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A NEW ITERATIVE SCHEME FOR ALTERING POINTS PROBLEM WITH APPLICATIONS

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Abstract. In this paper, we introduce a new parallel iterative scheme and employ the same to investigate an altering points problem. Some consequent results are also discussed. The obtained results extend and generalize some relevant results of the existing literature. The usefulness and efficiency of our scheme is illustrated using numerical examples.

1. Introduction

The theory of variational inequalities (VIs) is an illustrious research field which has been implemented and enriched diverse fields of science and engineering, such as gauge field theory in particle physics, mathematical biology, the general theory of relativity, etc. This theory has been made systematic advancement in the discipline of mathematics and engineering. It provides tools for dealing with systems of nonlinear equations, game theory, equilibrium, optimization theory, operations research, and complementarity problems, as special cases. Several techniques have been announced for exploring VIs in diverse directions. An fruitful generalized form of VI is called variational inclusion ($VarIncl$) which is to obtain $\theta \in \mathcal{H}$ such that

$$0 \in (\psi + G)\theta, \tag{1.1}$$

where $G : \mathcal{C} \subseteq \mathcal{H} \to \mathcal{H}$ and $\psi : \mathcal{H} \to 2^{\mathcal{H}}$ with $Dom(\psi) \subseteq \mathcal{C}$ are single-valued and set-valued maximal monotone mappings, respectively. $VarIncl$ (1.1) includes monotone inclusions, equilibrium problems (EPs), VIs, and saddle point problems as particular cases.

Most of methods for exploring VIs rely on projection techniques. Goldstein [11] studied a simplest version of the projection technique by generalizing gradient projection method for solving optimization problems. Projection technique enables us to reform VIs into fixed point problems (FPPs) and fixed point iterative methods can be employed to analyze the approximate solutions of VIs. Several schemes based on fixed points are constructed and implemented for exploring VIs, FPPs, initial and boundary value problems, image recovery, image restoration, image processing problems, and machine learning, etc., see, [3, 7, 12, 13, 17, 22, 25, 28, 30]. Agarwal et al. [1] came up with a noble iterative method, named S-iteration method. For some arbitrary point $\theta_0 \in \mathcal{C}$, the sequence $\{\theta_n\}$ is defined as follows:

$$\begin{cases} \theta_{n+1} = (1 - a_n)\psi\theta_n + a_n\psi\vartheta_n, \\ \vartheta_n = (1 - b_n)\theta_n + b_n\psi\theta_n, \end{cases} \tag{1.2}$$

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where the sequences $\{a_n\}, \{b_n\}$ are in (0,1). Since its inception, S-iterative scheme and its modified versions have attracted many researchers to investigate real world problems including FPPs, common fixed point problems and VIs, see, [4–6, 10, 14, 16, 20, 23]. In [18], the author further analyzed S-iterative method and proved by considering numerical example that it has efficient convergence rate than Picard as well as Mann iterative schemes for contraction mappings in metric spaces. By applying S-iteration method, the author also studied minimization and split feasibility problems.

In 2014, Sahu [19] brought forward the notion of altering points and outlined the convergence for the parallel S-iterative scheme as under: For initial point $(\theta_0, \vartheta_0) \in \mathcal{C}_1 \times \mathcal{C}_2$ and $\alpha \in (0,1)$, the sequence $\{(\theta_n, \vartheta_n)\}$ is approximated as follows:

$$\begin{cases} \theta_{n+1} = \psi_2[(1-\alpha)\vartheta_n + \alpha\psi_1\theta_n], \\ \vartheta_{n+1} = \psi_1[(1-\alpha)\theta_n + \alpha\psi_2\vartheta_n], \forall n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\psi_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $\psi_2 : \mathcal{C}_2 \to \mathcal{C}_1$ are two mappings. Recently, Zhao et al. [29] examined the system of variational inclusions for accretive mappings by carrying through the notion of altering points. Also, they executed these problems by proposing parallel Mann and parallel S-iterative schemes. Following this concept, several parallel iterative methods have been announced to deal with mathematical models including VIs, $VarIncl$, FPPs with numerous applications, see, for example; [2,14,27]. In [21], Sintunavarat and Pitea approximated fixed point of a self mapping by designing following iterative scheme:

$$\begin{cases} \theta_0 \in \mathcal{C}, \\ \xi_n = (1-b_n)\theta_n + b_n\psi\theta_n, \\ \vartheta_n = (1-c_n)\theta_n + c_n\xi_n, \\ \theta_{n+1} = (1-a_n)\psi\vartheta_n + a_n\psi\xi_n, n = 0, 1, 2, \dots, \end{cases} \tag{1.4}$$

where the real sequences $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty}$ are contained in $[0,1]$ and ψ is a nonlinear self mapping on a nonempty closed convex subset $\mathcal{C}$ of normed linear space $\mathcal{B}$. Following the attributed facts and methodologies, it is worthy to examine an altering points problem (APP). We design a new parallel iterative scheme based on (1.4) as follows.

Let $\psi_1 : \mathcal{C}_2 \to \mathcal{C}_1$ and $\psi_2 : \mathcal{C}_1 \to \mathcal{C}_2$ be two mappings. Then for initial point $(\theta_0, \vartheta_0) \in \mathcal{C}_1 \times \mathcal{C}_2$, we estimate the sequence $\{(\theta_n, \vartheta_n)\} \in \mathcal{C}_1 \times C_2$ as under:

$$\begin{cases} \theta_{n+1} = (1-a_n)\psi_2(\xi_n) + a_n\psi_2(q_n),\ \vartheta_{n+1} = (1-a_n)\psi_1(p_n) + a_n\psi_1(r_n), \\ \xi_n = (1-b_n)\theta_n + b_nq_n,\ p_n = (1-b_n)\vartheta_n + b_nr_n, \\ q_n = (1-c_n)\theta_n + c_n\psi_1(\vartheta_n),\ r_n = (1-c_n)\vartheta_n + c_n\psi_2(\theta_n), \end{cases} \tag{1.5}$$

where the sequences $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are in $[0,1]$. We shall analyze the convergence of scheme (1.5) to examine the altering points problem.

The paper is methodized in the succeeding order. The next section comprises preliminaries and requisite results. In Section 3, we commence the conceptualization of problem followed by some particular cases based on imposed conditions on the mappings. We bring forward a new parallel iterative scheme to achieve the approximate solution of altering points problem by proving existence and convergence results under some mild preassumptions. Our theoretical findings are verified by illustrative examples. At last, we explore some consequent systems of variational inclusions and inequalities.

2. Preliminaries

Let $\mathcal{B}^*$ be a dual space of a real Banach Space $\mathcal{B}$ with norm $\|\cdot\|$ and duality pairing $\langle\theta,\vartheta\rangle$ between $\theta\in\mathcal{B}$ and $\vartheta\in\mathcal{B}^*$. Let $\Xi=\{\theta\in\mathcal{B}:\|\theta\|=1\}$ be the unit sphere. $\mathcal{B}$ is called uniformly convex if $\exists\delta>0$ with $\theta,\vartheta\in\Xi$ satisfying $\|\theta-\vartheta\|\geq\epsilon$ implies $\left\|\frac{\theta-\vartheta}{2}\right\|\leq 1-\delta,\forall\epsilon\in(0,2]$ and $\mathcal{B}$ is called smooth, if for each $\theta\in\Xi$,

$$\lim_{\tau\to 0}\frac{\|\theta+\tau\vartheta\|-\|\theta\|}{\tau}, \tag{2.1}$$

exists. If the limit (2.1) is attained uniformly for all $\theta,\vartheta\in\Xi$ then $\mathcal{B}$ is known as uniformly smooth and $\lim_{\tau\to 0}\frac{\rho_{\mathcal{B}}(\tau)}{\tau}=0$. The modulus of smoothness $\rho_{\mathcal{B}}:[0,\infty)\to[0,\infty)$ is given as

$$\rho_{\mathcal{B}}(\tau)=\sup\left\{\frac{1}{2}(\|\theta+\vartheta\|+\|\theta-\vartheta\|)-1:\theta\in\Xi,\|\vartheta\|\leq\tau\right\}.$$

The normalized duality mapping $\mathcal{J}:\mathcal{B}\to\mathcal{B}^*$ is defined by

$$\mathcal{J}(\theta)=\{\vartheta\in\mathcal{B}^*:\langle\theta,\vartheta\rangle=\|\theta\|^2=\|\vartheta\|^2\},\ \forall\theta\in\mathcal{B}.$$

If $\mathcal{B}$ is smooth then $\mathcal{J}$ is single-valued.

Lemma 2.1 ([26])**.** *Let $\mathcal{J}:\mathcal{B}\to\mathcal{B}^*$ be a normalized duality mapping. Then*

$$\|\theta+\vartheta\|^2\leq\|\theta\|^2+2c^2\|\vartheta\|^2+2\langle\vartheta,\mathcal{J}(\theta)\rangle,\forall\theta,\vartheta\in\mathcal{B},$$

where $c>0$ is a real constant.

Next, we recall following notions. Let $\mathcal{C}$ be a nonempty closed convex subset of a Banach space $\mathcal{B}$. For each point $\theta\in\mathcal{B}$, $\exists$ a unique nearest point $\Pi_{\mathcal{C}}\in\mathcal{C}$, such that $\|\theta-\Pi_{\mathcal{C}}\theta\|=\inf_{\vartheta\in\mathcal{C}}\{\|\theta-\vartheta\|:\vartheta\in\mathcal{C}\}$. Note that the metric projection $\Pi_{\mathcal{C}}:\mathcal{B}\to\mathcal{C}$ has the following attributes [24]:

- $\|\Pi_{\mathcal{C}}(\theta)-\Pi_{\mathcal{C}}(\vartheta)\|\leq\|\theta-\vartheta\|,\forall\theta,\vartheta\in\mathcal{B}$;
- $\langle\theta-\Pi_{\mathcal{C}}(\theta),\vartheta-\Pi_{\mathcal{C}}(\theta)\rangle\leq 0,\forall\theta\in\mathcal{B},\vartheta\in\mathcal{C}$;
- $\|\Pi_{\mathcal{C}}(\theta)-\Pi_{\mathcal{C}}(\vartheta)\|^2\leq\langle\Pi_{\mathcal{C}}(\theta)-\Pi_{\mathcal{C}}(\vartheta),\theta-\vartheta\rangle,\forall\theta\in\mathcal{B},\vartheta\in\mathcal{C}$.

A mapping $Q_{\mathcal{C}}:\mathcal{B}\to\mathcal{C}$ is said to be sunny, if

$$Q_{\mathcal{C}}(Q_{\mathcal{C}}(\theta)+t(\theta-Q_{\mathcal{C}}(\theta)))=Q_{\mathcal{C}}(\theta),\forall\theta\in\mathcal{B},t\geq 0.$$

$Q_{\mathcal{C}}$ is called retraction, if $Q^2_{\mathcal{C}}=Q_{\mathcal{C}}$. Furthermore, $Q_{\mathcal{C}}$ is a sunny nonexpansive retraction, if a retraction $Q_{\mathcal{C}}:\mathcal{B}\to\mathcal{C}$ is sunny as well as nonexpansive. The following lemma pertinent to the sunny nonexpansive retraction plays a decisive role to derive our main results.

