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A-QUASICONVEXITY AND ITS CHARACTERIZATIONS

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Abstract. In this paper, we introduce A-quasiconvexity for functions in order to study a notion between convexity and quasiconvexity. We investigate some properties of A-quasiconvex functions. We show characterizations of A-quasiconvexity in terms of subdifferentials. Additionally, we show applications of our results. In particular, we investigate quasiconvexity of fractional functions.

1. Introduction

Convexity plays a central and essential role in convex analysis and optimization. Convex functions have so many nice properties, for example, a local minimizer for a convex function is also a global minimizer, the sum of two convex functions is also convex, and so on. Additionally, there are various characterizations of convexity. In this paper, the following characterization is important:

(1.1) f is convex if and only if $f - \langle v, \cdot \rangle$ is quasiconvex for each $v \in \mathbb{R}^n$.

By the characterization (1.1), if all affine perturbations of f are quasiconvex, then f is convex.

Generalized convexity is also used in mathematical economics, engineering, and optimization. In particular, quasiconvexity of functions plays an important role in economics. However, quasiconvex functions do not have the above nice properties in general. Actually, a local minimizer for a quasiconvex function is not always a global minimizer, and the sum of two quasiconvex functions is not always quasiconvex. In other words, even if f is quasiconvex, $f - \langle v, \cdot \rangle$ is not quasiconvex in general. There is a big difference from the characterization (1.1).

Many researchers have investigated a notion between convexity and quasiconvexity in terms of (1.1), for example, see [1–3, 7, 20] and references therein. In [7], Crouzeix shows the above characterization (1.1). In [1–3], Apetrii defines M-convexity of functions. Apetrii considers M-convexity as a generalization of convexity, and shows some important results. In [20], Seto, Kuroiwa, and Popovici study characterizations of convex and quasiconvex set-valued maps in terms of affine perturbations.

On the other hand, convexity and quasiconvexity of functions are closely related to these subdifferentials. For example, it is well known that a real-valued function is convex if and only if the subdifferential of the function is maximal monotone. In [4], Aussel, Corvellec, and Lassonde show a characterization of quasiconvexity in terms of Clarke subdifferential. In [21], Suzuki shows a characterization of quasiconvexity in terms of Greenberg-Pierskalla subdifferential.

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Following the previous results, in this paper, we study affine perturbations of quasiconvex functions, and show characterizations of the generalized quasiconvexity in terms of subdifferentials. We introduce A-quasiconvex functions as a generalized quasiconvexity. We investigate some properties of A-quasiconvex functions. We show characterizations of A-quasiconvexity in terms of subdifferentials. Additionally, we show applications of our results. In particular, we study quasiconvexity of fractional functions.

The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries and previous results. In Section 3, we introduce A-quasiconvexity of functions. We show some important properties of A-quasiconvex functions. Additionally, we show characterizations of A-quasiconvexity in terms of Clarke subdifferential and Greenberg-Pierskalla subdifferential. In Section 4, we discuss about our results. We show the characterization (1.1) as a corollary of our results. We investigate quasiconvexity of a fractional function.

2. Preliminaries

Let $\langle v, x\rangle$ denote the inner product of two vectors v and x in the n-dimensional Euclidean space $\mathbb{R}^n$. We denote $B(z,r)$ the open ball centered at $z \in \mathbb{R}^n$ with radius $r > 0$. We denote the closure, the convex hull, the conical hull, and the interior generated by A, by $\mathrm{cl}A$, $\mathrm{co}A$, $\mathrm{cone}A$, and $\mathrm{int}A$, respectively. The indicator function δ_A is defined by

$$\delta_A(x) = \begin{cases} 0 & x \in A, \\ \infty & \text{otherwise.} \end{cases}$$

Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. We denote the domain of f by $\mathrm{dom}f$, that is,

$$\mathrm{dom}f = \{x \in \mathbb{R}^n \mid f(x) < \infty\}.$$

A function f is said to be proper if $\mathrm{dom}f$ is nonempty and $f(x) > -\infty$ for each $x \in \mathbb{R}^n$. The epigraph of f is defined as

$$\mathrm{epi}f = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq \alpha\}.$$

f is said to be convex if $\mathrm{epi}f$ is convex. Fenchel conjugate of f, $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined as

$$f^*(v) = \sup\{\langle v, x\rangle - f(x) \mid x \in \mathbb{R}^n\}.$$

The subdifferential of f at x is defined as

$$\partial f(x) = \{v \in \mathbb{R}^n \mid \forall y \in \mathbb{R}^n, f(y) \geq f(x) + \langle v, y - x\rangle\}.$$

