Journal of Nonlinear and Convex Analysis Volume 25, Number 11, 2024, 2843–2862



COMPLEMENTARY RESULTS FOR *h*-CONVEX FUNCTIONS WITH APPLICATIONS

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ABSTRACT. In this paper, we present several new properties of h-convex functions in a way that complements those known properties for convex functions.

The obtained results include, but are not limited to, Mercer-type inequalities, gradient inequalities, Jensen-type inequalities, Mean-like bounds, Hermite-Hadamard inequalities, external behavior, and super-additive inequalities.

The obtained results, then, are employed to obtain some applications related to matrix inequalities, including unital positive mappings, weak majorization, and trace inequalities that generalize the celebrated Klein inequality.

1. INTRODUCTION

In [31], Varošanec introduced the notion of h-convexity. Let I, J be intervals in \mathbb{R} , $[0,1] \subseteq J$ and let $h: J \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. We say that $f: I \to \mathbb{R}$ is an h-convex function, if f is non-negative and for all $a, b \in I$ and $0 < \nu < 1$, we have

(1.1)
$$f((1-\nu)a+\nu b) \le h(1-\nu)f(a) + h(\nu)f(b).$$

We point out that the original definition in [31] required $(0,1) \subseteq J$. However, we will need h to be defined on [0,1], following [23].

For $\nu = \frac{1}{2}$, (1.1) reduces to the following inequality

(1.2)
$$f\left(\frac{a+b}{2}\right) \le h\left(\frac{1}{2}\right)(f(a)+f(b))$$

If $0 < h\left(\frac{1}{2}\right) \leq \frac{1}{2}$ and $f: I \to \mathbb{R}$ is a continuous *h*-convex function, then *f* is convex. This follows from Jensen's theorem [22]. When $h(\nu) = \nu$, the *h*-convex *f* becomes convex and positive. For $h(t) = \frac{1}{t}$ in (1.1) we say that *f* is a Godunova-Levin function [10]. For $h(t) = t^s$, $s \in (0, 1]$ and $t \in [0, 1]$ in (1.1) we say that $f : [0, \infty) \to [0, \infty)$ is *s*-convex (in the second sense) function [10]. This concept was introduced by Breckner [6].

If the inequality in (1.1) is reversed, then f is said to be h-concave.

A function $h: J \to \mathbb{R}$ is said to be a supermultiplicative function if $h(xy) \ge h(x)h(y)$ for all $x, y \in J$ with $xy \in J$. A function $h: J \to \mathbb{R}$ is said to be a superadditive function if $h(x+y) \ge h(x) + h(y)$ for all $x, y \in J$ with $x+y \in J$.

²⁰²⁰ Mathematics Subject Classification. Primary 26B25, 26A51, 26D15; Secondary 47B15.

Key words and phrases. h-convex function, Jensen inequality, Hermite–Hadamard inequality, majorization.

^{*}This research is supported by a grant (JSPS KAKENHI, Grant Number: 21K03341).

In [12], Ighachane and Bouchangour found a result that generalizes another important result due to Sababheh [26], as follows. If f is a positive h-convex function for a non-negative supermultiplicative and superadditive function h, then we have

(1.3)
$$h\left(\frac{\nu}{\mu}\right) \leq \frac{h(1-\nu)f(a) + h(\nu)f(b) - f((1-\nu)a + \nu b)}{h(1-\mu)f(a) + h(\mu)f(b) - f((1-\nu)a + \nu b)} \leq h\left(\frac{1-\nu}{1-\mu}\right),$$

where $0 < \nu \leq \mu < 1$.

Let w_1, \ldots, w_n be positive real numbers $(n \ge 2)$ with $\sum_{i=1}^n w_i = 1$ and h a non-negative supermultiplicative function. If f is an h-convex function, then

(1.4)
$$f\left(\sum_{i=1}^{n} w_i x_i\right) \le \sum_{i=1}^{n} h\left(w_i\right) f(x_i)$$

for all $x_1, \ldots, x_n \in I$. This is the Jensen-type inequality for an *h*-convex function [31]. We know that Jensen's classical inequality is the following:

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \le \sum_{i=1}^{n} w_i f(x_i)$$

for all $x_1, \ldots, x_n \in I$, where f is a convex function on I. In [21], Mercer showed an inequality of Jensen type given by

(1.5)
$$f\left(a+b-\sum_{i=1}^{n}w_{i}x_{i}\right) \leq f(a)+f(b)-\sum_{i=1}^{n}w_{i}f(x_{i})$$

for all $x_1, \ldots, x_n \in [a, b]$, where f is a convex function on [a, b].

In [28], Sarikaya *et al.* or Bombardelli and Varošanec [5] proved the Hermite– Hadamard–Fejér inequalities for an h–convex function. Namely, when $f : [a, b] \to \mathbb{R}$ and h is Riemann integrable on [0, 1] with $h\left(\frac{1}{2}\right) > 0$, we obtain the Hermite– Hadamard inequality for an h–convex function f, as follows

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \left(f(a)+f(b)\right)\int_0^1 h(t)dt.$$

We remark that, if h is a non-negative function such that $h(\nu) \geq \nu$ for any $\nu \in [0,1]$ and f is a non-negative convex function on I, then f is an h-convex function on I. If $h(\nu) \leq \nu$ for any $\nu \in [0,1]$, then any non-negative function f, which is h-convex on I is a convex function on I.

In [23], Olbrýs gave a characterization of h-convex functions under the condition $h(\nu)+h(1-\nu) = 1$, where $\nu \in [0, 1]$. In [25], Rostamian Delavar *et al.* investigated a characterization of an h-convex function via Hermite-Hadamard inequality related to the h-convex functions. It is determined under what conditions a function is h-convex if it satisfies the h-convex version of Hermite-Hadamard inequality. Other properties of h-convexity are given in [2] and [8]. Recently, in [13], Jin *et al.* gave other characterizations of h-convex functions and provided some basic applications.

We refer the interested reader to [2-5, 7, 8, 11-14, 16-21, 23, 25-30] as a list of references that treated *h*-convex functions and other types of convex functions, with related applications like the Hermite-Hadamard inequality and others.

Our target in this paper is to discuss further properties of h-convex functions in a way that complements those existing properties and aligns with our knowledge about convex functions. For example, we will extend the Mercer inequality (1.5) to the context of h-convex functions as follows:

$$\frac{1}{h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) - f\left(\sum_{i=1}^{n} w_{i}x_{i}\right)$$

$$\leq f\left(a+b-\sum_{i=1}^{n} w_{i}x_{i}\right)$$

$$\leq (f(a)+f(b))\sum_{i=1}^{n} h(w_{i})\left(h\left(\frac{b-x_{i}}{b-a}\right) + h\left(\frac{x_{i}-a}{b-a}\right)\right) - \sum_{i=1}^{n} h(w_{i})f(x_{i}).$$

Another interesting result will be the h-convex version of the well-known gradient inequality

(1.6)
$$f(a) + f'(a)(b-a) \le f(b),$$

valid for the differentiable convex function $f : I \to \mathbb{R}$, where $a, b \in I$. Using the obtained gradient inequality, we will be able to present a Jensen inequality for *h*-convex functions that is simpler than (1.4). Many other results will be shown too.

As applications of the obtained results, we discuss possible matrix versions that include unital positive mappings, weak majorization, and traces.