Proposition 2.2 ([9])**.** *A mapping $Q_{\mathcal{C}}:\mathcal{B}\to\mathcal{C}$ is a retraction sunny nonexpansive if and only if*

$$\langle\theta-Q_{\mathcal{C}}(\theta),\mathcal{J}(\xi-Q_{\mathcal{C}}(\theta))\rangle\leq 0,\forall\theta\in\mathcal{C}\text{ and }\xi\in\mathcal{B}.$$

Proposition 2.3. *Suppose $m = m(\theta) : \mathcal{B} \to \mathcal{B}$ and $Q_{\mathcal{C}} : \mathcal{C} \to \mathcal{B}$ be a retraction sunny nonexpansive. Then*

$$Q_{\mathcal{C}+m(\theta)}(\theta) = m(\theta) + Q_{\mathcal{C}}(\theta - m(\theta)), \forall \theta \in \mathcal{B}.$$

Remark 2.4. In a real Hilbert space $\mathcal{H}$, a sunny nonexpansive retraction of $\mathcal{B}$ onto $\mathcal{C}$ is the nearest point projection $\Pi_{\mathcal{C}}$ from $\mathcal{B}$ onto $\mathcal{C}$. But all Banach spaces do not carry this fact, since in Banach spaces nearest point projections are sunny but not nonexpansive. In [9], Bruck proved that if the Banach space is uniformly smooth, then for a nonexpansive retraction there exists a nonexpansive projection.

Lemma 2.5 ([8]). *If $\{\theta_n\}$ and $\{\vartheta_n\}$ are nonnegative real sequences comply with the inequality*

$$\theta_{n+1} \leq \varrho\theta_n + \vartheta_n, \ \forall n \in \mathbb{N},$$

where $\varrho \in (0, 1)$, $\lim_{n\to 0} \vartheta_n = 0$. Then $\lim_{n\to\infty} \theta_n = 0$.

If $\|\cdot\|$ is a norm on a real Banach space $\mathcal{B}$, then the norm $\|\cdot\|_*$ on $\mathcal{B} \times \mathcal{B}$ defined by

$$\|(\theta, \vartheta)\|_* = \|\theta\| + \|\vartheta\|, \forall \theta, \vartheta \in \mathcal{B} \tag{2.2}$$

is a Banach space.

2.1. **Generalized accretive mapping.** We collect the relevant background material which will be utilized to accomplish the goal. A mapping $\psi : \mathcal{C} \to \mathcal{B}$ is called

- accretive, if
$$\langle \psi(\theta) - \psi(\vartheta), \mathcal{J}(\theta - \vartheta)\rangle \geq 0, \forall \theta, \vartheta \in \mathcal{C};$$
- μ-strongly accretive, if $\exists \mu > 0$ such that
$$\langle \psi(\theta) - \psi(\vartheta), \mathcal{J}(\theta - \vartheta)\rangle \geq \mu\|\theta - \vartheta\|^2, \forall \theta, \vartheta \in \mathcal{C};$$
- η-expansive, if $\exists \eta > 0$ such that
$$\|\psi(\theta) - \psi(\vartheta)\| \geq \eta\|\theta - \vartheta\|, \forall \theta, \vartheta \in \mathcal{C};$$
- Lipschitz continuous, if $\exists \delta > 0$ such that
$$\|\psi(\theta) - \psi(\vartheta)\| \leq \delta\|\theta - \vartheta\|, \forall \theta, \vartheta \in \mathcal{C}.$$

If $\delta \in [0, 1)$ then the mapping ψ is called contraction and nonexpansive if $\delta = 1$.

Definition 2.6. Let $H : \mathcal{B} \to \mathcal{B}$ be the single-valued mapping. A set-valued mapping $M : \mathcal{B} \to 2^{\mathcal{B}}$ is called

(i) accretive, if
$$\langle z' - z'', \mathcal{J}(\theta - \vartheta)\rangle \geq 0, \ \forall \theta, \vartheta \in \mathcal{B}, z' \in M(\theta), z'' \in M(\vartheta);$$
(ii) m-accretive, if M is accretive and $(I + \varrho M)(\mathcal{B}) = \mathcal{B}, \forall \varrho > 0$;
(iii) H-accretive, if M is accretive and range$(H + \varrho M) = \mathcal{B}, \forall \varrho > 0$.

Remark 2.7. H-accretive mapping need not be m-accretive. Consider $H(\theta) = -\theta^3$ and $M(\theta) = sgn(\theta), \forall \theta \in \mathcal{B}$, then M is H-accretive but not m-accretive, see, [15].

Definition 2.8. Let $\mathcal{C}_1 \neq \emptyset$ be a closed convex subset of $\mathcal{B}$; $\varphi_1, \psi_1 : \mathcal{C}_1 \to \mathcal{B}$ and $H : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be the single-valued mappings. Then

(i) $H(\varphi_1, \cdot)$ is called μ_1-cocoercive regarding φ_1 if $\exists \mu_1 > 0$ such that

$$\langle H(\varphi_1(\theta), \upsilon) - H(\varphi_1(\vartheta), \upsilon), \mathcal{J}(\theta - \vartheta)\rangle \geq \mu_1 \|\theta - \vartheta\|^2, \forall \theta, \vartheta, \upsilon \in \mathcal{C}_1;$$

(ii) $H(\cdot, \psi_1)$ is called γ_1-relaxed cocoercive regarding ψ_1 if $\exists \gamma_1 > 0$ such that

$$\langle H(\upsilon, \psi_1(\theta)) - H(\upsilon, \psi_1(\vartheta)), \mathcal{J}(\theta - \vartheta)\rangle \geq (-\gamma_1) \|\theta - \vartheta\|^2, \forall \theta, \vartheta, \upsilon \in \mathcal{C}_1;$$

(iii) $H(\cdot, \cdot)$ is called (μ_1, γ_1)-symmetric cocoercive, if $H(\varphi_1, \cdot)$ is μ_1-cocoercive regarding φ_1 and γ_1-relaxed cocoercive regarding ψ_1;

(iv) $H(\cdot, \cdot)$ is called ν_1-mixed lipschitz continuous if $\exists \nu_1 > 0$ such that

$$\|H(\varphi_1(\theta), \psi_1(\theta)) - H(\varphi_1(\vartheta), \psi_1(\vartheta))\| \leq \nu_1 \|\theta - \vartheta\|, \forall \theta, \vartheta \in \mathcal{C}_1.$$

Definition 2.9. Let $\zeta_1, \xi_1 : \mathcal{C}_1 \to \mathcal{B}$ be the single-valued mappings and $M : \mathcal{B} \times \mathcal{B} \to 2^{\mathcal{B}}$ be a set-valued mapping. Then

(i) $M(\zeta_1, \cdot)$ is called ϵ_1-accretive regarding ζ_1 if $\exists \epsilon_1 > 0$ such that

$$\langle m - n, \mathcal{J}(\theta - \vartheta)\rangle \geq \epsilon_1 \|\theta - \vartheta\|^2, \forall \theta, \vartheta, \upsilon \in \mathcal{C}_1, m \in M(\zeta_1(\theta), \upsilon), n \in M(\zeta_1(\vartheta), \upsilon);$$

(ii) $M(\cdot, \xi_1)$ is called ω_1-relaxed accretive regarding ξ_1 if $\exists \omega_1 > 0$ such that

$$\langle m - n, \mathcal{J}(\theta - \vartheta)\rangle \geq (-\omega_1) \|\theta - \vartheta\|^2, \forall \theta, \vartheta, \upsilon \in \mathcal{C}_1, m \in M(\upsilon, \xi_1(\theta)), n \in M(\upsilon, \xi_1(\vartheta));$$

(iii) $M(\cdot, \cdot)$ is called (ϵ_1, ω_1)-symmetric accretive, if $M(\zeta_1, \cdot)$ is ϵ_1-accretive regarding ζ_1 and ω_1-relaxed accretive regarding ξ_1.

Definition 2.10. Let $\varphi_1, \psi_1, \zeta_1, \xi_1 : \mathcal{C}_1 \to \mathcal{B}$ and $H : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be the single-valued mappings. A multi-valued mapping $M : \mathcal{B} \times \mathcal{B} \to 2^{\mathcal{B}}$ is called $H(\cdot, \cdot)$-co-accretive if $H(\cdot, \cdot)$ is (μ_1, γ_1)-symmetric cocoercive; $M(\cdot, \cdot)$ is (ϵ_1, ω_1)-symmetric accretive and $[H(\varphi_1, \psi_1) + \varrho M(\zeta_1, \xi_1)](\mathcal{B}) = \mathcal{B}, \forall \varrho > 0$.

Remark 2.11. Every symmetric cocoercive and symmetric accretive mapping need not be $H(\cdot, \cdot)$-co-accretive.

Definition 2.12. Let $\varphi_1, \psi_1, \zeta_1, \xi_1 : \mathcal{C}_1 \to \mathcal{B}$ and $H : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be the single-valued mappings. Let $M : \mathcal{B} \times \mathcal{B} \to 2^{\mathcal{B}}$ be $H(\cdot, \cdot)$-co-accretive mapping. Then the resolvent operator $R^{H(\cdot,\cdot)}_{\varrho_1, M(\cdot,\cdot)} : \mathcal{B} \to \mathcal{B}$ is defined as

$$R^{H(\cdot,\cdot)}_{\varrho_1, M(\cdot,\cdot)}(\theta) = (H(\varphi_1, \psi_1) + \varrho_1 M(\zeta_1, \xi_1))^{-1}(\theta), \forall \theta \in \mathcal{B}, \varrho_1 > 0.$$

Proposition 2.13. *Let $H : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $\varphi_1, \psi_1, \zeta_1, \xi_1 : \mathcal{C}_1 \to \mathcal{B}$ be the single-valued mappings such that φ_1 is η_1-expansive and ψ_1 is σ_1-Lipschitz continuous with $\varepsilon_1 > \omega_1, \mu_1 > \gamma_1$ and $\eta_1 > \sigma_1$. Let $M : \mathcal{B} \times \mathcal{B} \to 2^{\mathcal{B}}$ be $H(\cdot, \cdot)$-co-accretive mapping. Then the mapping $R^{H(\cdot,\cdot)}_{\varrho_1, M(\cdot,\cdot)} : \mathcal{B} \to \mathcal{B}$ is Υ_1-Lipschitz continuous, i.e.,*

$$\|R^{H(\cdot,\cdot)}_{\varrho_1, M(\cdot,\cdot)}(\theta) - R^{H(\cdot,\cdot)}_{\varrho_1, M(\cdot,\cdot)}(\vartheta)\| \leq \Upsilon_1 \|\theta - \vartheta\|, \tag{2.3}$$

where $\Upsilon_1 = \frac{1}{\varrho_1(\varepsilon_1 - \omega_1) + (\mu_1 \eta_1^2 - \gamma_1 \sigma_1^2)}$.