The Clarke subdifferential of f at $x \in \mathrm{dom}f$ is defined as

$$\partial^C f(x) = \{v \in \mathbb{R}^n \mid \forall y \in \mathbb{R}^n, \langle v, y\rangle \leq f^\uparrow(x; y)\},$$

where

$$f^\uparrow(x;y) = \sup_{\varepsilon>0} \inf_{\substack{\gamma>0 \\ \delta>0 \\ \lambda>0}} \sup_{\substack{z\in B(x,\gamma) \\ f(z)\leq f(x)+\delta \\ t\in(0,\lambda)}} \inf_{w\in B(y,\varepsilon)} \frac{f(z+tw)-f(z)}{t}$$

is the Rockafellar directional derivative. If $x \notin \mathrm{dom}f$, we define $\partial^C f(x) = \emptyset$. If f is locally Lipschitzian on $\mathbb{R}^n$, then the Rockafellar directional derivative is equal to the generalized Clarke derivative, see [6, 19]. We need the following proposition of Clarke subdifferential.

Proposition 2.1 ([4, 6, 19]). *Let f be an extended real-valued proper lower semicontinuous (lsc) function on $\mathbb{R}^n$. Then, the following statements hold:*

(i) *for each $v \in \mathbb{R}^n$ and $x \in \mathrm{dom} f$, $\partial^C(f - \langle v, \cdot\rangle)(x) = \partial^C f(x) - v$,*
(ii) *if f is convex, $\partial f(x) = \partial^C f(x)$ for each $x \in \mathbb{R}^n$,*
(iii) *f is convex if and only if $\partial^C f$ is monotone, that is, for each x, $y \in \mathbb{R}^n$, $u \in \partial^C f(x)$ and $w \in \partial^C f(y)$,*

$$\langle u - w, x - y\rangle \geq 0.$$

Define the level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$L(f, \diamond, \alpha) := \{x \in \mathbb{R}^n \mid f(x) \diamond \alpha\}$$

for any $\alpha \in \mathbb{R}$. A function f is said to be quasiconvex if for all $\alpha \in \mathbb{R}$, $L(f, \leq, \alpha)$ is convex. A function f is said to be semistrictly quasiconvex if for all x, $y \in \mathbb{R}^n$ satisfying $f(x) \neq f(y)$, and $\lambda \in (0, 1)$,

$$f((1 - \lambda)x + \lambda y) < \max\{f(x), f(y)\}.$$

It is known that a semistrictly quasiconvex function is not always quasiconvex, and a lsc semistrictly quasiconvex function is quasiconvex. A function f is said to be explicitly quasiconvex if f is quasiconvex and semistrictly quasiconvex. A function f is said to be essentially quasiconvex if f is quasiconvex and each local minimizer $x \in \mathrm{dom} f$ of f in $\mathbb{R}^n$ is a global minimizer of f in $\mathbb{R}^n$. Clearly, all explicitly quasiconvex functions (in particular convex functions) are essentially quasiconvex. By Proposition 4 in [15], upper semicontinuous (usc) essentially quasiconvex function is explicitly quasiconvex. However, explicitly quasiconvexity and essentially quasiconvexity are not equivalent in general, see the following example.

Example 2.2. Let f be the following function from $\mathbb{R}^2$ to $\mathbb{R}$:

$$f(x_1, x_2) = \begin{cases} x_1 & x_1 > 0, \\ 0 & x_1 = 0, x_2 > 1 \\ |x_2| - 1 & x_1 = 0, -1 \leq x_2 \leq 1 \\ 0 & x_1 = 0, x_2 < -1 \\ -1 & x_1 < 0. \end{cases}$$

We can easily check that f is quasiconvex. Let $x = (x_1, x_2)$ be a local minimizer of f in $\mathbb{R}^n$. Then $x_1 < 0$ or $(x_1, x_2) = 0$, hence x is a global minimizer of f in $\mathbb{R}^n$. This shows that f is essentially quasiconvex. However, f is not semistrictly quasiconvex on $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0\}$. Hence, f is not explicitly quasiconvex.