2. Main results

In this section, we present our results on h-convex functions. Then, we present the possible application in the matrix setting.

First, we have the following simple Merer–type inequality that will enable us to obtain the general form of Mercer–type.

We emphasize that according to the definition of h-convex functions, if f is hconvex, then $f, h \ge 0$ and h is defined on an interval that contains [0, 1].

Lemma 2.1. Let $f : [a,b] \to [0,\infty)$ be an h-convex function, a < x < b and $h\left(\frac{1}{2}\right) > 0$. Then

(2.1)
$$\frac{1}{h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) - f(x) \le f\left(a+b-x\right) \le \left(h\left(\frac{b-x}{b-a}\right) + h\left(\frac{x-a}{b-a}\right)\right)(f(a)+f(b)) - f(x).$$

Proof. Using inequality (1.2) we deduce

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{a+b-x+x}{2}\right) \le h\left(\frac{1}{2}\right) \left(f\left(a+b-x\right) + f\left(x\right)\right).$$

This proves the first desired inequality. For the second inequality, let a + b - x = y, so that a + b = x + y. Since a < x < b, there is $\nu = \frac{b-x}{b-a} \in (0,1)$ such that $x = \nu a + (1 - \nu)b$, which implies also $y = (1 - \nu)a + \nu b$. Using *h*-convexity of the function *f*, we have

$$f(x) \le h(\nu) f(a) + h(1 - \nu) f(b)$$

and

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 $f(y) \le h(1-\nu) f(a) + h(\nu) f(b),$

where $\nu \in (0, 1)$. Adding the above inequalities, we get the second desired inequality of the statement. This completes the proof.

Now we are ready to present the Mercer-type inequality for h-convex functions.

Theorem 2.2. Let $f : [a, b] \to [0, \infty)$ be an *h*-convex function, where *h* is a non-negative super-multiplicative function with $h\left(\frac{1}{2}\right) > 0$. If $a < x_i < b$ for i = 1, ..., n and $n \ge 2$, then

$$\frac{1}{h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) - f\left(\sum_{i=1}^{n} w_{i}x_{i}\right)$$

$$\leq f\left(a+b-\sum_{i=1}^{n} w_{i}x_{i}\right)$$

$$\leq (f(a)+f(b))\sum_{i=1}^{n} h(w_{i})\left(h\left(\frac{b-x_{i}}{b-a}\right) + h\left(\frac{x_{i}-a}{b-a}\right)\right) - \sum_{i=1}^{n} h(w_{i})f(x_{i})$$

for all $w_i \in [0, 1]$, with $\sum_{i=1}^{n} w_i = 1$.

Proof. In (2.1), let $x = \sum_{i=1}^{n} w_i x_i$. Then we obtain

$$\frac{1}{h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) - f\left(\sum_{i=1}^{n} w_i x_i\right) \le f\left(a+b-\sum_{i=1}^{n} w_i x_i\right).$$

Further, letting $x = x_i$ in (2.1), we deduce

(2.2)
$$f(a+b-x_i) \le (h(\nu_i)+h(1-\nu_i))(f(a)+f(b))-f(x_i),$$

where $\nu_i = \frac{b-x_i}{b-a} \in (0,1)$ and $x_i = \nu_i a + (1-\nu_i)b$, $i = 1, \ldots, n$ and $n \ge 2$. Thus, we have

$$f\left(a+b-\sum_{i=1}^{n} w_{i}x_{i}\right) = f\left(\sum_{i=1}^{n} w_{i}(a+b-x_{i})\right)$$

$$\leq \sum_{i=1}^{n} h\left(w_{i}\right) f(a+b-x_{i}) \quad (by (1.4))$$

$$\leq (f(a)+f(b)) \sum_{i=1}^{n} h(w_{i}) \left(h(\nu_{i})+h(1-\nu_{i})\right)$$

$$-\sum_{i=1}^{n} h(w_{i}) f(x_{i}) \quad (by (2.2)).$$

This completes the proof.

In what follows, we will give a result similar to inequality (1.3), but without the condition $\nu \leq \mu$. We should remark that such results are applied in matrix inequalities, as seen in [12, 26].

Theorem 2.3. Let $f : [a,b] \to [0,\infty)$ be an h-convex function, where h is a supermultiplicative and super-additive function. Then

$$\begin{split} h\left(m\right)\left(h\left(\mu\right)f\left(a\right)+h\left(1-\mu\right)f\left(b\right)-f\left(\mu a+(1-\mu)b\right)\right)\\ &\leq h\left(\nu\right)f\left(a\right)+h\left(1-\nu\right)f\left(b\right)-f\left(\nu a+(1-\nu)b\right)\\ &\leq \frac{1}{h\left(\frac{1}{M}\right)}\left(h\left(\mu\right)f\left(a\right)+h\left(1-\mu\right)f\left(b\right)-f\left(\mu a+(1-\mu)b\right)\right), \end{split}$$

where $0 < \nu, \mu < 1, m = \min\{\frac{\nu}{\mu}, \frac{1-\nu}{1-\mu}\}$ and $M = \max\{\frac{\nu}{\mu}, \frac{1-\nu}{1-\mu}\}$, provided that $h(\frac{1}{M}) \neq 0$.

Proof. Since $m = \min\left\{\frac{\nu}{\mu}, \frac{1-\nu}{1-\mu}\right\}$, we deduce that $\nu - \mu m \ge 0$ and $1-\nu - (1-\mu)m \ge 0$. Using the following equality $m + (\nu - \mu m) + (1 - \nu - (1 - \mu)m) = 1$ and inequality (1.4) for three terms $(x_1 = \mu a + (1 - \mu)b, x_2 = a, x_3 = b)$, we obtain

$$\begin{split} h(m)f(\mu a + (1-\mu)b) + h(\nu - \mu m)f(a) + h(1-\nu - (1-\mu)m)f(b) \\ \geq f(m(\mu a + (1-\mu)b) + (\nu - \mu m)a + (1-\nu - (1-\mu)m)b) \\ = f(\nu a + (1-\nu)b). \end{split}$$

It follows that

$$\begin{split} h\left(\nu\right) f\left(a\right) + h\left(1-\nu\right) f\left(b\right) - f\left(\nu a + (1-\nu)b\right) \\ \geq h\left(m\right) \left(h\left(\mu\right) f\left(a\right) + h\left(1-\mu\right) f\left(b\right) - f\left(\mu a + (1-\mu)b\right)\right) \\ + f(a) \left(h(\nu) - h(m)h(\mu) - h(\nu - \mu m)\right) \\ + f(b) \left(h(1-\nu) - h(m)h(1-\mu) - h(1-\nu - (1-\mu)m)\right). \end{split}$$

But, the function h is a non–negative super–multiplicative and super–additive function, so we find

$$h(\nu) = h(\nu - m\mu + m\mu) \ge h(\nu - m\mu) + h(m\mu) \ge h(\nu - m\mu) + h(m)h(\mu).$$