2.2. Altering points.

Definition 2.14. [18] Let $\mathcal{C}_1, \mathcal{C}_2 \neq \emptyset$ be subsets of a metric space E. The points $\theta \in \mathcal{C}_1$ and $\vartheta \in \mathcal{C}_2$ are called altering points of the mappings $\psi_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $\psi_2 : \mathcal{C}_2 \to \mathcal{C}_1$, if

$$\begin{cases} \psi_1(\theta) = \vartheta, \\ \psi_2(\vartheta) = \theta. \end{cases} \tag{2.4}$$

We designate the set of altering points by $Alt(\psi_1, \psi_2) = \{(\theta, \vartheta) \in \mathcal{C}_1 \times \mathcal{C}_2 : \psi_1(\theta) = \vartheta \text{ and } \psi_2(\vartheta) = \theta\}$.

Example 2.15. Let $\mathcal{B} = \mathbb{R}, \mathcal{C}_1 = \mathcal{C}_2 = R_+$. Define $\psi_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $\psi_2 : \mathcal{C}_2 \to \mathcal{C}_1$ as $\psi_1(\theta) = e^\theta$ and $\psi_2(\vartheta) = \ln \vartheta$. Then $\psi_1\psi_2(\vartheta) = \vartheta$ and $\psi_2\psi_1(\theta) = \theta$. Thus, (θ, ϑ) are altering points of ψ_1 and ψ_2.

Example 2.16. Let $\mathcal{B} = \mathbb{R}^2, \mathcal{C}_1 = \{(\theta, \vartheta) : \theta + \vartheta = 2\}, \mathcal{C}_2 = \{(\theta, \vartheta) : \theta + \vartheta = 6\}$. Define $\psi_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $\psi_2 : \mathcal{C}_2 \to \mathcal{C}_1$ as $\psi_1(\theta, \vartheta) = 3(\vartheta, \theta)$ and $\psi_2(\theta, \vartheta) = \frac{1}{3}(\vartheta, \theta)$. Setting $\theta^* = (\theta, \vartheta) \in \mathcal{C}_1$ and $\vartheta^* = 3(\vartheta, \theta) \in \mathcal{C}_2$. Then, $\psi_1(\theta^*) = \vartheta^*$ and $\psi_2(\vartheta^*) = \theta^*$. Thus, (θ^*, ϑ^*) are altering points of ψ_1 and ψ_2.

Now onward, $\Gamma = \{1, 2\}$, for each $i \in \Gamma$, we assume $\mathcal{C}_i(\neq \phi)$ be closed convex subsets of a real 2-uniformly smooth Banach space $\mathcal{B}$. Hereinafter, it is shown that APP and the system of generalized variational inequalities (SG$VarInequal$) are analogous.

Lemma 2.17. *For each $i \in \Gamma$, suppose that $Q_{\mathcal{C}_i} : \mathcal{B} \to \mathcal{C}_i$ are sunny nonexpansive retractions and $S_i : \mathcal{C}_i \to \mathcal{B}$ are nonlinear mappings, then following statements are identical:*

(i) *$\theta \in \mathcal{C}_1$ and $\vartheta \in \mathcal{C}_2$ are altering points of $Q_{\mathcal{C}_2}[I - \varrho_1 S_1]$ and $Q_{\mathcal{C}_1}[I - \varrho_2 S_2]$.*
(ii) *$(\theta, \vartheta) \in \mathcal{C}_1 \times \mathcal{C}_2$ solves the following SG$VarInequal$: Find $(\theta, \vartheta) \in \mathcal{C}_1 \times \mathcal{C}_2$ such that*

$$\begin{cases} \langle \varrho_1 S_1(\theta) + \vartheta - \theta, \mathcal{J}(\omega_2 - \vartheta) \rangle \geq 0, \ \forall \omega_2 \in \mathcal{C}_2, \\ \langle \varrho_2 S_2(\vartheta) + \theta - \vartheta, \mathcal{J}(\omega_1 - \eta) \rangle \geq 0, \ \forall \omega_1 \in \mathcal{C}_1. \end{cases} \tag{2.5}$$

Lemma 2.18. *Let the single-valued mappings $\varphi_2, \psi_2 : \mathcal{C}_2 \to \mathcal{B}$ be such that φ_2 is η_2-expansive and ψ_2 is σ_2-Lipschitz continuous; $G : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be (μ_2, γ_2)-symmetric cocoercive and ν_2-mixed Lipschitz continuous. Let $T : \mathcal{C}_2 \to \mathcal{B}$ be κ_2-strongly accretive and ς_2-Lipschitz continuous. Suppose that the constant $\varrho_1 > 0$ satisfies*

$$0 < \delta_{\mathcal{C}_1}\left(\sqrt{1 - 2(\mu_2\eta_2^2 - \gamma_2\sigma_2^2) + 2c_2^2\nu_2^2} + \sqrt{1 - 2\varrho_1\kappa_2 + 2c_2^2\varrho_1^2\varsigma_2^2}\right) < 1. \tag{2.6}$$

Then the mapping $\Pi_{\mathcal{C}_1}[G(\varphi_2, \psi_2) - \varrho_1 T] : \mathcal{C}_2 \to \mathcal{C}_1$ is Φ_1-contraction, where

$$\Phi_1 = \delta_{\mathcal{C}_1}(\Theta_2 + \Delta_2), \Theta_2 = \sqrt{1 - 2(\mu_2\eta_2^2 - \gamma_2\sigma_2^2) + 2c_2^2\nu_2^2}$$
$$\text{and } \Delta_2 = \sqrt{1 - 2\varrho_1\kappa_2 + 2c_2^2\varrho_1^2\varsigma_2^2}.$$

Proof. Let $\vartheta_1, \vartheta_2 \in \mathcal{C}_2$, then we have

$$\begin{aligned}
&\|G(\varphi_2(\vartheta_1), \psi_2(\vartheta_1)) - \varrho_1 T(\vartheta_1) - (G(\varphi_2(\vartheta_2), \psi_2(\vartheta_2)) - \varrho_1 T(\vartheta_2))\| \\
&\leq \|G(\varphi_2(\vartheta_1), \psi_2(\vartheta_1)) - G(\varphi_2(\vartheta_2), \psi_2(\vartheta_2)) - (\vartheta_1 - \vartheta_2)\| \\
&\quad + \|\vartheta_1 - \vartheta_2 - \varrho_1(T(\vartheta_1) - T(\vartheta_2))\|.
\end{aligned} \tag{2.7}$$

Since G is (μ_2, γ_2)-symmetric cocoercive, ν_2-mixed Lipschitz continuous, φ_2 is η_2-expansive and ψ_2 is σ_2-Lipschitz continuous, then referring Lemma 2.1, we obtain

$$\begin{aligned}
&\|G(\varphi_2(\vartheta_1), \psi_2(\vartheta_1)) - G(\varphi_2(\vartheta_2), \psi_2(\vartheta_2)) - (\vartheta_1 - \vartheta_2)\|^2 \\
&\leq \|\vartheta_1 - \vartheta_2\|^2 - 2\langle G(\varphi_2(\vartheta_1), \psi_2(\vartheta_1)) - G(\varphi_2(\vartheta_2), \psi_2(\vartheta_2)), \mathcal{J}(\vartheta_1 - \vartheta_2)\rangle \\
&\quad + 2c_2^2\|G(\varphi_2(\vartheta_1), \psi_2(\vartheta_1)) - G(\varphi_2(\vartheta_2), \psi_2(\vartheta_2))\|^2 \\
&\leq \|\vartheta_1 - \vartheta_2\|^2 - 2[\mu_2\|\varphi_2(\vartheta_1) - \varphi_2(\vartheta_2)\|^2 \\
&\quad - \gamma_2\|\psi_2(\vartheta_1) - \psi_2(\vartheta_2)\|^2] + 2c_2^2\nu_2^2\|\vartheta_1 - \vartheta_2\|^2 \\
&\leq \|\vartheta_1 - \vartheta_2\|^2 - 2[\mu_2\eta_2^2\|\vartheta_1 - \vartheta_2\|^2 - \gamma_2\sigma_2^2\|\vartheta_1 - \vartheta_2\|^2] + 2c_2^2\nu_2^2\|\vartheta_1 - \vartheta_2\|^2,
\end{aligned} \tag{2.8}$$

which implies

$$\|G(\varphi_2(\vartheta_1), \psi_2(\vartheta_1)) - G(\varphi_2(\vartheta_2), \psi_2(\vartheta_2)) - (\vartheta_1 - \vartheta_2)\| \leq \Theta_2\|\vartheta_1 - \vartheta_2\|, \tag{2.9}$$

where, $\Theta_2 = \sqrt{1 - 2(\mu_2\eta_2^2 - \gamma_2\sigma_2^2) + 2c_2^2\nu_2^2}$. Taking into account the ς_2-Lipschitz continuity and κ_2-strongly accretive property of T, we acquire

$$\begin{aligned}
&\|\vartheta_1 - \vartheta_2 - \varrho_1(T(\vartheta_1) - T(\vartheta_2))\|^2 \\
&\leq \|\vartheta_1 - \vartheta_2\|^2 - 2\varrho_1\langle T(\vartheta_1) - T(\vartheta_2), \mathcal{J}(\vartheta_1 - \vartheta_2)\rangle + 2c_2^2\varrho_1^2\|T(\vartheta_1) - T(\vartheta_2)\|^2 \\
&\leq \|\vartheta_1 - \vartheta_2\|^2 - 2\varrho_1\kappa_2\|\vartheta_1 - \vartheta_2\|^2 + 2c_2^2\varrho_1^2\varsigma_2^2\|\vartheta_1 - \vartheta_2\|^2,
\end{aligned} \tag{2.10}$$

which implies

$$\|\vartheta_1 - \vartheta_2 - \varrho_1(T(\vartheta_1) - T(\vartheta_2))\| \leq \Delta_2\|\vartheta_1 - \vartheta_2\|, \tag{2.11}$$

where, $\Delta_2 = \sqrt{1 - 2\varrho_1\kappa_2 + 2c_2^2\varrho_1^2\varsigma_2^2}$. Employing the Lipschitz continuity of $\Pi_{\mathcal{C}_1}$, we obtain

$$\|\Pi_{\mathcal{C}_1}[G(\varphi_2, \psi_2) - \varrho_1 T](\vartheta_1) - \Pi_{\mathcal{C}_1}[G(\varphi_2, \psi_2) - \varrho_1 T](\vartheta_2) \leq \Phi_1\|\vartheta_1 - \vartheta_2\|. \tag{2.12}$$