In Proposition 2.1, a characterization of convexity in terms of Clarke subdifferential have been introduced. Additionally, it is well known that a real-valued function f is convex if and only if ∂f is monotone, see [16–18]. For quasiconvex functions, similar results have been given in terms of the usual derivative and subdifferentials, for example, see [4, 11, 12]. In this paper, we need the following proposition.

Proposition 2.3 ([4])**.** *Let f be an extended real-valued proper lsc function on $\mathbb{R}^n$. Then, f is quasiconvex if and only if $\partial^C f$ is quasimonotone, that is, for each x, $y \in \mathbb{R}^n$, $u \in \partial^C f(x)$ and $v \in \partial^C f(y)$,*

$$\min\{\langle u, y-x\rangle, \langle v, x-y\rangle\} \leq 0.$$

In [8], Greenberg and Pierskalla introduce the Greenberg-Pierskalla subdifferential of f at $x_0 \in \mathbb{R}^n$ as follows:

$$\partial^{GP} f(x_0) = \{v \in \mathbb{R}^n \mid \langle v, x\rangle \geq \langle v, x_0\rangle \text{ implies } f(x) \geq f(x_0)\}.$$

The Greenberg-Pierskalla subdifferential is closely related to Moreau's generalized conjugation and λ-quasiconjugate, for detail, see [8, 13, 14]. In [21], we show the following characterization of quasiconvexity of the sum of real-valued usc functions in terms of Greenberg-Pierskalla subdifferential.

Proposition 2.4 ([21])**.** *Let f and g be real-valued usc functions. Then, $f+g$ is quasiconvex if and only if for each $x \in \mathbb{R}^n$, $\partial^{GP}(f+g)(x) \neq \emptyset$.*

3. A-quasiconvexity and its characterizations

In this section, we investigate A-quasiconvexity of functions. We show some important properties of A-quasiconvex functions. We introduce characterizations of A-quasiconvexity in terms of Clarke subdifferential and Greenberg-Pierskalla subdifferential.

At first, we introduce a definition of a generalized quasiconvexity.

Definition 3.1. Let f be an extended real-valued function on $\mathbb{R}^n$, and $A \subset \mathbb{R}^n$. A function f is said to be A-quasiconvex if $f - \langle v, \cdot\rangle$ is quasiconvex for each $v \in A$.

In [1–3], Apetrii defines M-convexity of functions as follows: a function f is said to be M-convex if for each $v \in M \subset \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, the following set is convex:

$$\{x \in \mathbb{R}^n \mid f(x) \leq \alpha + \langle v, x\rangle\}.$$

Clearly, M-convexity in the sense of [1–3] is equivalent to A-quasiconvexity in Definition 3.1 if $M = A$. In this paper, we consider the notion as a generalization of quasiconvexity, and denote by A-quasiconvexity. By (1.1), f is convex if and only if $f - \langle v, \cdot\rangle$ is quasiconvex for each $v \in \mathbb{R}^n$, for detail, see [7, 20]. Hence, f is $\mathbb{R}^n$-quasiconvex if and only if f is convex. In the following theorem, we show some important properties of A-quasiconvex functions.

Theorem 3.2. *Let f be an extended real-valued function on $\mathbb{R}^n$, and $A \subset \mathbb{R}^n$. Assume that f is A-quasiconvex. Then, the following statements hold:*

(i) *if $0 \in A$, then f is quasiconvex,*
(ii) *if $0 \in \text{int}A$, then f is explicitly quasiconvex,*
(iii) *if f is lsc and proper, then f is $\text{cl}A$-quasiconvex.*

Proof. The statement (i) is clear, but important.

We prove the statement (ii). Since $0 \in A$, f is quasiconvex. Assume that f is not semistrictly quasiconvex. Then, there exist x, $y \in \mathbb{R}^n$, and $\lambda_0 \in (0, 1)$ such that

$f(x) = f((1-\lambda_0)x + \lambda_0 y) > f(y)$. This shows that for sufficiently small $\varepsilon > 0$, $f + \langle \varepsilon(y-x), \cdot \rangle$ is not quasiconvex. Actually, we can find $\varepsilon > 0$ such that

$$\begin{aligned} & \max\{f(x) + \langle \varepsilon(y-x), x \rangle, f(y) + \langle \varepsilon(y-x), y \rangle\} \\ < \ & f((1-\lambda_0)x + \lambda_0 y) + \langle \varepsilon(y-x), (1-\lambda_0)x + \lambda_0 y \rangle . \end{aligned}$$

This contradicts that f is A-quasiconvex and $0 \in \text{int}A$.