Similarly, we obtain

$$h(1-\nu) - h(m)h(1-\mu) - h(1-\nu - (1-\mu)m) \ge 0.$$

Because, by assumption, the function f is non-negative, we deduce

$$h(\nu) f(a) + h(1-\nu) f(b) - f(\nu a + (1-\nu)b)$$

$$\geq h(m) (h(\mu) f(a) + h(1-\mu) f(b) - f(\mu a + (1-\mu)b)).$$

Further, since $M = \max\left\{\frac{\nu}{\mu}, \frac{1-\nu}{1-\mu}\right\}$, we have $\mu - \frac{\nu}{M} \ge 0$ and $1 - \mu - \frac{1-\nu}{M} \ge 0$. Using the following equality $\frac{1}{M} + (\mu - \frac{\nu}{M}) + (1 - \mu - \frac{1-\nu}{M}) = 1$ and inequality (1.4) for

three terms $(x_1 = \nu a + (1 - \nu)b, x_2 = a, x_3 = b)$, we deduce

$$h\left(\frac{1}{M}\right)f(\nu a + (1-\nu)b) + h\left(\mu - \frac{\nu}{M}\right)f(a) + h\left(1 - \mu - \frac{1-\nu}{M}\right)f(b)$$

$$\geq f\left(\frac{1}{M}(\nu a + (1-\nu)b) + \left(\mu - \frac{\nu}{M}\right)a + \left(1 - \mu - \frac{1-\nu}{M}\right)b\right)$$

$$= f(\mu a + (1-\mu)b).$$

This means that

$$\begin{split} h\left(\mu\right) f\left(a\right) + h\left(1-\mu\right) f\left(b\right) - f\left(\mu a + (1-\mu)b\right) \\ &\geq h\left(\frac{1}{M}\right) \left(h\left(\nu\right) f\left(a\right) + h\left(1-\nu\right) f\left(b\right) - f\left(\nu a + (1-\nu)b\right)\right) \\ &+ f(a)\left(h(\mu) - h\left(\frac{1}{M}\right) h(\nu) - h\left(\mu - \frac{\nu}{M}\right)\right) \\ &+ f(b)\left(h(1-\mu) - h\left(\frac{1}{M}\right) h(1-\nu) - h\left(1-\mu - \frac{1-\nu}{M}\right)\right) \end{split}$$

We know that h is a non–negative super–multiplicative and super–additive function. Thus, we have

$$h(\mu) = h\left(\mu - \frac{\nu}{M} + \frac{\nu}{M}\right) \ge h\left(\mu - \frac{\nu}{M}\right) + h\left(\frac{\nu}{M}\right) \ge h\left(\mu - \frac{\nu}{M}\right) + h\left(\frac{1}{M}\right)h(\nu).$$

Similarly, we find

$$h(1-\mu) - h\left(\frac{1}{M}\right)h(1-\nu) - h\left(1-\mu - \frac{1-\nu}{M}\right) \ge 0$$

Since the function f is non–negative, we deduce

$$h(\mu) f(a) + h(1-\mu) f(b) - f(\mu a + (1-\mu)b)$$

$$\geq h\left(\frac{1}{M}\right) (h(\nu) f(a) + h(1-\nu) f(b) - f(\nu a + (1-\nu)b)).$$

This completes the proof.

Remark 2.4. It is easy to see that if $\nu \leq \mu$ in Theorem 2.3, then we have $h(m) = h\left(\frac{\nu}{\mu}\right)$ and $h\left(\frac{1}{M}\right) = h\left(\frac{1-\mu}{1-\nu}\right)$.

The gradient inequality (1.6) has been an important inequality that characterizes convex functions. We present a possible gradient inequality for *h*-convex functions in the following.

Theorem 2.5. Let $f : I \to [0, \infty)$ be a differentiable *h*-convex function on *I*, where $h: J \supseteq [0,1] \to \mathbb{R}$ is differentiable at 0 and 1. Then for all $a, b \in I$,

$$(b-a) f'(a) + h'(1) f(a) \le h'(0) f(b)$$

and

$$h'(1) f(b) - (b - a) f'(b) \le h'(0) f(a)$$

provided that h(0) = 0 and h(1) = 1. The above inequalities are reversed if f is h-concave.

Proof. It follows from (1.1) that

$$\frac{f\left(a+\nu\left(b-a\right)\right)-h\left(1-\nu\right)f\left(a\right)}{\nu} \leq \frac{h\left(\nu\right)}{\nu}f\left(b\right),$$

when $\nu \in (0, 1)$. Now, by the assumptions h(0) = 0 and h(1) = 1, one can write

$$\lim_{\nu \to 0} \frac{f(a + \nu (b - a)) - h(1 - \nu) f(a)}{\nu}$$

=
$$\lim_{\nu \to 0} \left(f'(a + \nu (b - a)) (b - a) + h'(1 - \nu) f(a) \right)$$

=
$$f'(a) (b - a) + h'(1) f(a)$$

$$\leq \lim_{\nu \to 0} \frac{h(\nu)}{\nu} f(b) = h'(0) f(b),$$

which completes the proof of the first inequality. In the same way, from (1.1) we obtain

$$\frac{f\left(a+\nu\left(b-a\right)\right)-h\left(\nu\right)f\left(b\right)}{1-\nu}\leq\frac{h\left(1-\nu\right)}{1-\nu}f\left(a\right),$$

when $\nu \in (0,1)$. Taking the limit for $\nu \to 1$ above inequality, we deduce the second inequality of the statement.

We employ Theorem 2.5 to show a Jensen–type inequality for h–convex functions. We remark here that the Jensen inequality (1.4) is given for super–multiplicative functions. In the following, we present Jensen inequality with a possible reverse without imposing the super–multiplicativity condition on h.

Theorem 2.6. Let f and h be as in Theorem 2.5 and let $x_1, x_2, \ldots, x_n \in I$. If w_1, w_2, \ldots, w_n are positive scalars such that $\sum_{i=1}^n w_i = 1$, then

$$h'(1) f\left(\sum_{i=1}^{n} w_i x_i\right) \le h'(0) \sum_{i=1}^{n} w_i f(x_i),$$

and

$$\left(\left(\sum_{i=1}^{n} w_i x_i\right) \left(\sum_{i=1}^{n} w_i f'\left(x_i\right)\right) - \sum_{i=1}^{n} w_i x_i f'\left(x_i\right)\right) + h'\left(1\right) \sum_{i=1}^{n} w_i f\left(x_i\right)$$
$$\leq h'\left(0\right) f\left(\sum_{i=1}^{n} w_i x_i\right).$$

Proof. For $i = 1, \ldots, n$, if $x_i \in I$, then

(2.3)
$$f'(a)(x_i - a) + h'(1) f(a) \le h'(0) f(x_i)$$

for any $a \in I$, thanks to the first inequality from Theorem 2.5. Multiplying (2.3) by w_i (i = 1, ..., n), then adding over *i* from 1 to *n*, we infer that

$$f'(a)\left(\sum_{i=1}^{n} w_i x_i - a\right) + h'(1) f(a) \le h'(0) \sum_{i=1}^{n} w_i f(x_i).$$

Letting $a = \sum_{i=1}^{n} w_i x_i$ in the above inequality yields

$$h'(1) f\left(\sum_{i=1}^{n} w_i x_i\right) \le h'(0) \sum_{i=1}^{n} w_i f(x_i),$$

which proves the second inequality. For the first inequality, we have

(2.4)
$$bf'(x_i) - f'(x_i) x_i + h'(1) f(x_i) \le h'(0) f(b), 1 \le i \le n,$$

due to Theorem 2.5. Multiplying (2.4) by w_i (i = 1, ..., n), then adding over *i* from 1 to *n*, we get

$$b\sum_{i=1}^{n} w_i f'(x_i) - \sum_{i=1}^{n} w_i f'(x_i) x_i + h'(1) \sum_{i=1}^{n} w_i f(x_i) \le h'(0) f(b).$$

Allowing $b = \sum_{i=1}^{n} w_i x_i$ in the above inequality, it makes

$$\left(\left(\sum_{i=1}^{n} w_i x_i\right) \left(\sum_{i=1}^{n} w_i f'\left(x_i\right)\right) - \sum_{i=1}^{n} w_i x_i f'\left(x_i\right)\right) + h'\left(1\right) \sum_{i=1}^{n} w_i f\left(x_i\right)$$
$$\leq h'\left(0\right) f\left(\sum_{i=1}^{n} w_i x_i\right).$$

This completes the proof of the theorem.

