. \tag{3.4}$$

Let $\{c_n\}$ be the sequence defined by (1.3) and $c_0 \in \mathcal{P}$ arbitrary. Then

$$\begin{aligned} \|e_n - c^*\| &= \|\mathcal{P}c_n - c^*\| = \|\mathcal{P}c^* - \mathcal{P}c_n\|, \\ &\leq \delta\|c^* - c_n\| + 2\delta\|p^* - \mathcal{P}c^*\| = \delta\|c_n - c^*\|. \end{aligned} \tag{3.5}$$

Further,

$$\|d_n - c^*\| = \|\mathcal{P}e_n - c^*\| \leq \delta\|e_n - c^*\|,$$

$$\leq \delta^2\|c_n - c^*\|. \tag{3.6}$$

Now,

$$\begin{aligned}
\|c_{n+1} - c^*\| &= \|(1-\alpha_n)d_n + \alpha_n \mathcal{P}d_n - c^*\|,\\
&\leq (1-\alpha_n)\|d_n - c^*\| + \alpha_n\|\mathcal{P}d_n - c^*\|,\\
&\leq (1-(1-\delta)\alpha_n)\|d_n - c^*\|, \qquad (3.7)
\end{aligned}$$

using (3.6) in (3.7), we get

$$\|c_{n+1} - c^*\| \leq \delta^2(1-(1-\delta)\alpha_n)\|c_n - c^*\|.$$

Since $0 < \delta < 1$ and $\{\alpha_n\} \in (0,1)$, then using fact $(1-(1-\delta)\alpha_n) < 1$, we get

$$\|c_{n+1} - c^*\| \leq \delta^2\|c_n - c^*\|.$$

Inductively, we get

$$\|c_{n+1} - c^*\| \leq \delta^{2(n+1)}\|c_0 - c^*\| \longrightarrow 0 \ as \ n \longrightarrow \infty.$$

Hence, $\{c_n\}$ converges strongly to c^*. □

Now, we construct the following example to compare (numerically) the rate of convergence of different iterative schemes with iterative scheme (1.3) for the Zamfirescu operator.

Example 3.2. Let $\mathcal{X} = \mathbb{R}$ be a Banach space and $\mathcal{K} = [0,2] \subset \mathbb{R}$. Define a mapping $\mathcal{P} : \mathcal{K} \to \mathcal{K}$ such that $\mathcal{P}x = \frac{sin(x)}{2}$, for all $x \in \mathcal{K}$. It can easily verify that $\mathcal{P}$ is a Zamfirescu operator.

Now, using $\mathcal{P}$, we demonstrate that the iterative algorithm (1.3) achieves a faster convergence rate compared to other iterative methods [2, 3, 11, 13, 15]. By taking parameters $\alpha_n = \beta_n = \frac{3}{4}$ for all $n \in \mathbb{N}$ and initial $c_0 = 1.2$. We obtain Table 1 and Table 2 of iteration values and Fig 1, which clearly demonstrate that the scheme (1.3) has the fastest convergence rate.

4. Application to BVPs

In this section, we approximate the solution of a second order boundary value problem. Now, consider the following second order nonlinear BVP:

$$\begin{cases} L[u] \equiv u''(t) = \psi(t, u(t), u'(t)), \\ u(0) = A, \ u(1) = B, \end{cases} \tag{4.1}$$

where $L(u)$ is a linear term and $\psi(t, u, u')$ is a nonlinear term. The existence and uniqueness result for the solution of problems (4.1) can be found in [9,16,19]. $\mathcal{G}(t,s)$ is Green's function corresponding to the linear term $L[u]$, satisfying homogeneous boundary conditions, defined by

$$\mathcal{G}(t,s) = \begin{cases} s(1-t), \ 0 \leq s < t, \\ t(1-s), \ t \leq s \leq 1. \end{cases} \tag{4.2}$$

It is well known that to solve the boundary value problem (4.1) is equivalent to solve the following integral equation:

Step	Picard	Mann	Ishikawa
1	1.200000000000000	1.200000000000000	1.200000000000000
2	0.466019542983613	0.649514657237710	0.526799985361230
3	0.224666928866521	0.389178649670657	0.249747406413433
4	0.111390836992146	0.239580392145287	0.120374425235551
5	0.055580312561007	0.148880732212207	0.058239019689571
.	.	.	.
.	.	.	.
.	.	.	.
12	0.000433923284305	0.005526170799849	0.000364193691092
13	0.000216961635344	0.003453846202339	0.000176406317253
.	.	.	.
.	.	.	.
.	.	.	.
18	0.000006780051034	0.000329383802665	0.000004703517144
19	0.000003390025517	0.000205864874432	0.000002278266117
20	0.000001695012758	0.000128665545975	0.000001103535150

TABLE 1. Iterative values for different iteration process of Example 3.1.