It follows from (2.6) that $0 < \Phi_1 < 1$, where $\Phi_1 = \delta_{\mathcal{C}_1}\Big(\sqrt{1 - 2(\mu_2\eta_2^2 - \gamma_2\sigma_2^2) + 2c_2^2\nu_2^2} + \sqrt{1 - 2\varrho_1\kappa_2 + 2c_2^2\varrho_1^2\varsigma_2^2}\Big)$. Therefore, the mapping $\Pi_{\mathcal{C}_1}[G(\varphi_2, \psi_2) - \varrho_1 T] : \mathcal{C}_2 \to \mathcal{C}_1$ is Φ_1-contraction. Similarly, we can verify that $\Pi_{\mathcal{C}_2}[H(\varphi_1, \psi_1) - \varrho_2 S] : \mathcal{C}_1 \to \mathcal{C}_2$ is Φ_2-contraction. □

Proposition 2.19. *Let $\psi_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $\psi_2 : \mathcal{C}_2 \to \mathcal{C}_1$ be κ_1 and κ_2-contraction mappings, respectively. Then $(\theta, \vartheta) \in \mathcal{C}_1 \times \mathcal{C}_2$ is the unique solution of the problem (2.4), i.e.,*

$$\begin{cases} \psi_1(\theta) = \vartheta, \\ \psi_2(\vartheta) = \theta. \end{cases} \tag{2.13}$$

Proof. Since $\psi_1 : \mathcal{C}_1 \to \mathcal{C}_2$ is κ_1-contraction and $\psi_2 : \mathcal{C}_2 \to \mathcal{C}_1$ is κ_2-contraction mapping. It yields that $\psi_2\psi_1 : \mathcal{C}_1 \to \mathcal{C}_1$ is a contraction mapping. Thus, by appealing to PBC, $\psi_2\psi_1$ has a unique element $\theta \in \mathcal{C}_1$ such that $\theta = \psi_2\psi_1(\theta)$. Further, there exists a unique element $\vartheta \in \mathcal{C}_2$ such that $\vartheta = \psi_1(\theta)$. Thus, we have $\theta = \psi_2(\vartheta)$. □

Lemma 2.20. *For each $i \in \Gamma$; let the single-valued mappings $\varphi_i, \psi_i, \zeta_i, \xi_i : \mathcal{C}_i \to \mathcal{B}$ be such that φ_i is η_i-expansive and ψ_i is σ_i-Lipschitz continuous; $G : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be (μ_2, γ_2)-symmetric cocoercive and ν_2-mixed Lipschitz continuous with $\varepsilon_1 > \omega_1, \mu_1 > \gamma_1$ and $\eta_1 > \sigma_1$. Let $T : \mathcal{C}_2 \to \mathcal{B}$ be κ_2-strongly accretive and ς_2-Lipschitz continuous. Let $M : \mathcal{B} \times \mathcal{B} \to 2^{\mathcal{B}}$ be $H(\cdot,\cdot)$-co-accretive mapping such that $\overline{Dom(M)} \subseteq \mathcal{C}_1$. Suppose that the constant $\varrho_1 > 0$ satisfies*

$$0 < \frac{\sqrt{1 - 2(\mu_2\eta_2^2 - \gamma_2\sigma_2^2) + 2c_2^2\nu_2^2} + \sqrt{1 - 2\varrho_1\kappa_2 + 2c_2^2\varrho_1^2\varsigma_2^2}}{\varrho_1(\varepsilon_1 - \omega_1) + (\mu_1\eta_1^2 - \gamma_1\sigma_1^2)} < 1. \tag{2.14}$$

Then the mapping $R^{H(\cdot,\cdot)}_{\varrho_1,M(\cdot,\cdot)}[G(\varphi_2, \psi_2) - \varrho_1 T] : \mathcal{C}_2 \to \mathcal{C}_1$ is Ω_1-contraction, where $\Omega_1 = \Upsilon_1(\Theta_2 + \Delta_2), \Theta_2 = \sqrt{1 - 2(\mu_2\eta_2^2 - \gamma_2\sigma_2^2) + 2c_2^2\nu_2^2}, \Delta_2 = \sqrt{1 - 2\varrho_1\kappa_2 + 2c_2^2\varrho_1^2\varsigma_2^2}$ and $\Upsilon_1 = \frac{1}{\varrho_1(\varepsilon_1 - \omega_1) + (\mu_1\eta_1^2 - \gamma_1\sigma_1^2)}$.

Proof. Following the steps as from (2.7)-(2.11) and employing the Lipschitz continuity of resolvent operator $R^{H(\cdot,\cdot)}_{\varrho_1,M(\cdot,\cdot)}$, we obtain

$$\|R^{H(\cdot,\cdot)}_{\varrho_1,M(\cdot,\cdot)}[G(\varphi_2, \psi_2) - \varrho_1 T](\vartheta_1) - R^{H(\cdot,\cdot)}_{\varrho_1,M(\cdot,\cdot)}[G(\varphi_2, \psi_2) - \varrho_1 T](\vartheta_2) \leq \Omega_1\|\vartheta_1 - \vartheta_2\|, \tag{2.15}$$

where $\Omega_1 = \Upsilon_1(\Theta_2 + \Delta_2)$. From the assumption (2.14), we have $0 < \Omega_1 < 1$. Thus, $R^{H(\cdot,\cdot)}_{\varrho_1,M(\cdot,\cdot)}[G(\varphi_2, \psi_2) - \varrho_1 T] : \mathcal{C}_2 \to \mathcal{C}_1$ is Ω_1-contraction mapping. □

3. Problem formulation and convergence

For each $i \in \Gamma$, let $\Pi_{\mathcal{C}_i} : \mathcal{B} \to \mathcal{C}_i$ be operators; $\varphi_i, \psi_i, \zeta_i, \xi_i : \mathcal{C}_i \to \mathcal{B}; H, G : \mathcal{B} \times \mathcal{B} \to \mathcal{B}; S : \mathcal{C}_1 \to \mathcal{B}$ and $T : \mathcal{C}_2 \to \mathcal{B}$ be the single-valued mappings. We examine the following altering points problem (APP): Find $(\theta, \vartheta) \in \mathcal{C}_1 \times \mathcal{C}_2$ such that

$$\begin{cases} \Pi_{\mathcal{C}_2}[H(\varphi_1, \psi_1) - \varrho_2 S](\theta) = \vartheta, \\ \Pi_{\mathcal{C}_1}[G(\varphi_2, \psi_2) - \varrho_1 T](\vartheta) = \theta. \end{cases} \tag{3.1}$$

If $H(\cdot,\cdot) = H, G(\cdot,\cdot) = G, \varphi_i = \psi_i = I$, then (3.1) is identical to the following APP: Find $(\theta, \vartheta) \in \mathcal{C}_1 \times \mathcal{C}_2$ such that

$$\begin{cases} \Pi_{\mathcal{C}_2}[H - \varrho_2 S](\theta) = \vartheta, \\ \Pi_{\mathcal{C}_1}[G - \varrho_1 T](\vartheta) = \theta. \end{cases} \tag{3.2}$$

If $H = G = I$, then APP (3.2) is equivalent to the problem of finding $(\theta, \vartheta) \in \mathcal{C}_1 \times \mathcal{C}_2$ such that

$$\begin{cases} \Pi_{\mathcal{C}_2}[I - \varrho_2 S](\theta) = \vartheta, \\ \Pi_{\mathcal{C}_1}[I - \varrho_1 T](\vartheta) = \theta. \end{cases} \tag{3.3}$$

APP (3.1) includes several problems existing in the literature. A few particular cases of APP (3.1) are enumerated as under:

(i) If $\Pi_{\mathcal{C}_1} = R^{H(\cdot,\cdot)}_{\varrho_1,M(\cdot,\cdot)}$ and $\Pi_{\mathcal{C}_2} = R^{G(\cdot,\cdot)}_{\varrho_2,N(\cdot,\cdot)}$, where $M : \mathcal{B}\times\mathcal{B} \to 2^{\mathcal{B}}$ is $H(\cdot,\cdot)$-co-accretive with $\overline{Dom(M)} \subseteq \mathcal{C}_1$ and $N : \mathcal{B}\times\mathcal{B} \to 2^{\mathcal{B}}$ is $G(\cdot,\cdot)$-co-accretive mapping with $\overline{Dom(N)} \subseteq \mathcal{C}_2$, then APP (3.1) turns into the system of generalized variational inclusions (SG$VarIncl$): Find $(\theta,\vartheta) \in \mathcal{C}_1\times\mathcal{C}_2$ such that

$$\begin{cases} 0 \in G(\varphi_2(\vartheta),\psi_2(\vartheta)) - H(\varphi_1(\theta),\psi_1(\theta)) + \varrho_2(N(\zeta_2(\vartheta),\xi_2(\vartheta)) + S(\theta)), \\ 0 \in H(\varphi_1(\theta),\psi_1(\theta)) - G(\varphi_2(\vartheta),\psi_2(\vartheta)) + \varrho_1(M(\zeta_1(\theta),\xi_1(\theta)) + T(\vartheta)). \end{cases} \tag{3.4}$$

(ii) If $\varphi_i = \psi_i = \zeta_i = \xi_i = I, \Pi_{\mathcal{C}_1} = R^H_{\varrho_1,M}$ and $\Pi_{\mathcal{C}_2} = R^G_{\varrho_2,N}$, where $M : \mathcal{B} \to 2^{\mathcal{B}}$ is H-accretive with $\overline{Dom(M)} \subseteq \mathcal{C}_1$ and $N : \mathcal{B} \to 2^{\mathcal{B}}$ is G-accretive mapping with $\overline{Dom(N)} \subseteq \mathcal{C}_2$, then APP (3.1) is identical to the problem: Find $(\theta,\vartheta) \in \mathcal{C}_1\times\mathcal{C}_2$ such that

$$\begin{cases} 0 \in G(\vartheta) - H(\theta) + \varrho_2(N(\vartheta) + S(\theta)), \\ 0 \in H(\theta) - G(\vartheta) + \varrho_1(M(\theta) + T(\vartheta)). \end{cases} \tag{3.5}$$

(iii) If $H = G = I$, $\Pi_{\mathcal{C}_1} = R^M_{\varrho_1}$ and $\Pi_{\mathcal{C}_2} = R^N_{\varrho_2}$, where $M, N : \mathcal{B} \to 2^{\mathcal{B}}$ are m-accretive mappings with $\overline{Dom(M)} \subseteq \mathcal{C}_1$ and $\overline{Dom(N)} \subseteq \mathcal{C}_2$, then SG$VarIncl$ (3.5) turns to the system of variational inclusions: Find $(\theta,\vartheta) \in \mathcal{C}_1\times\mathcal{C}_2$ such that

$$\begin{cases} 0 \in \vartheta - \theta + \varrho_2(N(\vartheta) + S(\theta)), \\ 0 \in \theta - \vartheta + \varrho_1(M(\theta) + T(\vartheta)). \end{cases} \tag{3.6}$$