For a, b > 0 and $0 \le \nu \le 1$, the weighted arithmetic mean is denoted by $a\nabla_{\nu}b$, where $a\nabla_{\nu}b := (1 - \nu)a + \nu b$. We use the symbols ∇ instead of $\nabla_{1/2}$. We also introduce the *h*-quasi-weighted arithmetic mean $a\nabla_{\nu}^{h(\nu)}b := h(1 - \nu)a + h(\nu)b$ to simplify the expressions of our results. We see that $\nabla_{\nu}^{\nu} = \nabla_{\nu}$.

In the following theorem, we present an integral inequality that extends many results from the literature.

Theorem 2.7. Let $a, b \in I$ such that a, b > 0 and $0 \le \nu \le 1$ Let $f : I \to [0, \infty)$ be an integrable *h*-convex function and let $h : [0, 1] \to [0, \infty)$. Then

$$\begin{split} f\left((1-\nu)\,a+\nu b\right) &\leq \left(\int_{0}^{1}f\left(a\nabla_{\nu t}b\right)dt\right)\nabla_{\nu}^{h(\nu)}\left(\int_{0}^{1}f\left(b\nabla_{(1-\nu)t}a\right)dt\right) \\ &\leq f(a)\int_{0}^{1}h(1-\nu t)\nabla_{\nu}^{h(\nu)}h((1-\nu)t)dt \\ &\quad +f(b)\int_{0}^{1}h(\nu t)\nabla_{\nu}^{h(\nu)}h(1-(1-\nu)t)dt. \end{split}$$

Proof. Since

$$(1 - \nu) a + \nu b = ((1 - \nu) \nu (tb + (1 - t) a)) + (1 - \nu)^2 a$$
$$+ \nu (1 - \nu) ((1 - t) b + ta) + \nu^2 b$$

with the definition of h-convex function, we have

$$\begin{split} f\left((1-\nu)\,a+\nu b\right) &\leq h\left(1-\nu\right) \int_{0}^{1} f\left(\nu\left(tb+(1-t)\,a\right)+(1-\nu)\,a\right) dt \\ &\quad + h\left(\nu\right) \int_{0}^{1} f\left((1-\nu)\left((1-t)\,b+ta\right)+\nu b\right) dt \\ &= h\left(1-\nu\right) \int_{0}^{1} f\left(\nu\left(b-a\right)t+a\right) dt \\ &\quad + h\left(\nu\right) \int_{0}^{1} f\left((1-\nu)(a-b)t+b\right) dt \\ &= h\left(1-\nu\right) \int_{0}^{1} f\left(a\nabla_{\nu t}b\right) dt + h\left(\nu\right) \int_{0}^{1} f\left(b\nabla_{(1-\nu)t}a\right) dt \end{split}$$

Now, using the fact that f is h-convex, and noting the definition of ∇_t , we can write

$$f((1-\nu)a+\nu b) \le h(1-\nu) \int_0^1 (h(t\nu)f(b) + h(1-t\nu)f(a)) dt + h(\nu) \int_0^1 (h((1-\nu)t)f(a) + h(1-(1-\nu)t)f(b)) dt = f(a) \int_0^1 (h(1-\nu)h(1-\nu t) + h(\nu)h((1-\nu)t)) dt + f(b) \int_0^1 (h(1-\nu)h(\nu t) + h(\nu)h(1-(1-\nu)t)) dt.$$

This completes the proof.

Remark 2.8.

(i) The right hand side of the inequalities in Theorem 2.7 can be written by

$$f(a)R_1(\nu) + f(b)R_2(\nu), \quad (0 < \nu < 1),$$

where

$$R_1(\nu) := \frac{h(\nu)}{1-\nu} \int_0^{1-\nu} h(t)dt + \frac{h(1-\nu)}{\nu} \int_{1-\nu}^1 h(t)dt$$

and

$$R_2(\nu) := \frac{h(1-\nu)}{\nu} \int_0^{\nu} h(t)dt + \frac{h(\nu)}{1-\nu} \int_{\nu}^1 h(t)dt$$

Note that $R_1(\nu) = 1 - \nu$ and $R_2(\nu) = \nu$ if h(x) := x. Thus, if we take h(x) := x in Theorem 2.7, then the inequalities in Theorem 2.7 are reduced to [24, Theorem 2.1]:

$$f((1-\nu)a + \nu b) \le C_{f,\nu}(a,b) \le (1-\nu)f(a) + \nu f(b),$$

where

$$C_{f,\nu}(a,b) := \left(\int_0^1 f(a\nabla_{\nu t}b)dt\right)\nabla_{\nu}\left(\int_0^1 f(b\nabla_{(1-\nu)t}a)dt\right).$$

(ii) If we take $\nu := \frac{1}{2}$ in Theorem 2.7, then the inequalities are reduced to

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right) \le \int_0^1 f\left((1-t)a+tb\right)dt \le (f(a)+f(b))\int_0^1 h(t)dt,$$

which gives the original Hermite–Hadamard inequality for the case h(x) := x.

Allowing the derivatives, we have the following interesting Hermite–Hadamard inequality for h-convex functions. When h(x) = x, this reduces to the original version of Hermite–Hadamard inequality for convex functions.

Proposition 2.9. Let $f: I \to [0, \infty)$ be a differentiable *h*-convex function, where the function $h: J \to \mathbb{R}$ is differentiable at 0 and 1 with h(0) = 0 and h(1) = 1. If h'(0), h'(1) > 0, then for all $a, b \in I$,

$$\frac{h'(1)}{h'(0)}f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_{a}^{b} f(t) \, d\nu \le \frac{h'(0)}{h'(1)} \frac{f(a)+f(b)}{2}.$$

Proof. It follows from the first inequality in Theorem 2.6 that

(2.5)
$$h'(1) f((1-\nu) a + \nu b) \le h'(0) ((1-\nu) f(a) + \nu f(b))$$

for $0 \le \nu \le 1$. This indicates, by taking integral over $0 \le \nu \le 1$,

(2.6)
$$h'(1) \int_0^1 f((1-\nu)a+\nu b) d\nu \le h'(0) \left(\frac{f(a)+f(b)}{2}\right).$$

Moreover, since

$$h'(1) f\left(\frac{a+b}{2}\right) \le h'(0) \left(\frac{f(a)+f(b)}{2}\right),$$

we conclude, by substituting a and b with $(1-\nu)a+\nu b$ and $(1-\nu)b+\nu a$, respectively,

(2.7)
$$h'(1) f\left(\frac{a+b}{2}\right) \le h'(0) \int_0^1 f\left((1-\nu) a + \nu b\right) d\nu,$$

due to

$$\int_{0}^{1} f((1-\nu)a + \nu b) d\nu = \int_{0}^{1} f((1-\nu)b + \nu a) d\nu$$

Therefore, if h'(0), h'(1) > 0 and using relations (2.6) and (2.7), then we deduce

$$\frac{h'(1)}{h'(0)}f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_{a}^{b} f(t) \, d\nu \le \frac{h'(0)}{h'(1)} \frac{f(a)+f(b)}{2}$$

as desired.