Step	Agarwal	Khan	Abbas et al.
1	1.200000000000000	1.200000000000000	1.200000000000000
2	0.343304871107133	0.302399980481640	0.139709859960740
3	0.120999854165697	0.093093352251751	0.021738512332836
4	0.043379046151442	0.029050102945964	0.003396298143930
5	0.015584486505938	0.009076892452300	0.000530670271481
.	.	.	.
.	.	.	.
.	.	.	.
12	0.000012064448341	0.000002641683566	0.000000001206604
13	0.000004335661123	0.000000825526115	0.000000000188532
.	.	.	.
.	.	.	.
.	.	.	.
18	0.000000025989303	0.000000002460260	0.000000000000018
19	0.000000009339906	0.000000000768831	0.000000000000003
20	0.000000003356529	0.000000000240260	0.000000000000000

TABLE 2. Iterative values for different iteration process of Example 3.1.

$$u(t) = (B - A)t + A - \int_0^1 \mathcal{G}(t,s)\psi(s, u(s), u'(s))ds. \tag{4.3}$$

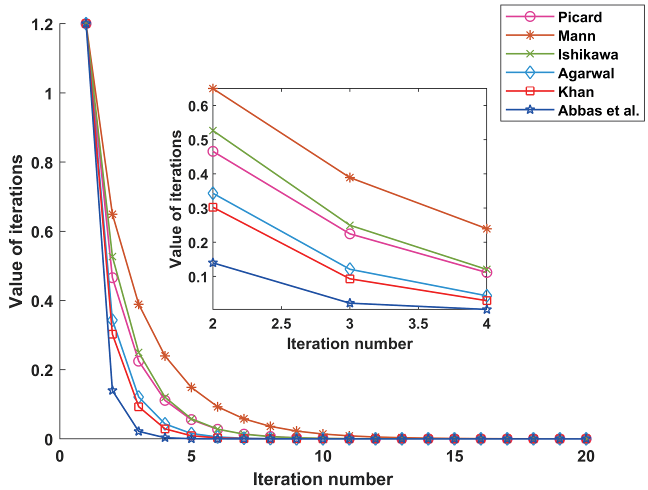


FIGURE 1. Convergence behaviour of different iterative process of Example 3.1.

$u_0 = (B - A)t + A$ satisfies the homogeneous linear differential equation $u'' = 0$ and corresponding boundary conditions (4.1). Now, we define the integral operator, $\mathcal{P} : C[0,1] \to C[0,1]$

$$\mathcal{P}(u) = u(t) + \int_0^1 \mathcal{G}(t,s)(u''(s) - \psi(s,u(s),u'(s)))ds. \tag{4.4}$$

Taking integration by part in (4.4), we obtain

$$\mathcal{P}(u) = (B - A)t + A - \int_0^1 \mathcal{G}(t,s)\psi(s,u(s),u'(s))ds. \tag{4.5}$$

Hence, the fixed point of $\mathcal{P}$, i.e., $\mathcal{P}(u) = u$, is the solution of integral Equation (4.3) (BVP (4.1)). Now, using $\mathcal{P}$ in the iterative method (1.3), we get

$$\begin{cases} e_n = \mathcal{P}(c_n), \\ d_n = \mathcal{P}(e_n), \\ c_{n+1} = (1-\alpha_n)d_n + \alpha_n \mathcal{P}(d_n), \quad n \in \mathbb{N}. \end{cases} \tag{4.6}$$

Now, we prove the following main result of this section.

Theorem 4.1. *Let ψ be a continuous function, which appear in integral operator $\mathcal{P}$ defined in (4.4). Assume that*

$$\frac{1}{\sqrt{48}} \times \sup_{[0,1]\times\mathbb{R}^3} \left|\frac{\partial \psi}{\partial u}\right| < 1, \tag{4.7}$$

$\forall\ t \in [0,1]$, $\forall\ u \in C[0,1]$. *Then, $\mathcal{P}$ is Zamfirescu operator hence, the sequence defined by (4.6) converges to a solution $u \in C[0,1]$ of the problem (4.1).*

Proof. Observe that $u \in C[0,1]$ is a solution of problem (4.1) if and only if $u \in C[0,1]$ is a solution of the integral equation

$$u(t) = (B-A)t + A - \int_0^1 \mathcal{G}(t,s)\psi(s,u(s),u'(s))ds.$$

Now, let $u, v \in C[0,1]$ and $\forall\ t \in [0,1]$.