(iv) If $\Pi_{\mathcal{C}_i} = Q_{\mathcal{C}_i}$, the sunny nonexpansive retractions onto $\mathcal{C}_i$, then APP (3.1) becomes the system of generalized variational inequalities (SG$VarIneq$) of finding $(\theta,\vartheta) \in \mathcal{C}_1\times\mathcal{C}_2$ such that

$$\begin{cases} \langle \varrho_1 T(\vartheta) + \theta - G(\varphi_2(\vartheta),\psi_2(\vartheta)), \mathcal{J}(\omega_2 - \theta)\rangle \geq 0,\ \forall\omega_2 \in \mathcal{C}_2, \\ \langle \varrho_2 S(\theta) + \vartheta - H(\varphi_1(\theta),\psi_1(\theta)), \mathcal{J}(\omega_1 - \vartheta)\rangle \geq 0,\ \forall\omega_1 \in \mathcal{C}_1. \end{cases} \tag{3.7}$$

(v) If $\mathcal{B} = \mathcal{H}$, a real Hilbert space then SG$VarIneq$ (3.7) turns into the following SG$VarIneq$: Find $(\theta,\vartheta) \in \mathcal{C}_1\times\mathcal{C}_2$ such that

$$\begin{cases} \langle \varrho_1 T(\vartheta) + \theta - G(\varphi_2(\vartheta),\psi_2(\vartheta)), \omega_2 - \theta\rangle \geq 0,\ \forall\omega_2 \in \mathcal{C}_2, \\ \langle \varrho_2 S(\theta) + \vartheta - H(\varphi_1(\theta),\psi_1(\theta)), \omega_1 - \vartheta\rangle \geq 0,\ \forall\omega_1 \in \mathcal{C}_1. \end{cases} \tag{3.8}$$

Lemma 3.1. *For each $i \in \Gamma$; let $\varphi_i,\psi_i,\zeta_i,\xi_i : \mathcal{C}_i \to \mathcal{B}$ and $H, G : \mathcal{B}\times\mathcal{B} \to \mathcal{B}$ be the single-valued mappings. Let $S : \mathcal{C}_1 \to \mathcal{B}$ and $T : \mathcal{C}_2 \to \mathcal{B}$ be the single-valued mappings. Let $M : \mathcal{B}\times\mathcal{B} \to 2^{\mathcal{B}}$ be $H(\cdot,\cdot)$-co-accretive mapping with $\overline{Dom(M)} \subseteq \mathcal{C}_1$ and $N : \mathcal{B}\times\mathcal{B} \to 2^{\mathcal{B}}$ be $G(\cdot,\cdot)$-co-accretive mapping with $\overline{Dom(N)} \subseteq \mathcal{C}_2$. Then SG$VarIncl$ (3.4) has a solution (θ,ϑ), if and only if $(\theta,\vartheta) \in \mathcal{C}_1\times\mathcal{C}_2$ solves the* *following APP:*

$$\begin{cases} R^{G(\cdot,\cdot)}_{\varrho_2,N(\cdot,\cdot)}[H(\varphi_1,\psi_1)-\varrho_2 S](\theta)=\vartheta, \\ R^{H(\cdot,\cdot)}_{\varrho_1,M(\cdot,\cdot)}[G(\varphi_2,\psi_2)-\varrho_1 T](\vartheta)=\theta. \end{cases} \tag{3.9}$$

Next, we shall estimate the solution of APP (3.1) by implementing iterative method (1.5).

Theorem 3.2. *Let $\Pi_{\mathcal{C}_i}:\mathcal{B}\to\mathcal{C}_i$ be δ_{Ω_i}-Lipschitz continuous mappings; the single-valued mappings $\varphi_i,\psi_i,\zeta_i,\xi_i:\mathcal{C}_i\to\mathcal{B}$ be such that φ_i is η_i-expansive and ψ_i is σ_i-Lipschitz continuous; $H:\mathcal{B}\times\mathcal{B}\to\mathcal{B}$ be (μ_1,γ_1)-symmetric cocoercive and ν_1-mixed Lipschitz continuous and $G:\mathcal{B}\times\mathcal{B}\to\mathcal{B}$ be (μ_2,γ_2)symmetric cocoercive and ν_2-mixed Lipschitz continuous. Let $S:\mathcal{C}_1\to\mathcal{B}$ be κ_1-strongly accretive and ς_1-Lipschitz continuous and $T:\mathcal{C}_2\to\mathcal{B}$ be κ_2-strongly accretive and ς_2-Lipschitz continuous. Let $M:\mathcal{B}\times\mathcal{B}\to 2^{\mathcal{B}}$ be $H(\cdot,\cdot)$-co-accretive mapping with $\overline{Dom(M)}\subseteq\mathcal{C}_1$ and $N:\mathcal{B}\times\mathcal{B}\to 2^{\mathcal{B}}$ be $G(\cdot,\cdot)$-co-accretive mapping with $\overline{Dom(N)}\subseteq\mathcal{C}_2$. Suppose that the constants $\varrho_i>0$ comply with*

$$\begin{aligned} &0<\delta_{\mathcal{C}_1}\Big(\sqrt{1-2(\mu_2\eta_2^2-\gamma_2\sigma_2^2)+2c_2^2\nu_2^2}+\sqrt{1-2\varrho_1\kappa_2+2c_2^2\varrho_1^2\varsigma_2^2}\Big)<1. \\ &0<\delta_{\mathcal{C}_2}\Big(\sqrt{1-2(\mu_1\eta_1^2-\gamma_1\sigma_1^2)+2c_1^2\nu_1^2}+\sqrt{1-2\varrho_2\kappa_1+2c_1^2\varrho_2^2\varsigma_1^2}\Big)<1. \\ &1+2c_i^2\nu_i^2>2(\mu_i\eta_i^2-\gamma_i\sigma_i^2), 1+2c_2^2\varrho_1^2\varsigma_2^2>2\varrho_1\kappa_2, 1+2c_1^2\varrho_2^2\varsigma_1^2>2\varrho_2\kappa_1. \end{aligned} \tag{3.10}$$

(i) *Then there exists a unique element $(\theta,\vartheta)\in\mathcal{C}_1\times\mathcal{C}_2$ such that (θ,ϑ) solves APP (3.1).*
(ii) *The sequence $\{(\theta_n,\vartheta_n)\}\in\mathcal{C}_1\times\mathcal{C}_2$ generated by parallel scheme (1.5) converges strongly to (θ,ϑ).*

Proof. (i) Define $\Lambda_1 =: \Pi_{\mathcal{C}_1}[G(\varphi_2,\psi_2)-\varrho_1 T]$ and $\Lambda_2=\Pi_{\mathcal{C}_2}[H(\varphi_1,\psi_1)-\varrho_2 S]$. Then, it follows from Lemma 2.18 that $\Lambda_1:\mathcal{C}_2\to\mathcal{C}_1$ is Φ_1-contraction mapping and $\Lambda_2:\mathcal{C}_1\to\mathcal{C}_2$ is Φ_2-contraction. Hence, the required proof can be obtained by utilizing the Proposition 2.19.

(ii) It follows from (1.5) that

$$\begin{aligned} \|q_n-q\| &= \|(1-c_n)\theta_n+c_n\Lambda_1(\vartheta_n)-[(1-c_n)\theta+c_n\Lambda_1(\vartheta)]\| \\ &\le (1-c_n)\|\theta_n-\theta\|+c_n\|\Lambda_1(\vartheta_n)-\Lambda_1(\vartheta)\| \\ &\le (1-c_n)\|\theta_n-\theta\|+c_n\Phi_1\|\vartheta_n-\vartheta\|. \end{aligned} \tag{3.11}$$

$$\begin{aligned} \|\xi_n-\xi\| &= \|(1-b_n)\theta_n+b_nq_n-[(1-b_n)\theta+b_nq]\| \\ &\le (1-b_n)\|\theta_n-\theta\|+b_n\|q_n-q\|. \end{aligned} \tag{3.12}$$

$$\begin{aligned} \|\theta_{n+1}-\theta\| &= \|(1-a_n)\Lambda_2(\xi_n)+a_n\Lambda_2(q_n)-[(1-a_n)\Lambda_2(\xi)+a_n\Lambda_2(q)]\| \\ &\le (1-a_n)\|\Lambda_2(\xi_n)-\Lambda_2(\xi)\|+a_n\|\Lambda_2(q_n)-\Lambda_2(q)\| \\ &\le (1-a_n)\Phi_2\|\xi_n-\xi\|+a_n\Phi_2\|q_n-q\| \\ &\le (1-a_n)\Phi_2(1-b_n)\|\theta_n-\theta\|+\Phi_2[b_n+a_n(1-b_n)]\|q_n-q\| \\ &\le \Phi_2[(1-a_n)(1-b_n)+(1-c_n)(b_n+a_n(1-b_n))]\|\theta_n-\theta\| \\ &\quad +\Phi_1\Phi_2c_n[b_n+a_n(1-b_n)]\|\vartheta_n-\vartheta\|. \end{aligned} \tag{3.13}$$

In the similar manner, we obtain

$$\begin{aligned}\|r_n - r\| &= \|(1-c_n)\vartheta_n + c_n\Lambda_2(\theta_n) - [(1-c_n)\vartheta + c_n\Lambda_2(\theta)]\| \\ &\le (1-c_n)\|\vartheta_n - \vartheta\| + c_n\|\Lambda_2(\theta_n) - \Lambda_2(\theta)\| \\ &\le (1-c_n)\|\vartheta_n - \vartheta\| + c_n\Phi_2\|\theta_n - \theta\|.\end{aligned} \tag{3.14}$$

$$\begin{aligned}\|p_n - p\| &= \|(1-b_n)\vartheta_n + b_n r_n - [(1-b_n)\vartheta + b_n r]\| \\ &\le (1-b_n)\|\vartheta_n - \vartheta\| + b_n\|r_n - r\|.\end{aligned} \tag{3.15}$$

$$\begin{aligned}\|\vartheta_{n+1} - \vartheta\| &= \|(1-a_n)\Lambda_1(p_n) + a_n\Lambda_1(r_n) - [(1-a_n)\Lambda_1(p) + a_n\Lambda_1(r)]\| \\ &\le (1-a_n)\|\Lambda_1(p_n) - \Lambda_1(p)\| + a_n\|\Lambda_1(r_n) - \Lambda_1(r)\| \\ &\le (1-a_n)\Phi_1\|p_n - p\| + a_n\Phi_1\|r_n - r\| \\ &\le (1-a_n)\Phi_2(1-b_n)\|\theta_n - \theta\| + \Phi_2(b_n + a_n(1-b_n))\|q_n - q\|] \\ &\le \Phi_1[(1-a_n)(1-b_n) + (1-c_n)(a_n + b_n(1-a_n))]\|\vartheta_n - \vartheta\| \\ &\quad + \Phi_1\Phi_2 c_n[a_n + b_n(1-a_n)]\|\theta_n - \theta\|.\end{aligned} \tag{3.16}$$