We give supplemental inequalities for (2.5).

Lemma 2.10. Let $f : I \to [0, \infty)$ be a differentiable h-convex function, where the function $h : J \to \mathbb{R}$ is differentiable at 0 and 1, with h(0) = 0 and h(1) = 1. Then for all $a, b \in I$,

(2.8)
$$h'(1)(1-\nu)f(a) + h'(0)\nu f(b) \le h'(0)f((1-\nu)a + vb), \quad (\nu < 0)$$

and

(2.9)
$$h'(0)(1-\nu)f(a) + h'(1)\nu f(b) \le h'(0)f((1-\nu)a + vb), \quad (\nu > 1).$$

Proof. Assume that $\mu > 0$. Then by (2.5), we have

$$h'(1) f(a) = h'(1) f\left(\frac{1}{1+\mu} \left((1+\mu) a - \mu b\right) + \frac{\mu}{1+\mu} b\right)$$

$$\leq h'(0) \left(\frac{1}{1+\mu} f\left((1+\mu) a - \mu b\right) + \frac{\mu}{1+\mu} f(b)\right).$$

Then, we prove that

$$h'(1)(1+\mu)f(a) - h'(0)\mu f(b) \le h'(0)f((1+\mu)a - \mu b).$$

Putting $\mu := -\nu$ with $\nu < 0$, we have (2.8). Assume that $\mu < -1$. Then by (2.5), we have

$$h'(1)f(b) = h'(1)f\left(-\frac{1}{\mu}\left((1+\mu)a - \mu b\right) + \frac{1+\mu}{\mu}a\right)$$

$$\leq h'(0)\left(-\frac{1}{\mu}f\left((1+\mu)a - \mu b\right) + \frac{1+\mu}{\mu}f(a)\right),$$

which implies

$$\mu h'(1)f(b) \ge h'(0)\left((1+\mu)f(a) - f\left((1+\mu)a - \mu b\right)\right), \quad (\mu < -1).$$

Putting $\mu := -\nu$ with $\nu > 1$, we have (2.9).

Note that we both inequalities (2.8) and (2.9) recover the following inequality for convex function f:

$$(1-\nu)f(a) + \nu f(b) \le f((1-\nu)a + \nu b), \quad \nu \notin [0,1],$$

when h(x) := x.

When dealing with convex functions, it is interesting to find refinements and reverses of the existing inequalities. In the following, we present possible refinement and reverse for the Jensen inequality shown in Theorem 2.6 when n = 2.

Proposition 2.11. Let $f: I \to [0, \infty)$ be a differentiable *h*-convex function, where the function $h: J \to \mathbb{R}$ is differentiable at 0 and 1 with h(0) = 0 and h(1) = 1. Then for all $a, b \in I$,

$$h'(1) f((1-\nu) a + \nu b) \le h'(0) \left((1-\nu) f(a) + \nu f(b) - 2r \left(\frac{f(a) + f(b)}{2} - f \left(\frac{a+b}{2} \right) \right) \right),$$

and

(2.10)
$$(1-\nu) h'(0) f(a) + \nu f(b) - 2R\left(\frac{h'(0) f(a) + f(b)}{2} - h'(1) f\left(\frac{a+b}{2}\right)\right) \\ \leq h'(0) f((1-\nu) a + \nu b)$$

where $R = \max\{\nu, 1 - \nu\}$, $r = \min\{\nu, 1 - \nu\}$, and $0 \le \nu \le 1$.

Proof. Let $0 \le \nu \le 1/2$. In this case, we have by (2.5) that

$$h'(1) f((1 - \nu) a + \nu b)$$

= $h'(1) f\left((1 - 2\nu) a + 2\nu \frac{a + b}{2}\right)$
 $\leq h'(0) \left((1 - 2\nu) f(a) + 2\nu f\left(\frac{a + b}{2}\right)\right)$
= $h'(0) \left((1 - \nu) f(a) + \nu f(b) - 2r\left(\frac{f(a) + f(b)}{2} - f\left(\frac{a + b}{2}\right)\right)\right).$

The same inequality holds when $1/2 \le \nu \le 1$. This completes the proof of the first inequality.

To prove the second inequality, notice that by Lemma 2.10,

$$(1 - \nu) h'(0) f(a) + \nu f(b) - 2R\left(\frac{h'(0) f(a) + f(b)}{2} - h'(1) f\left(\frac{a + b}{2}\right)\right)$$

= $2\nu h'(1) f\left(\frac{a + b}{2}\right) + (1 - 2\nu) h'(0) f(a)$
 $\leq h'(0) f((1 - \nu) a + \nu b)$

for $1/2 \leq \nu \leq 1$. The same approach shows the inequality (2.10) holds when $0 \leq \nu \leq 1/2$.

Remark 2.12. From the second inequality in Theorem 2.6, we have for any $0 \le \nu \le 1$,

(2.11)

$$((1 - \nu) a + \nu b) ((1 - \nu) f'(a) + \nu f'(b)) - ((1 - \nu) a f'(a) + \nu b f'(b)) + h'(1) ((1 - \nu) f(a) + \nu f(b)) \le h'(0) f ((1 - \nu) a + \nu b).$$

In particular, we deduce

(2.12)
$$\frac{(a-b)(f'(b)-f'(a))}{4} + h'(1)\left(\frac{f(a)+f(b)}{2}\right) \le h'(0)f\left(\frac{a+b}{2}\right).$$

Taking integral over $0 \le \nu \le 1$, in (2.11), we obtain

$$\frac{(f'(b) - f'(a))(a - b)}{6} + h'(1)\left(\frac{f(a) + f(b)}{2}\right) \le h'(0)\int_0^1 f((1 - \nu)a + \nu b)\,d\nu.$$

By (2.12), we also have

$$h'(1) \int_0^1 f((1-\nu)a+\nu b) d\nu + \frac{1}{4} \int_0^1 ((2\nu-1)(b-a)) f'((1-\nu)b+\nu a) d\nu - \frac{1}{4} \int_0^1 ((2\nu-1)(b-a)) f'((1-\nu)a+\nu b) d\nu \le h'(0) f\left(\frac{a+b}{2}\right).$$

Proposition 2.13. Let f and h be as in Theorem 2.5 and let $x_1, x_2, \ldots, x_n \in I$, $y_1, y_2, \ldots, y_n \in I$. If w_1, w_2, \ldots, w_n are positive scalars, then

(2.13)
$$\sum_{i=1}^{n} w_i (x_i - y_i) f'(y_i) \le h'(0) \sum_{i=1}^{n} w_i f(x_i) - h'(1) \sum_{i=1}^{n} w_i f(y_i).$$

Proof. If we employ (1.1) for the selection $b = x_i$, $a = y_i$ (i = 1, ..., n), we may write

(2.14)
$$f'(y_i)(x_i - y_i) + h'(1) f(y_i) \le h'(0) f(x_i).$$

Multiplying (2.14) by w_i (i = 1, ..., n) and summing over *i* from 1 to *n* we may deduce (2.13).