$$\begin{aligned}|\mathcal{P}u(t) - \mathcal{P}v(t)| &= |\int_0^1 \mathcal{G}(t,s)\psi(s,v(s),v'(s))ds - \int_0^1 \mathcal{G}(t,s)\psi(s,u(s),u'(s))ds| \\ &= |\int_0^1 \mathcal{G}(t,s)\left(\psi(s,v(s),v'(s)) - \psi(s,u(s),u'(s))ds\right)| \\ &\le \left(\int_0^1 |\mathcal{G}(t,s)|^2 ds\right)^{\frac{1}{2}} \left(\int_0^1 |\psi(s,v(s),v'(s)) - \psi(s,u(s),u'(s))|^2 ds\right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{3}}(1-t)t\left(\int_0^1 |\psi(s,v(s),v'(s)) - \psi(s,u(s),u'(s))|^2 ds\right)^{\frac{1}{2}} \\ &\le \frac{1}{\sqrt{48}}\left(\int_0^1 |\psi(s,v(s),v'(s)) - \psi(s,u(s),u'(s))|^2 ds\right)^{\frac{1}{2}}\end{aligned}$$

Applying the mean value theorem to the function ψ, we obtain

$$\le \quad \frac{1}{\sqrt{48}} \times \sup_{[0,1]\times\mathbb{R}^3} |\frac{\partial\psi}{\partial u}| \times \sup_{[0,1]} |v(t) - u(t)|.$$

Therefore,

$$\|\mathcal{P}(u) - \mathcal{P}(v)\| \quad \le \quad \delta\|v-u\|,$$

where $\delta = \frac{1}{\sqrt{48}} \times \sup_{[0,1]\times\mathbb{R}^3} |\frac{\partial\psi}{\partial u}| < 1$. Thus, $\mathcal{P}$ is contraction. Hence, iterative scheme defined by (4.6) converges to the solution of (4.1). □

Examples 4.1. Consider the second order nonlinear BVP as follows:

$$u''(t) = -u(t)(1-u(t)) - t^4 - t^2 + 2, \tag{4.8}$$

where $0 \le t \le 1$ and subject to

$$u(0) = 1,\ u(1) = 2. \tag{4.9}$$

$u_0 = t + 1 \in C[0,1]$ is the initial guess that satisfies $u'' = 0$ and BCs (4.9) The exact solution is $u(t) = t^2 + 1$. We taking values of the sequences $\alpha_n = \beta_n = \frac{3}{4}$, for all $n \in \mathbb{N}$. We obtain Table 3 and Table 4, in which it is shown that the Abbas et al.-Green approach exhibits both high accuracy and high effectiveness, as observed from the results in the lierature [4, 5, 14, 21].

$$\begin{aligned}e_n &= c_n + \int_0^t s(1-t)\left[c_n''(s) + c_n(s)(1-c_n(s)) + s^4 + s^2 - 2\right] ds \\ &\quad + \int_t^1 t(1-s)\left[c_n''(s) + c_n(s)(1-c_n(s)) + s^4 + s^2 - 2\right] ds, \\ d_n &= r_n + \int_0^t s(1-t)\left[e_n''(s) + e_n(s)(1-e_n(s)) + s^4 + s^2 - 2\right] ds\end{aligned}$$

$$
\begin{aligned}
& +\int_t^1 t(1-s)\left[e_n''(s)+e_n(s)(1-e_n(s))+s^4+s^2-2\right] ds, \\
+c_{n+1} \quad = \quad & (1-\alpha_n)d_n+\int_0^t s(1-t)\left[d_n''(s)+d_n(s)(1-d_n(s))+s^4+s^2-2\right] ds \\
& +\alpha_n d_n+\int_t^1 t(1-s)\left[d_n''(s)+d_n(s)(1-d_n(s))+s^4+s^2-2\right] ds.
\end{aligned}
$$

TABLE 3. The absolute errors for the 3rd iteration of Example 4.1 are compared with different iterative methods.