Choose $\Phi = \max\{\Phi_1, \Phi_2\}$, taking the assumptions that $a_n, b_n, c_n \in [0,1]$ and Φ_1, Φ_2 are contractions into consideration and combining (3.13) and (3.16), we obtain

$$\begin{aligned}\|\theta_{n+1} - \theta\| + \|\vartheta_{n+1} - \vartheta\| &\le \Phi_1[1 - c_n(1-\Phi_2)]\|\theta_n - \theta\| \\ &\quad + \Phi_2[1 - c_n(1-\Phi_1)]\|\vartheta_n - \vartheta\| \\ &\le \Phi[\|\theta_n - \theta\| + \|\vartheta_n - \vartheta\|].\end{aligned} \tag{3.17}$$

Thus, from (2.2), (3.17) yields

$$\|(\theta_{n+1}, \vartheta_{n+1}) - (\theta, \vartheta)\|_* \le \Phi\|(\theta_n, \vartheta_n) - (\theta, \vartheta)\|_*. \tag{3.18}$$

Noting $\Phi \in (0,1)$, then by appealing the Lemma 2.5, we get

$$\lim_{n\to\infty} \|(\theta_n, \vartheta_n) - (\theta, \vartheta)\|_* = 0, \tag{3.19}$$

which implies $\lim_{n\to\infty}\|\theta_n - \theta\| = \lim_{n\to\infty}\|\vartheta_n - \vartheta\| = 0$. Thus, $\{(\theta_n, \vartheta_n)\}$ converges to $(\theta, \vartheta) \in \mathcal{C}_1 \times \mathcal{C}_2$. □

Example 3.3. Let $\mathcal{B} = \mathbb{R}, \mathcal{C}_1 = \mathcal{C}_2 = \mathbb{R}$ with usual norm and inner product. Define $\Pi_{\mathcal{C}_1} : \mathcal{B} \to \mathcal{C}_1$ and $\Pi_{\mathcal{C}_2} : \mathcal{B} \to \mathcal{C}_2$ by

$$\Pi_{\mathcal{C}_1}(\theta) = \frac{2\theta + 1}{5} \text{ and } \Pi_{\mathcal{C}_2}(\theta) = \frac{2\theta + 2}{6}, \forall \theta \in \mathcal{B}.$$

Then $\Pi_{\mathcal{C}_1}$ and $\Pi_{\mathcal{C}_2}$ are Lipschitz continuous with constant $\delta_{\mathcal{C}_1} = \frac{2}{5}$ and $\delta_{\mathcal{C}_2} = \frac{1}{3}$, respectively. Define the single-valued mappings $\varphi_1, \psi_1, \zeta_1, \xi_1, S : \mathcal{C}_1 \to \mathcal{B}$; $\varphi_2, \psi_2, \zeta_2, \xi_2, T : \mathcal{C}_2 \to \mathcal{B}$; $H, G : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and the set-valued mappings $M, N : \mathcal{B} \times \mathcal{B} \to 2^{\mathcal{B}}$ by

$$\varphi_1(\theta) = \frac{\theta + 1}{3}, \psi_1(\theta) = \frac{2\theta}{3} + \frac{1}{5}, \zeta_1(\theta) = \frac{\theta + 3}{8}, \xi_1(\theta) = -\frac{\theta - 5}{10}, S(\theta) = \frac{\theta + 1}{6}, \forall \theta \in \mathcal{C}_1,$$

$$\varphi_2(\theta) = \frac{\theta + 1}{2}, \psi_2(\theta) = \frac{2\theta + 1}{6}, \zeta_2(\theta) = \frac{\theta + 2}{10}, \xi_2(\theta) = -\frac{\theta - 8}{6}, T(\theta) = \frac{\theta}{10} + \frac{1}{2}, \forall \theta \in \mathcal{C}_2,$$

$$H(\varphi_1(\theta), \psi_1(\theta)) = \varphi_1(\theta) - \psi_1(\theta), G(\varphi_2(\vartheta), \psi_2(\vartheta)) = \varphi_2(\vartheta) - \psi_2(\vartheta), \forall \theta, \vartheta \in \mathcal{B},$$

$$M(\zeta_1(\theta), \xi_1(\theta)) = \zeta_1(\theta) + \xi_1(\theta), N(\zeta_2(\vartheta), \xi_2(\vartheta)) = \zeta_2(\vartheta) + \xi_2(\vartheta), \forall \theta, \vartheta \in \mathcal{B}.$$

Now, we calculate

$$\|\varphi_1(\theta)-\varphi_1(\vartheta)\| = \Big|\frac{\theta+1}{3}-\frac{\vartheta+1}{3}\Big| \geq \frac{1}{3}\|\theta-\vartheta\|, \forall\theta,\vartheta\in\mathcal{C}_1,$$
$$\|\varphi_2(\theta)-\varphi_2(\vartheta)\| = \Big|\frac{\theta+1}{2}-\frac{\vartheta+1}{2}\Big| \geq \frac{1}{2}\|\theta-\vartheta\|, \forall\theta,\vartheta\in\mathcal{C}_2.$$

Thus, φ_1 is $\frac{1}{3}$-expansive and φ_2 is $\frac{1}{2}$-expansive.

$$\|\psi_1(\theta)-\psi_1(\vartheta)\| = \Big|\frac{2\theta}{3}+\frac{1}{5}-\frac{2\vartheta}{3}-\frac{1}{5}\Big| \leq \frac{2}{3}\|\theta-\vartheta\|, \forall\theta,\vartheta\in\mathcal{C}_1,$$
$$\|\psi_2(\theta)-\psi_2(\vartheta)\| = \Big|\frac{2\theta+1}{6}-\frac{2\vartheta+1}{6}\Big| \leq \frac{1}{3}\|\theta-\vartheta\|, \forall\theta,\vartheta\in\mathcal{C}_2.$$

Thus, ψ_1 is $\frac{2}{3}$-Lipschitz continuous and ψ_2 is $\frac{1}{3}$-Lipschitz continuous.

$$\begin{aligned}\langle H(\varphi_1(\theta),\varpi)-H(\varphi_1(\vartheta),\varpi),\theta-\vartheta\rangle &= \Big\langle\frac{\theta+1}{3}-\frac{\vartheta+1}{3},\theta-\vartheta\Big\rangle\\ &= \frac{1}{3}(\theta-\vartheta)^2,\end{aligned}$$
$$\begin{aligned}\|\varphi_1(\theta)-\varphi_1(\vartheta)\|^2 &= \Big\langle\frac{\theta+1}{3}-\frac{\vartheta+1}{3},\frac{\theta+1}{3}-\frac{\vartheta+1}{3}\Big\rangle\\ &= \frac{1}{9}(\theta-\vartheta)^2\end{aligned}$$

which implies

$$H(\varphi_1(\theta),\varpi)-H(\varphi_1(\vartheta),\varpi),\theta-\vartheta\rangle \geq 3\|\varphi_1(\theta)-\varphi_1(\vartheta)\|^2,$$

and

$$\begin{aligned}\langle H(\varpi,\psi_1(\theta))-H(\varpi,\psi_1(\vartheta)),\theta-\vartheta\rangle &= -\Big\langle\frac{2\theta}{3}+\frac{1}{5}-\frac{2\vartheta}{3}+\frac{1}{5},\theta-\vartheta\Big\rangle\\ &= -\frac{2}{3}(\theta-\vartheta)^2,\end{aligned}$$
$$\begin{aligned}\|\psi_1(\theta)-\psi_1(\vartheta)\|^2 &= \Big\langle\frac{2\theta}{3}+\frac{1}{5}-\frac{2\vartheta}{3}+\frac{1}{5},\frac{2\theta}{3}+\frac{1}{5}-\frac{2\vartheta}{3}+\frac{1}{5}\Big\rangle\\ &= \frac{4}{9}(\theta-\vartheta)^2\end{aligned}$$

which implies

$$\Big\langle H(\varpi,\psi_1(\theta))-H(\varpi,\psi_1(\vartheta)),\theta-\vartheta\Big\rangle \geq -\frac{3}{2}\|\psi_1(\theta)-\psi_1(\vartheta)\|^2.$$

Thus, $H(\cdot,\cdot)$ is 3-cocoercive and $\frac{3}{2}$-relaxed cocoercive with respect to φ_1 and ψ_1, respectively.

$$\begin{aligned}\langle G(\varphi_2(\theta),\varpi)-G(\varphi_2(\vartheta),\varpi),\theta-\vartheta\rangle &= \Big\langle\frac{\theta+1}{2}-\frac{\vartheta+1}{2},\theta-\vartheta\Big\rangle\\ &= \frac{1}{2}(\theta-\vartheta)^2,\end{aligned}$$
$$\begin{aligned}\|\varphi_2(\theta)-\varphi_2(\vartheta)\|^2 &= \Big\langle\frac{\theta+1}{2}-\frac{\vartheta+1}{2},\frac{\theta+1}{2}-\frac{\vartheta+1}{2}\Big\rangle\\ &= \frac{1}{4}(\theta-\vartheta)^2\end{aligned}$$

which implies

$$\langle G(\varphi_2(\theta),\varpi)-G(\varphi_2(\vartheta),\varpi),\theta-\vartheta\rangle \geq 2\|\varphi_1(\theta)-\varphi_1(\vartheta)\|^2,$$

and

$$\begin{aligned}\langle G(\varpi,\psi_2(\theta))-G(\varpi,\psi_2(\vartheta)),\theta-\vartheta\rangle &= -\Big\langle \frac{2\theta+1}{6}-\frac{2\vartheta+1}{6},\theta-\vartheta\Big\rangle \\ &= -\frac{1}{3}(\theta-\vartheta)^2,\end{aligned}$$

$$\begin{aligned}\|\psi_2(\theta)-\psi_2(\vartheta)\|^2 &= \Big\langle \frac{2\theta+1}{6}-\frac{2\vartheta+1}{6},\frac{2\theta+1}{6}-\frac{2\vartheta+1}{6}\Big\rangle \\ &= \frac{1}{9}(\theta-\vartheta)^2\end{aligned}$$

which implies

$$\Big\langle G(\varpi,\psi_2(\theta))-G(\varpi,\psi_2(\vartheta)),\theta-\vartheta\Big\rangle \geq -3\|\psi_2(\theta)-\psi_2(\vartheta)\|^2,$$

Thus, $G(\cdot,\cdot)$ is 2-cocoercive and 3-relaxed cocoercive with respect to φ_2 and ψ_2, respectively. Also, M and N are $H(\cdot,\cdot)$ and $G(\cdot,\cdot)$-co-accretive mappings, respectively. Further,

$$\begin{aligned}\langle S(\theta)-S(\vartheta),\theta-\vartheta\rangle = \Big\langle \frac{\theta+1}{6}-\frac{\vartheta+1}{6},\theta-\vartheta\Big\rangle &\geq \frac{1}{6}\|\theta-\vartheta\|^2, \forall\theta,\vartheta\in\mathcal{B},\\ \|S(\theta)-S(\vartheta)\| = \Big|\frac{\theta+1}{6}-\frac{\vartheta+1}{6}\Big| &\leq \frac{1}{6}\|\theta-\vartheta\|^2, \forall\theta,\vartheta\in\mathcal{B}.\end{aligned}$$

i.e., S is $\frac{1}{6}$-strongly accretive and $\frac{1}{6}$-Lipschitz continuous.