Corollary 2.14. Let the assumptions of Proposition 2.13 hold. Let $x_1 - y_1$, $x_2 - y_2, \ldots, x_n - y_n$ and $f'(y_1), f'(y_2), \ldots, f'(y_n)$ be both non-decreasing or non-increasing. If $\sum_{i=1}^n w_i x_i = \sum_{i=1}^n w_i y_i$, then

$$h'(1) \sum_{i=1}^{n} w_i f(y_i) \le h'(0) \sum_{i=1}^{n} w_i f(x_i).$$

Proof. Chebyshev's inequality says that

(2.15)
$$\frac{1}{W_n} \left(\sum_{i=1}^n w_i a_i \right) \left(\sum_{i=1}^n w_i b_i \right) \le (\text{resp.} \ge) \sum_{i=1}^n w_i a_i b_i$$

provided that a_1, a_2, \ldots, a_n and b_1, b_2, \ldots, b_n are monotonic in the same (resp. opposite) sense, $W_n = \sum_{i=1}^n w_i > 0$. We reach

$$0 = \frac{1}{W_n} \left(\sum_{i=1}^n w_i \left(x_i - y_i \right) \right) \left(\sum_{i=1}^n w_i f'(y_i) \right) \quad \left(\text{since } \sum_{i=1}^n w_i x_i = \sum_{i=1}^n w_i y_i \right)$$

$$\leq \sum_{i=1}^n w_i f'(y_i) \left(x_i - y_i \right) \quad (by \ (2.15))$$

$$\leq h'(0) \sum_{i=1}^n w_i f(x_i) - h'(1) \sum_{i=1}^n w_i f(y_i) \quad (by \ \text{Proposition } 2.13).$$

This completes the proof.

Remark 2.15. Let $W_n = \sum_{i=1}^n w_i > 0$.

(i) If we take $y_1 = y_2 = \cdots = y_n = \sum_{i=1}^n w_i x_i / W_n$, in Proposition 2.13, we get

$$h'(1) f\left(\frac{\sum_{i=1}^{n} w_i x_i}{W_n}\right) \le \frac{h'(0)}{W_n} \sum_{i=1}^{n} w_i f(x_i).$$

(ii) If we take
$$x_1 = x_2 = \dots = x_n = \sum_{i=1}^n w_i y_i / W_n$$
, in Proposition 2.13, we get

$$\left(\frac{\sum_{i=1}^n w_i y_i}{W_n}\right) \left(\frac{\sum_{i=1}^n w_i f'(y_i)}{W_n}\right) - \frac{\sum_{i=1}^n w_i y_i f'(y_i)}{W_n}$$

$$\leq h'(0) f\left(\frac{\sum_{i=1}^n w_i y_i}{W_n}\right) - \frac{h'(1)}{W_n} \sum_{i=1}^n w_i f(y_i).$$

It is well–known that a convex function $f: [0, \infty) \to [0, \infty)$ with f(0) = 0 satisfies the super–additive behavior

$$f(a+b) \ge f(a) + f(b).$$

In the following, we present an interesting super–additive inequality for h–convex functions.

Theorem 2.16. Let $f: [0,\infty) \to [0,\infty)$ be h-convex such that f(0) = 0. If a, b > 0, then

$$f(a) + f(b) \le \left(h\left(\frac{a}{a+b}\right) + h\left(\frac{b}{a+b}\right)\right) f(a+b).$$

Proof. By (1.1), one can write

(2.16)
$$f(\nu b) \le h(1-\nu)f(0) + h(\nu)f(b) = h(\nu)f(b).$$

Using (2.16), we reach

$$f(a) = f\left(\frac{a}{a+b} \cdot (a+b)\right) \le h\left(\frac{a}{a+b}\right) f(a+b).$$

Similarly, we find

$$f(b) \le h\left(\frac{b}{a+b}\right) f(a+b)$$

Adding the last two inequalities together shows the desired inequality.

The inequality (1.1) can be written in the following format

$$f(\nu_1 a + \nu_2 b) \le h(\nu_1) f(a) + h(\nu_2) f(b),$$

provided that $\nu_1 + \nu_2 = 1$. In the following result, we present this inequality under the assumption $\nu_1 + \nu_2 \leq 1$.

Theorem 2.17. Let $f: [0,\infty) \to [0,\infty)$ be h-convex such that f(0) = 0, and let $\nu_1, \nu_2 > 0$ be such that $\nu_1 + \nu_2 \leq 1$. If a, b > 0, then

$$f(\nu_1 a + \nu_2 b) \le h(\nu_1 + \nu_2) \left(h\left(\frac{\nu_1}{\nu_1 + \nu_2}\right) f(a) + h\left(\frac{\nu_2}{\nu_1 + \nu_2}\right) f(b) \right)$$

Proof. Employing (1.1) two times, we obtain

$$f(\nu_{1}a + \nu_{2}b) = f\left((\nu_{1} + \nu_{2})\left(\frac{\nu_{1}}{\nu_{1} + \nu_{2}}a + \frac{\nu_{2}}{\nu_{1} + \nu_{2}}b\right) + (1 - (\nu_{1} + \nu_{2})) \cdot 0\right)$$

$$\leq h(\nu_{1} + \nu_{2})f\left(\frac{\nu_{1}}{\nu_{1} + \nu_{2}}a + \frac{\nu_{2}}{\nu_{1} + \nu_{2}}b\right) + h(1 - (\nu_{1} + \nu_{2}))f(0)$$

$$= h(\nu_{1} + \nu_{2})f\left(\frac{\nu_{1}}{\nu_{1} + \nu_{2}}a + \frac{\nu_{2}}{\nu_{1} + \nu_{2}}b\right)$$

$$\leq h(\nu_{1} + \nu_{2})\left(h\left(\frac{\nu_{1}}{\nu_{1} + \nu_{2}}\right)f(a) + h\left(\frac{\nu_{2}}{\nu_{1} + \nu_{2}}\right)f(b)\right),$$

desired.

as desired.

3. MATRIX INEQUALITIES

Let \mathcal{H} be a complex Hilbert space, endowed with the inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$. Let $\mathcal{B}(\mathcal{H})$ denote the C^* -algebra of all bounded linear operators on \mathcal{H} , with identity operator $I_{\mathcal{H}}$. We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is positive if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$, and then we write $T \geq 0$. If a bounded linear operator T on \mathcal{H} is positive, then there exists a unique positive bounded linear operator denoted by $T^{1/2}$ such that $T = (T^{1/2})^2$. Furthermore, the absolute value of T, denoted by $|\tilde{T}|$, is defined by $|T| = (T^*T)^{1/2}$. We remark that $|T| \ge 0$.