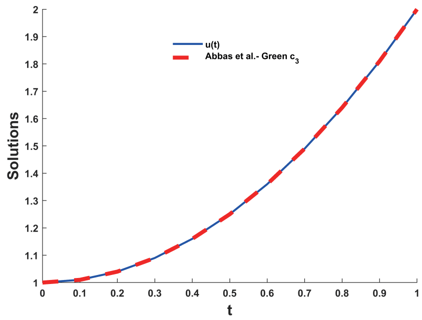
t	u(t)	Picard-Green	Mann-Green	Ishikawa-Green
0.0	1.0	0.0	0.0	0.0
0.1	1.01	3.193313×10^{-4}	3.954429×10^{-4}	1.211811×10^{-3}
0.2	1.04	6.182624×10^{-4}	5.649865×10^{-4}	2.129198×10^{-3}
0.3	1.09	8.754274×10^{-4}	5.842919×10^{-4}	2.766515×10^{-3}
0.4	1.16	1.068170×10^{-3}	5.187407×10^{-4}	3.135293×10^{-3}
0.5	1.25	1.173365×10^{-3}	4.225513×10^{-4}	3.244795×10^{-3}
0.6	1.16	1.169867×10^{-3}	3.356147×10^{-4}	3.101900×10^{-3}
0.7	1.49	1.042877×10^{-3}	2.782957×10^{-4}	2.710444×10^{-3}
0.8	1.64	7.900476×10^{-4}	2.437190×10^{-4}	2.069884×10^{-3}
0.9	1.81	4.283933×10^{-4}	1.858549×10^{-4}	1.172776×10^{-3}
1.0	2.0	0.0	0.0	0.0

TABLE 4. The absolute errors for the 3rd iteration of Example 4.1 are compared with different iterative methods.

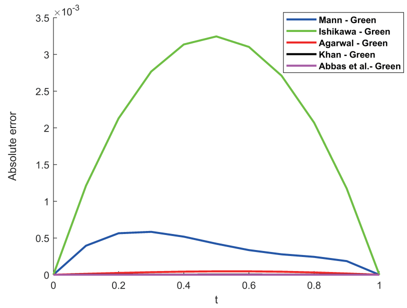
t	Agarwal-Green	Khan-Green	Abbas et al.-Green
0.0	0.0	0.0	0.0
0.1	1.273466×10^{-5}	5.515071×10^{-7}	2.851889×10^{-9}
0.2	2.464973×10^{-5}	1.066659×10^{-6}	5.522761×10^{-9}
0.3	3.489011×10^{-5}	1.507853×10^{-6}	7.822956×10^{-9}
0.4	4.255281×10^{-5}	1.835969×10^{-6}	9.550644×10^{-9}
0.5	4.672004×10^{-5}	2.011958×10^{-6}	1.049855×10^{-8}
0.6	4.655673×10^{-5}	2.001037×10^{-6}	1.047552×10^{-8}
0.7	4.148291×10^{-5}	1.779787×10^{-6}	9.345895×10^{-9}
0.8	3.141354×10^{-5}	1.345853×10^{-6}	7.085198×10^{-9}
0.9	1.702919×10^{-5}	7.289156×10^{-7}	3.843856×10^{-9}
1.0	0.0	0.0	0.0

5. CONCLUSION

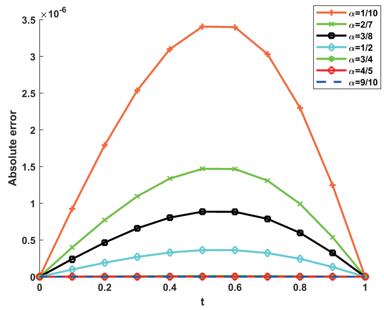
In this paper, the convergence behaviour of the iterative scheme (1.3) has been performed for the Zamfirescu operator in an arbitary Banach space. Further, the



(A) Plot of $u(t)$ and Abbas et al.-Green's for $n = 3$ of Example 4.1.



(B) Behaviour of absolute errors of different number iterative methods for $n = 3$ of Example 4.1.



(C) Behaviour of absolute errors of Abbas et al.-Green's for $n = 3$ of Example 4.1.

iterative scheme (1.3) has been successfully applied to the solution of nonlinear BVPs. This iterative strategy is based on a modified integral operator expressed in terms of Green's function in the Abbas et al. iteration process (1.3). Moreover, we compared our result with numerical solutions obtained by different iterative methods existing in the literature. Our method outperforms other methods and resulted in very high accuracy. The proposed method is a very powerful mathematical tool for finding the solutions to BVPs.

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Barakah Almarri
Department of General Studies, Jubail Industrial College, 8244, Rd Number 6, Al Huwaylat, Al Jubail 35718, Saudi Arabia
E-mail address: marribj@rcjy.edu.sa

Mohammad Saif
Department of Mathematics, Jamia Millia Islamia, New Delhi-110025, India
E-mail address: saifsrd1997@gmail.com

M. Akram
Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah 42351, Saudi Arabia
E-mail address: akramkhan20@rediffmail.com

Izhar Uddin
Department of Mathematics, Jamia Millia Islamia, New Delhi-110025, India.
E-mail address: izharuddin1@jmi.ac.in