$$\begin{aligned}\langle T(\theta)-T(\vartheta),\theta-\vartheta\rangle = \Big\langle \frac{\theta}{10}+\frac{1}{2}-\frac{\vartheta}{10}-\frac{1}{2},\theta-\vartheta\Big\rangle &\geq \frac{1}{10}\|\theta-\vartheta\|^2, \forall\theta,\vartheta\in\mathcal{B},\\ \|T(\theta)-T(\vartheta)\| = \Big|\frac{\theta}{10}+\frac{1}{2}-\frac{\vartheta}{10}-\frac{1}{2}\Big| &\leq \frac{1}{10}\|\theta-\vartheta\|^2, \forall\theta,\vartheta\in\mathcal{B}.\end{aligned}$$

i.e., T is $\frac{1}{10}$-strongly accretive and $\frac{1}{10}$-Lipschitz continuous. Also, for constants $\varrho_1=1,\delta_{\mathcal{C}_1}=\frac{2}{5},\mu_1=3,\eta_1=\frac{1}{3},\gamma_1=\frac{3}{2},\sigma_1=\frac{2}{3},\nu_1=\frac{1}{3},\kappa_1=\frac{1}{6},\varsigma_1=\frac{1}{6},\varrho_2=1,\delta_{\mathcal{C}_2}=\frac{1}{3},\mu_2=2,\eta_2=\frac{1}{2},\gamma_2=3,\sigma_2=\frac{1}{3},\nu_2=\frac{1}{6},\kappa_2=\frac{1}{10},\varsigma_2=\frac{1}{10}$, the conditions

$$\begin{aligned}0<\delta_{\mathcal{C}_1}\Big(\sqrt{1-2(\mu_1\eta_1^2-\gamma_1\sigma_1^2)+2c_1^2\nu_1^2}+\sqrt{1-2\varrho_1\kappa_1+2c_1^2\varrho_1^2\varsigma_1^2}\Big) &= 0.71241<1\\ 0<\delta_{\mathcal{C}_2}\Big(\sqrt{1-2(\mu_2\eta_2^2-\gamma_2\sigma_2^2)+2c_2^2\nu_2^2}+\sqrt{1-2\varrho_2\kappa_2+2c_2^2\varrho_2^2\varsigma_2^2}\Big) &= 0.68012<1\end{aligned}$$

are also satisfied. Further,

$$\begin{aligned}\Pi_{\mathcal{C}_2}[H(\varphi_1,\psi_1)-\varrho_2 S](-0.16344) &= 0.04838,\\ \Pi_{\mathcal{C}_1}[G(\varphi_2,\psi_2)-\varrho_1 T](0.04838) &= -0.16344,\end{aligned}$$

i.e., $(-0.16344, 0.04838)$ is the altering point of APP (3.1).

Consequences and applications

Here, we shall look over the convergence of proposed iterative scheme (1.5) to investigate the SG$VarIncl$ (3.4) and SG$VarIneq$ (3.7). We can re-design scheme (1.5) by taking $\psi_1 =: R^{H(\cdot,\cdot)}_{\varrho_1,M(\cdot,\cdot)}[G(\varphi_2,\psi_2)-\varrho_1 T]$ and $\psi_2 = R^{G(\cdot,\cdot)}_{\varrho_2,N(\cdot,\cdot)}[H(\varphi_1,\psi_1)-\varrho_2 S]$. For initial point $(\theta_0,\vartheta_0) \in \mathcal{C}_1 \times \mathcal{C}_2$, we estimate the sequence $(\theta_n,\vartheta_n) \in \mathcal{C}_1 \times \mathcal{C}_2$ as under:

$$\begin{cases} \theta_{n+1} = (1-a_n)R^{G(\cdot,\cdot)}_{\varrho_2,N(\cdot,\cdot)}[H(\varphi_1,\psi_1)-\varrho_2 S](\xi_n) \\ \quad +a_n R^{G(\cdot,\cdot)}_{\varrho_2,N(\cdot,\cdot)}[H(\varphi_1,\psi_1)-\varrho_2 S](q_n), \\ \vartheta_{n+1} = (1-a_n)R^{H(\cdot,\cdot)}_{\varrho_1,M(\cdot,\cdot)}[G(\varphi_2,\psi_2)-\varrho_1 T](p_n) \\ \quad +a_n R^{H(\cdot,\cdot)}_{\varrho_1,M(\cdot,\cdot)}[G(\varphi_2,\psi_2)-\varrho_1 T](r_n), \\ \xi_n = (1-b_n)\theta_n + b_n q_n,\ p_n = (1-b_n)\vartheta_n + b_n r_n, \\ q_n = (1-c_n)\theta_n + c_n R^{H(\cdot,\cdot)}_{\varrho_1,M(\cdot,\cdot)}[G(\varphi_2,\psi_2)-\varrho_1 T](\vartheta_n), \\ r_n = (1-c_n)\vartheta_n + c_n R^{G(\cdot,\cdot)}_{\varrho_2,N(\cdot,\cdot)}[H(\varphi_1,\psi_1)-\varrho_2 S](\theta_n), \end{cases} \tag{3.20}$$

where the sequences $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are in $[0,1]$.

Theorem 3.4. *For each $i \in \Gamma$; let the single-valued mappings $\varphi_i,\psi_i,\zeta_i,\xi_i : \mathcal{C}_i \to \mathcal{B}$ be such that φ_i is η_i-expansive and ψ_i is σ_i-Lipschitz continuous; $H : \mathcal{B}\times\mathcal{B} \to \mathcal{B}$ be (μ_1,γ_1)-symmetric cocoercive and ν_1-mixed Lipschitz continuous and $G : \mathcal{B}\times\mathcal{B} \to \mathcal{B}$ be (μ_2,γ_2)-symmetric cocoercive and ν_2-mixed Lipschitz continuous. Let $S : \mathcal{C}_1 \to \mathcal{B}$ be κ_1-strongly accretive and ς_1-Lipschitz continuous and $T : \mathcal{C}_2 \to \mathcal{B}$ be κ_2-strongly accretive and ς_2-Lipschitz continuous. Let $M : \mathcal{B}\times\mathcal{B} \to 2^{\mathcal{B}}$ be $H(\cdot,\cdot)$-co-accretive mapping with $\overline{Dom(M)} \subseteq \mathcal{C}_1$ and $N : \mathcal{B}\times\mathcal{B} \to 2^{\mathcal{B}}$ be $G(\cdot,\cdot)$-co-accretive mapping with $\overline{Dom(N)} \subseteq \mathcal{C}_2$. Suppose that the constants ϱ_i comply with the following inequalities:*

$$\begin{gathered} 0 < \frac{\sqrt{1-2(\mu_2\eta_2^2-\gamma_2\sigma_2^2)+2c_2^2\nu_2^2} + \sqrt{1-2\varrho_1\kappa_2+2c_2^2\varrho_1^2\varsigma_2^2}}{\varrho_1(\varepsilon_1-\omega_1)+(\mu_1\eta_1^2-\gamma_1\sigma_1^2)} < 1, \\ 0 < \frac{\sqrt{1-2(\mu_1\eta_1^2-\gamma_1\sigma_1^2)+2c_1^2\nu_1^2} + \sqrt{1-2\varrho_2\kappa_1+2c_1^2\varrho_2^2\varsigma_1^2}}{\varrho_2(\varepsilon_2-\omega_2)+(\mu_2\eta_2^2-\gamma_2\sigma_2^2)} < 1, \\ 1+2c_i^2\nu_i^2 > 2(\mu_i\eta_i^2-\gamma_i\sigma_i^2), 1+2c_2^2\varrho_1^2\varsigma_2^2 > 2\varrho_1\kappa_2, 1+2c_1^2\varrho_2^2\varsigma_1^2 > 2\varrho_2\kappa_1. \end{gathered} \tag{3.21}$$

(i) *Then there exists a unique element $(\theta,\vartheta) \in \mathcal{C}_1 \times \mathcal{C}_2$ such that (θ,ϑ) solves SG$VarIncl$ (3.4).*

(ii) *The sequence $\{(\theta_n,\vartheta_n)\} \in \mathcal{C}_1 \times \mathcal{C}_2$ generated by the parallel scheme (3.20) converges strongly to (θ,ϑ).*

Proof. (i) Define $\Psi_1 =: R^{H(\cdot,\cdot)}_{\varrho_1,M(\cdot,\cdot)}[G(\varphi_2,\psi_2)-\varrho_1 T]$ and $\Psi_2 =: R^{G(\cdot,\cdot)}_{\varrho_2,N(\cdot,\cdot)}[H(\varphi_1,\psi_1)-\varrho_2 S]$. Then taking the Lemma 2.20 into account, one can achieve that $\Psi_1 : \mathcal{C}_2 \to \mathcal{C}_1$ is Ω_1-contraction mapping. Similarly $\Psi_2 : \mathcal{C}_1 \to \mathcal{C}_2$ is Ω_2-contraction mapping. By utilizing Proposition 2.19, we infer that there exists a unique element $(\theta,\vartheta) \in \mathcal{C}_1 \times \mathcal{C}_2$

such that $\Psi_1(\vartheta)=\theta$ and $\Psi_2(\theta)=\vartheta$ and the definitions of Ψ_1 and Ψ_2 lead to

$$\begin{cases} R^{H(\cdot,\cdot)}_{\varrho_1,M(\cdot,\cdot)}[G(\varphi_2,\psi_2)-\varrho_1 T](\vartheta)=\theta,\\ R^{G(\cdot,\cdot)}_{\varrho_2,N(\cdot,\cdot)}[H(\varphi_1,\psi_1)-\varrho_2 S](\theta)=\vartheta. \end{cases}$$