To study the eigenvalues of a Hermitian matrix T, we use the notation $\lambda_i(T)$ to mean the *i*-th eigenvalue of T, when written in decreasing order. A way to compare matrices is given by the Löwner partial order " \leq ". In other words, when T and S are Hermitian such that $T \leq S$, we have that $\lambda_i(T) \leq \lambda_i(S)$ for all $i \in \{1, \ldots, n\}$, which is another perspective to compare between T and S. We remark that the relation $\lambda_i(T) \leq \lambda_i(S)$, for all $i \in \{1, \ldots, n\}$ prove the inequality $\sum_{i=1}^{p} \lambda_i(T) \leq \sum_{i=1}^{p} \lambda_i(S)$ for all $p \in \{1, \ldots, n\}$. The last comparison is what we call weak majorization and is denoted by " \prec_w ". Thus, if we have $T \leq S$, then $\lambda_i(T) \leq \lambda_i(S)$ for all $i \in \{1, \ldots, n\}$, which then implies $T \prec_w S$. When \mathcal{H} is finite dimensional of dimension n, we identify $\mathcal{B}(\mathcal{H})$ with the algebra \mathcal{M}_n of all $n \times n$ complex matrices.

In the sequel, a positive linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a linear map that satisfies $\Phi(T) \geq 0$ whenever $T \geq 0$.

In the following result, we present a possible *h*-convex version of the celebrated Jensen inequality that asserts $f(\langle Tx, x \rangle) \leq \langle f(T)x, x \rangle$ whenever T is a self-adjoint operator with spectrum in the domain of the convex function f, and where $x \in \mathcal{H}$ is a unit vector.

Lemma 3.1. Let $f: I \to [0, \infty)$ be a differentiable *h*-convex function on an interval I that contains $[0, \infty)$, where $h: J \supseteq [0, 1] \to \mathbb{R}$ is differentiable at 0 and 1. Let $T_1, T_2, \ldots, T_n \in \mathcal{B}(\mathcal{H})$ be positive operators and let $\Phi_1, \Phi_2, \ldots, \Phi_n$ be positive linear map on $\mathcal{B}(\mathcal{H})$ such that $\sum_{i=1}^n \Phi_i(I_{\mathcal{H}}) = I_{\mathcal{H}}$. Then for any unit vector $x \in \mathcal{H}$,

$$h'(1) f\left(\left\langle \sum_{i=1}^{n} \Phi_i(T_i)x, x \right\rangle \right) \le h'(0) \left\langle \sum_{i=1}^{n} \Phi_i(f(T_i))x, x \right\rangle.$$

In particular, we have

$$h'(1) f(\langle Tx, x \rangle) \le h'(0) \langle f(T) x, x \rangle$$

Proof. By employing functional calculus for the self-adjoint operators T_i , we have from Theorem 2.5 that

$$f'(a) (T_i - aI_{\mathcal{H}}) + h'(1) f(a) I_{\mathcal{H}} \le h'(0) f(T_i).$$

Applying positive linear maps Φ_i and adding, it follows that

$$f'(a)\left(\sum_{i=1}^{n} \Phi_{i}(T_{i}) - aI_{\mathcal{H}}\right) + h'(1) f(a) I_{\mathcal{H}} \le h'(0) \sum_{i=1}^{n} \Phi_{i}(f(T_{i}))$$

Hence for any unit vector $x \in \mathcal{H}$,

$$f'(a)\left(\left\langle\sum_{i=1}^{n}\Phi_{i}(T_{i})x,x\right\rangle-a\right)+h'(1)f(a)\leq h'(0)\left\langle\sum_{i=1}^{n}\Phi_{i}(f(T_{i}))x,x\right\rangle.$$

Putting $a = \langle \sum_{i=1}^{n} \Phi_i(T_i) x, x \rangle$ in the above inequality, gives the desired result. \Box

Theorem 3.2. Let f and h be as in Lemma 3.1. Let $T_1, T_2, \ldots, T_n \in \mathcal{M}_n$ be positive semidefinite matrices and let $\Phi_1, \Phi_2, \ldots, \Phi_n$ be positive linear map on \mathcal{M}_n

such that $\sum_{i=1}^{n} \Phi_i(I_{\mathcal{M}_n}) = I_{\mathcal{M}_n}$. Then

$$\lambda\left(h'(1) f\left(\sum_{i=1}^{n} \Phi_{i}(T_{i})\right)\right) \prec_{w} \lambda\left(h'(0) \sum_{i=1}^{n} \Phi_{i}(f(T_{i}))\right).$$

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $\sum_{i=1}^n \Phi_i(T_i)$ and let u_1, u_2, \ldots, u_n be the corresponding orthonormal eigenvectors arranged such that $f(\lambda_1) \geq$ $f(\lambda_2) \geq \cdots \geq f(\lambda_n)$. Let $k = 1, \ldots, n$. Then using Lemma 3.1, we deduce

$$\sum_{j=1}^{k} \lambda_{j} \left(h'(1) f\left(\sum_{i=1}^{n} \Phi_{i}(T_{i})\right) \right) = \sum_{j=1}^{k} h'(1) f\left(\left\langle \left(\sum_{i=1}^{n} \Phi_{i}(T_{i})\right) u_{j}, u_{j} \right\rangle \right) \right)$$
$$\leq \sum_{j=1}^{k} h'(0) \left\langle \sum_{i=1}^{n} \Phi_{i}(f(T_{i})) u_{j}, u_{j} \right\rangle$$
$$= \sum_{j=1}^{k} \left\langle \left(h'(0) \sum_{i=1}^{n} \Phi_{i}(f(T_{i})) \right) u_{j}, u_{j} \right\rangle$$
$$\leq \sum_{j=1}^{k} \lambda_{j} \left(h'(0) \sum_{i=1}^{n} \Phi_{i}(f(T_{i})) \right),$$

where the last inequality follows from the fact that when $X \in \mathcal{M}_n$ is Hermitian, one has

$$\sum_{j=1}^{k} \lambda_j(X) = \sup \sum_{j=1}^{k} \langle X x_j, x_j \rangle,$$

where the supremum is taken over all possible choices of orthonormal vectors $\{x_1, \cdots, x_k\} \subset \mathbb{C}^n.$ That is, for $1 \le k \le n$,

$$\sum_{j=1}^{k} \lambda_j \left(h'(1) f\left(\sum_{i=1}^{n} \Phi_i(T_i)\right) \right) \leq \sum_{j=1}^{k} \lambda_j \left(h'(0) \sum_{i=1}^{n} \Phi_i(f(T_i)) \right).$$

This proves the desired result.

The following extends an outstanding result about weak majorization under convex functions to the context of h-convex functions. We refer the reader to [3] for the original version for convex functions.