We prove the statement (iii). Let $v \in \text{cl}A$, and $\{v_k\} \subset A$ satisfying $v_k \to v$. By the assumption, $f - \langle v_k, \cdot \rangle$ is quasiconvex. By Proposition 2.1 and Proposition 2.3, for each x, $y \in \mathbb{R}^n$, $u \in \partial^C f(x)$ and $w \in \partial^C f(y)$,

$$\min\{\langle u - v_k, y - x \rangle, \langle w - v_k, x - y \rangle\} \leq 0.$$

Without loss of generality, there exists a subsequence $\{v_{k_i}\}$ of $\{v_k\}$ such that

$$\langle u - v_{k_i}, y - x \rangle \leq 0$$

for each $i \in \mathbb{N}$. Therefore,

$$\langle u - v, y - x \rangle \leq 0$$

since $v_{k_i} \to v$. This shows that $\partial^C(f - \langle v, \cdot \rangle)$ is quasimonotone, that is, $f - \langle v, \cdot \rangle$ is quasiconvex. This completes the proof. □

Remark 3.3. In the statement (iii) of Theorem 3.2, we show that if f is proper lsc A-quasiconvex, then f is cl$A$-quasiconvex. However, even if f is A-quasiconvex, f is not co$A$-quasiconvex in general. Actually, we show that following example such that f is A-quasiconvex, and f is not co$A$-quasiconvex. Let $f(x) = \frac{x^4}{4} + \frac{x^3}{3}$. Then f is $\{0, \frac{4}{27}\}$-quasiconvex, and f is not $[0, \frac{4}{27}]$-quasiconvex. Actually, $F(x) = f(x) - \frac{x}{10}$ is not quasiconvex since

$$\max\left\{F\left(-\frac{2}{3}\right), F(0)\right\} = \max\left\{\frac{7}{405}, 0\right\} < \frac{13}{540} = F\left(-\frac{1}{3}\right).$$

Next, we study characterizations of A-quasiconvexity. At first, we show the following characterization in terms of Clarke subdifferential.

Theorem 3.4. *Let f be an extended real-valued proper lsc function on $\mathbb{R}^n$, and A is a convex subset of $\mathbb{R}^n$. Then, f is A-quasiconvex if and only if for each x, $y \in \mathbb{R}^n$, at least one of the following statements holds:*

(i) *for each $u \in \partial^C f(x)$ and $w \in \partial^C f(y)$, $\langle u - w, y - x \rangle \leq 0$,*
(ii) *for each $v \in A$ and $u \in \partial^C f(x)$, $\langle u - v, y - x \rangle \leq 0$,*
(iii) *for each $v \in A$ and $w \in \partial^C f(y)$, $\langle w - v, x - y \rangle \leq 0$.*

Proof. Assume that f is A-quasiconvex, and let x, $y \in \mathbb{R}^n$. By Proposition 2.1 and Proposition 2.3,

$$\min\{\langle u - v, y - x \rangle, \langle w - v, x - y \rangle\} \leq 0$$

for each $v \in A$, $u \in \partial^C f(x)$ and $w \in \partial^C f(y)$. This shows that

$$\langle u, y - x \rangle \leq \langle v, y - x \rangle, \text{ or } \langle w, y - x \rangle \geq \langle v, y - x \rangle .$$

Therefore, A is contained in the union of two closed half spaces, that is,

$$A \subset \{v \mid \langle u, y - x \rangle \leq \langle v, y - x \rangle\} \cup \{v \mid \langle w, y - x \rangle \geq \langle v, y - x \rangle\}.$$