Thus by implementing Lemma 3.1, we deduce that $(\theta,\vartheta)\in\mathcal{C}_1\times\mathcal{C}_2$ solves SG$VarIncl$ (3.4).
(ii) Since $\Psi_1:\mathcal{C}_2\to\mathcal{C}_1$ is Ω_1-contraction mapping and $\Psi_2:\mathcal{C}_1\to\mathcal{C}_2$ is Ω_2-contraction mapping. Then from (3.20), we have

$$\begin{aligned} \|q_n-q\| &= \|(1-c_n)\theta_n+c_n\Psi_1(\vartheta_n)-[(1-c_n)\theta+c_n\Psi_1(\vartheta)]\|\\ &\le (1-c_n)\|\theta_n-\theta\|+c_n\|\Psi_1(\vartheta_n)-\Psi_1(\vartheta)\|\\ &\le (1-c_n)\|\theta_n-\theta\|+c_n\Omega_1\|\vartheta_n-\vartheta\|. \end{aligned} \tag{3.22}$$

$$\begin{aligned} \|\xi_n-\xi\| &= \|(1-b_n)\theta_n+b_nq_n-[(1-b_n)\theta+b_nq]\|\\ &\le (1-b_n)\|\theta_n-\theta\|+b_n\|q_n-q\|. \end{aligned} \tag{3.23}$$

$$\begin{aligned} \|\theta_{n+1}-\theta\| &= \|(1-a_n)\Psi_2(\xi_n)+a_n\Psi_2(q_n)-[(1-a_n)\Psi_2(\xi)+a_n\Psi_2(q)]\|\\ &\le (1-a_n)\|\Psi_2(\xi_n)-\Psi_2(\xi)\|+a_n\|\Psi_2(q_n)-\Psi_2(q)\|\\ &\le (1-a_n)\Omega_2\|\xi_n-\xi\|+a_n\Omega_2\|q_n-q\|\\ &\le (1-a_n)\Omega_2(1-b_n)\|\theta_n-\theta\|+\Omega_2[b_n+a_n(1-b_n)]\|q_n-q\|\\ &\le \Omega_2[(1-a_n)(1-b_n)+(1-c_n)(b_n+a_n(1-b_n))]\|\theta_n-\theta\|\\ &\quad+\Omega_1\Omega_2c_n[b_n+a_n(1-b_n)]\|\vartheta_n-\vartheta\|. \end{aligned} \tag{3.24}$$

In the similar manner, we obtain

$$\begin{aligned} \|r_n-r\| &= \|(1-c_n)\vartheta_n+c_n\Psi_2(\theta_n)-[(1-c_n)\vartheta+c_n\Psi_2(\theta)]\|\\ &\le (1-c_n)\|\vartheta_n-\vartheta\|+c_n\|\Psi_2(\theta_n)-\Psi_2(\theta)\|\\ &\le (1-c_n)\|\vartheta_n-\vartheta\|+c_n\Omega_2\|\theta_n-\theta\|. \end{aligned} \tag{3.25}$$

$$\begin{aligned} \|p_n-p\| &= \|(1-b_n)\vartheta_n+b_nr_n-[(1-b_n)\vartheta+b_nr]\|\\ &\le (1-b_n)\|\vartheta_n-\vartheta\|+b_n\|r_n-r\|. \end{aligned} \tag{3.26}$$

$$\begin{aligned} \|\vartheta_{n+1}-\vartheta\| &= \|(1-a_n)\Psi_1(p_n)+a_n\Psi_1(r_n)-[(1-a_n)\Psi_1(p)+a_n\Psi_1(r)]\|\\ &\le (1-a_n)\|\Psi_1(p_n)-\Psi_1(p)\|+a_n\|\Psi_1(r_n)-\Psi_1(r)\|\\ &\le (1-a_n)\Omega_1\|p_n-p\|+a_n\Omega_1\|r_n-r\|\\ &\le (1-a_n)\Omega_2(1-b_n)\|\theta_n-\theta\|+\Omega_2[b_n+a_n(1-b_n)]\|q_n-q\|]\\ &\le \Omega_1[(1-a_n)(1-b_n)+(1-c_n)(a_n+b_n(1-a_n))]\|\vartheta_n-\vartheta\|\\ &\quad+\Omega_1\Omega_2c_n[a_n+b_n(1-a_n)]\|\theta_n-\theta\|. \end{aligned} \tag{3.27}$$

Choose $\Omega=\max\{\Omega_1,\Omega_2\}$, where Ω_1,Ω_2 are contractions and $a_n,b_n,c_n\in[0,1]$. Combining (3.24) and (3.27), we obtain

$$\begin{aligned} \|\theta_{n+1}-\theta\|+\|\vartheta_{n+1}-\vartheta\| &\le \Omega_1[1-c_n(1-\Omega_2)]\|\theta_n-\theta\|\\ &\quad+\Omega_2[1-c_n(1-\Omega_1)]\|\vartheta_n-\vartheta\|\\ &\le \Omega[\|\theta_n-\theta\|+\|\vartheta_n-\vartheta\|]. \end{aligned} \tag{3.28}$$

Thus, from (2.2), (3.28) yields

$$\|(\theta_{n+1},\vartheta_{n+1})-(\theta,\vartheta)\|_* \le \Omega\|(\theta_n,\vartheta_n)-(\theta,\vartheta)\|_*. \tag{3.29}$$

Noting $\Omega\in(0,1)$, then by appealing the Lemma 2.5, we get

$$\lim_{n\to\infty}\|(\theta_n,\vartheta_n)-(\theta,\vartheta)\|_*=0, \tag{3.30}$$

which implies $\lim_{n\to\infty}\|\theta_n-\theta\|=\lim_{n\to\infty}\|\vartheta_n-\vartheta\|=0$. Thus, $\{(\theta_n,\vartheta_n)\}$ converges to $(\theta,\vartheta)\in\mathcal{C}_1\times\mathcal{C}_2$. □

Theorem 3.5. *For each $i\in\Gamma$; let $Q_{\mathcal{C}_i}:\mathcal{B}\to\mathcal{C}_i$ be sunny nonexpansive retractions and the mappings $\varphi_i,\psi_i,\zeta_i,\xi_i,H,G,S,T$ are identical as in Theorem 3.4. Suppose that the constants $\varrho_i>0$ comply with the following inequalities:*

$$\begin{aligned}&\sqrt{1-2(\mu_1\eta_1^2-\gamma_1\sigma_1^2)+2c_1^2\nu_1^2}+\sqrt{1-2\varrho_2\kappa_1+2c_1^2\varrho_2^2\varsigma_1^2}<1,\\&\sqrt{1-2(\mu_2\eta_2^2-\gamma_2\sigma_2^2)+2c_2^2\nu_2^2}+\sqrt{1-2\varrho_1\kappa_2+2c_2^2\varrho_1^2\varsigma_2^2}<1,\\&1+2c_i^2\nu_i^2>2(\mu_i\eta_i^2-\gamma_i\sigma_i^2),1+2c_2^2\varrho_1^2\varsigma_2^2>2\varrho_1\kappa_2,1+2c_1^2\varrho_2^2\varsigma_1^2>2\varrho_2\kappa_1.\end{aligned} \tag{3.31}$$

(i) *Then $(\theta,\vartheta)\in\mathcal{C}_1\times\mathcal{C}_2$ is the unique solution of $SGVarIneq$ (3.7).*

(ii) *The sequence $\{(\theta_n,\vartheta_n)\}\in\mathcal{C}_1\times\mathcal{C}_2$ generated by the following parallel iterative scheme:*

$$\begin{cases}(\theta_0,\vartheta_0)\in\mathcal{C}_1\times\mathcal{C}_2,\\ \theta_{n+1}=(1-a_n)Q_{\mathcal{C}_2}[H(\varphi_1,\psi_1)-\varrho_2S](\xi_n)\\ \quad +a_nQ_{\mathcal{C}_2}[H(\varphi_1,\psi_1)-\varrho_2S](q_n),\\ \vartheta_{n+1}=(1-a_n)Q_{\mathcal{C}_1}[G(\varphi_2,\psi_2)-\varrho_1T](p_n)\\ \quad +a_nQ_{\mathcal{C}_1}[G(\varphi_2,\psi_2)-\varrho_1T](r_n),\\ \xi_n=(1-b_n)\theta_n+b_nq_n,\ \ p_n=(1-b_n)\vartheta_n+b_nr_n,\\ q_n=(1-c_n)\theta_n+c_nQ_{\mathcal{C}_1}[G(\varphi_2,\psi_2)-\varrho_1T](\vartheta_n),\\ r_n=(1-c_n)\vartheta_n+c_nQ_{\mathcal{C}_2}[H(\varphi_1,\psi_1)-\varrho_2S](\theta_n),\end{cases} \tag{3.32}$$

converges strongly to (θ,ϑ).

Proof. (i) Define $g_1=:Q_{\mathcal{C}_1}[G(\varphi_2,\psi_2)-\varrho_1T]$ and $g_2=:Q_{\mathcal{C}_2}[H(\varphi_1,\psi_1)-\varrho_2S]$. Since $Q_{\mathcal{C}_1}$ and $Q_{\mathcal{C}_2}$ are sunny nonexpansive, then one can deduce from Lemma 2.18 and (3.31) that $g_1:\mathcal{C}_2\to\mathcal{C}_1$ is L_1-contraction mapping, where

$$L_1=\sqrt{1-2(\mu_2\eta_2^2-\gamma_2\sigma_2^2)+2c_2^2\nu_2^2}+\sqrt{1-2\varrho_1\kappa_2+2c_2^2\varrho_1^2\varsigma_2^2}, \tag{3.33}$$

and $g_2:\mathcal{C}_1\to\mathcal{C}_2$ is L_2-contraction mapping, where

$$L_2=\sqrt{1-2(\mu_1\eta_1^2-\gamma_1\sigma_1^2)+2c_1^2\nu_1^2}+\sqrt{1-2\varrho_2\kappa_1+2c_1^2\varrho_2^2\varsigma_1^2}. \tag{3.34}$$

From Proposition 2.19, it follows that there exists unique element $(\theta,\vartheta)\in\mathcal{C}_1\times\mathcal{C}_2$ such that $g_2(\theta)=\vartheta$ and $g_1(\vartheta)=\theta$. Thus, the required result can be obtained by invoking Lemma 2.17.

(ii) Since $g_1:\mathcal{C}_2\to\mathcal{C}_1$ is L_1-contraction and $g_2:\mathcal{C}_1\to\mathcal{C}_2$ is L_2-contraction. Then following the steps as in (3.22)-(3.27) and choosing $L=\max\{L_1,L_2\}$, we obtain

$$\|\theta_{n+1}-\theta\|+\|\vartheta_{n+1}-\vartheta\|\le L[\|\theta_n-\theta\|+\|\vartheta_n-\vartheta\|]. \tag{3.35}$$

Thus, from (2.2), (3.35) yields

$$\|(\theta_{n+1}, \vartheta_{n+1}) - (\theta, \vartheta)\|_* \le L\|(\theta_n, \vartheta_n) - (\theta, \vartheta)\|_*. \tag{3.36}$$

Noting $L \in (0,1)$, then by appealing the Lemma 2.5, we get

$$\lim_{n\to\infty} \|(\theta_n, \vartheta_n) - (\theta, \vartheta)\|_* = 0, \tag{3.37}$$

which implies $\lim_{n\to\infty} \|\theta_n - \theta\| = \lim_{n\to\infty} \|\vartheta_n - \vartheta\| = 0$. Thus, $\{(\theta_n, \vartheta_n)\}$ converges to $(\theta, \vartheta) \in \mathcal{C}_1 \times \mathcal{C}_2$. □

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