Theorem 3.3. Let f and h be as in Lemma 3.1. Let $A, B \in \mathcal{M}_n$ be positive semidefinite matrices and let $0 \le \nu \le 1$. If h'(1) > 0, then

$$\lambda\left(f\left((1-\nu)A+\nu B\right)\right) \prec_{w} \lambda\left(\frac{h'(0)}{h'(1)}\left(h\left(1-\nu\right)f\left(A\right)+h\left(\nu\right)f\left(B\right)\right)\right).$$

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $(1 - \nu) A + \nu B$ and let u_1, u_2, \ldots, u_n be the corresponding orthonormal eigenvectors arranged such that $f(\lambda_1) \ge f(\lambda_2) \ge$

 $\cdots \geq f(\lambda_n)$. Let $k = 1, \ldots, n$. Then, implementing Lemma 3.1,

$$\begin{split} &\sum_{j=1}^{k} \lambda_{j} \left(f\left(\left(1 - \nu \right) A + \nu B \right) \right) \\ &= \sum_{j=1}^{k} f\left(\left(\left(\left(1 - \nu \right) A + \nu B \right) u_{j}, u_{j} \right) \right) \\ &= \sum_{j=1}^{k} f\left(\left(1 - \nu \right) \left\langle A u_{j}, u_{j} \right\rangle + \nu \left\langle B u_{j}, u_{j} \right\rangle \right) \\ &\leq \sum_{j=1}^{k} \left(h\left(1 - \nu \right) f\left(\left\langle A u_{j}, u_{j} \right\rangle \right) + h\left(\nu \right) f\left(\left\langle B u_{j}, u_{j} \right\rangle \right) \right) \\ &\leq \sum_{j=1}^{k} \left(\frac{h'\left(0 \right)}{h'\left(1 \right)} \left(h\left(1 - \nu \right) \left\langle f\left(A \right) u_{j}, u_{j} \right\rangle + h\left(\nu \right) \left\langle f\left(B \right) u_{j}, u_{j} \right\rangle \right) \right) \\ &= \sum_{j=1}^{k} \left(\left(\left(\frac{h'\left(0 \right)}{h'\left(1 \right)} \left(h\left(1 - \nu \right) f\left(A \right) + h\left(\nu \right) f\left(B \right) \right) \right) u_{j}, u_{j} \right\rangle \right) \right) \\ &= \sum_{j=1}^{k} \lambda_{j} \left(\frac{h'\left(0 \right)}{h'\left(1 \right)} \left(h\left(1 - \nu \right) f\left(A \right) + h\left(\nu \right) f\left(B \right) \right) \right) . \end{split}$$

Thus, for $1 \le k \le n$,

$$\sum_{j=1}^{k} \lambda_j \left(f \left((1-\nu) A + \nu B \right) \right) \le \sum_{j=1}^{k} \lambda_j \left(\frac{h'(0)}{h'(1)} \left(h \left(1-\nu \right) f \left(A \right) + h \left(\nu \right) f \left(B \right) \right) \right).$$

This completes the proof.

Proposition 3.4. Let $f: I \to [0, \infty)$ be a differentiable *h*-convex function on an interval I, where $h: J \supseteq [0,1] \to \mathbb{R}$ is differentiable at 0 and 1, and let $A, B \in \mathcal{M}_n$ be Hermitian matrices with spectra in I. If h'(1) > 0, then

$$h'(1)Tr\left[f\left((1-\nu)f(A)+\nu f(B)\right)\right] \le h'(0)Tr\left[h(1-\nu)f(A)+h(\nu)f(B)\right],$$

for any $0 \le \nu \le 1$.

Proof. Let $\{\phi_k\}_{k \in \{1,...,n\}}$ be an orthonormal basis of eigenvectors for $(1 - \nu)A + \nu B$. Since $Sp((1 - \nu)A + \nu B) \subset (1 - \nu)Sp(A) + \nu Sp(B)$, where $Sp(\cdot)$ denotes the

spectrum, we have

$$\begin{aligned} h'(1)Tr\left[f\left((1-\nu)f(A)+\nu f(B)\right)\right] \\ &= h'(1)\sum_{k=1}^{n} f\left(\langle ((1-\nu)A+\nu B)\phi_{k},\phi_{k}\rangle\right) \\ &= h'(1)\sum_{k=1}^{n} f\left((1-\nu)\langle A\phi_{k},\phi_{k}\rangle+\nu\langle B\phi_{k},\phi_{k}\rangle\right) \\ &\leq \sum_{k=1}^{n} \left(h'(1)h(1-\nu)f\left(\langle A\phi_{k},\phi_{k}\rangle\right)+h'(1)h(\nu)f\left(\langle B\phi_{k},\phi_{k}\rangle\right)\right) \\ &\leq \sum_{k=1}^{n} \left(h'(0)h(1-\nu)\langle f(A)\phi_{k},\phi_{k}\rangle+h'(0)h(\nu)\langle f(B)\phi_{k},\phi_{k}\rangle\right) \\ &= h'(0)\sum_{k=1}^{n} \langle (h(1-\nu)f(A)+h(\nu)f(B)x)\phi_{k},\phi_{k}\rangle \\ &= h'(0)Tr\left[h(1-\nu)f(A)+h(\nu)f(B)\right]. \end{aligned}$$

In the first and second inequality, we used (1.1) and Lemma 3.1, respectively, noting that h'(1) > 0. This completes the proof.

Remark 3.5. We call the function f a quasi-h-convex, if we have

$$h'(1)f((1-\nu)a+\nu b) \le h'(0)(h(1-\nu)f(a)+h(\nu)f(b)),$$

for any $0 \leq \nu \leq 1$. Then Proposition 3.4 states that $A \mapsto Tr[f(A)]$ is quasih-convex whenever f is h-convex, provided that h is differentiable at 0,1 and h'(1) > 0.

One can show the following proposition by the standard method [9, 14].

Proposition 3.6. Let $f: I \to [0, \infty)$ be a differentiable *h*-convex function on an interval I, where $h: J \supseteq [0,1] \to \mathbb{R}$ is differentiable at 0 and 1, and let $A, B \in \mathcal{M}_n$ be Hermitian matrices with spectra in I. Then

$$Tr\left[(B-A)f'(A) + h'(1)f(A) - h'(0)f(B)\right] \le 0.$$

The above inequality is reversed if f is h-concave.

Proof. Let $A := \sum_{i=1}^{n} \lambda_j P_i$ and $B := \sum_{j=1}^{n} \mu_j Q_j$ be spectral decompositions. Since $\sum_{i=1}^{n} P_i = \sum_{j=1}^{n} Q_j = I_{\mathcal{M}_n}, P_i P_j = \delta_{ij} P_i$ and $Q_i Q_j = \delta_{ij} Q_i$, where $I_{\mathcal{M}_n}$ is the

identity matrix in \mathcal{M}_n and δ_{ij} is Kronecker delta, we have from Theorem 2.5,

$$Tr\left[(B-A)f'(A) + h'(1)f(A) - h'(0)f(B)\right]$$

= $Tr\left[f'(A)(B-A) + h'(1)f(A) - h'(0)f(B)\right]$
= $\sum_{i=1}^{n}\sum_{j=1}^{n}Tr\left[P_i\left(f'(A)(B-A) + h'(1)f(A) - h'(0)f(B)\right)Q_j\right]$
= $\sum_{i=1}^{n}\sum_{j=1}^{n}Tr\left[P_i\left(f'(\lambda_i)(\mu_j - \lambda_i) + h'(1)f(\lambda_i) - h'(0)f(\mu_j)\right)Q_j\right]$
= $\sum_{i=1}^{n}\sum_{j=1}^{n}\left((\mu_j - \lambda_i)f'(\lambda_i) + h'(1)f(\lambda_i) - h'(0)f(\mu_j)\right)Tr[P_iQ_j] \le 0,$

because $Tr[P_iQ_j] \ge 0$.

For the case h(x) := x in Proposition 3.6 is often called Klein inequality [1, Lemma 9.5]. Proposition 3.6 can be regarded as a generalization of Klein inequality. We should note that the usual Klein inequality is given for a differentiable convex function as an opposite inequality.

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Manuscript received November 22, 2023 revised November 24, 2024

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