If $\langle u-w, x-y\rangle \geq 0$ for each $u \in \partial^C f(x)$ and $w \in \partial^C f(y)$, then the statement (i) holds. Hence, we assume that there exists $u_0 \in \partial^C f(x)$ and $w_0 \in \partial^C f(y)$ such that $\langle u_0 - w_0, x-y\rangle < 0$. Since $\langle u_0, y-x\rangle > \langle w_0, y-x\rangle$, $\{v \mid \langle u_0, y-x\rangle \leq \langle v, y-x\rangle\}$ and $\{v \mid \langle w_0, y-x\rangle \geq \langle v, y-x\rangle\}$ are closed half spaces and the intersection is empty. Since A is convex, A is contained in one of these closed half spaces. Assume that $A \subset \{v \in \mathbb{R}^n \mid \langle u_0, y-x\rangle \leq \langle v, y-x\rangle\}$, and let $u \in \partial^C f(x)$. If $\langle u, y-x\rangle \leq \langle u_0, y-x\rangle$, then it is clear that for each $v \in A$, $\langle u-v, y-x\rangle \leq 0$. Additionally, if $\langle u, y-x\rangle > \langle u_0, y-x\rangle$, then $\langle u, y-x\rangle > \langle w_0, y-x\rangle$. Since

$$A \subset \{v \mid \langle u, y-x\rangle \leq \langle v, y-x\rangle\} \cup \{v \mid \langle w_0, y-x\rangle \geq \langle v, y-x\rangle\},$$

we can prove that

$$A \subset \{v \in \mathbb{R}^n \mid \langle u, y-x\rangle \leq \langle v, y-x\rangle\}.$$

in the similar way of the first half of the proof. This shows that (ii) holds. If $A \subset \{v \in \mathbb{R}^n \mid \langle w_0, y-x\rangle \geq \langle v, y-x\rangle\}$, then we can prove that (iii) holds.

Let $v \in A$, x, $y \in \mathbb{R}^n$, and assume that one of the statements (i), (ii) and (iii) holds. If (ii) or (iii) holds, it is clear that for each $u \in \partial^C f(x)$ and $w \in \partial^C f(y)$,

$$\min\{\langle u-v, y-x\rangle, \langle w-v, x-y\rangle\} \leq 0.$$

This shows that $\partial^C(f - \langle v, \cdot\rangle)$ is quasimonotone, that is, $f - \langle v, \cdot\rangle$ is quasiconvex. Assume that (i) holds, that is, for each $u \in \partial^C f(x)$ and $w \in \partial^C f(y)$,

$$\langle u-w, y-x\rangle \leq 0.$$

For each $v \in A$,

$$\langle u-v, y-x\rangle + \langle w-v, x-y\rangle = \langle u-v+v-w, y-x\rangle \leq 0.$$

This shows that

$$\min\{\langle u-v, y-x\rangle, \langle w-v, x-y\rangle\} \leq 0.$$

This completes the proof. □

Remark 3.5. If f is a real-valued differentiable function on $\mathbb{R}^n$, then $\partial^C f(x) = \{\nabla f(x)\}$. Hence, we can show the following characterization of A-quasiconvexity in terms of the usual derivative: f is A-quasiconvex if and only if for each x, $y \in \mathbb{R}^n$, at least one of the following statements holds:

(i) $\langle \nabla f(x) - \nabla f(y), y-x\rangle \leq 0$,
(ii) for each $v \in A$, $\langle \nabla f(x) - v, y-x\rangle \leq 0$,
(iii) for each $v \in A$, $\langle \nabla f(y) - v, x-y\rangle \leq 0$.

Next, we show the following characterization of A-quasiconvexity in terms of Greenberg-Pierskalla subdifferential and the usual subdifferential.

Theorem 3.6. *Let f be an usc real-valued function on $\mathbb{R}^n$, and $A \subset \mathbb{R}^n$. Then, the following statements are equivalent:*

(i) *f is A-quasiconvex,*
(ii) *for each $v \in A$ and $x \in \mathbb{R}^n$, $\partial^{GP}(f - \langle v, \cdot\rangle)(x)$ is nonempty.*
(iii)

$$A \subset \bigcap_{x \in \mathbb{R}^n} \bigcup_{w \in \mathbb{R}^n} \partial(f + \delta_{L(w, \geq, \langle w, x\rangle)})(x).$$

Proof. By Proposition 2.4, (i) and (ii) are equivalent. Assume that (ii) holds, and let $v \in A$. Then, for each $x \in \mathbb{R}^n$, $\partial^{GP}(f - \langle v, \cdot \rangle)(x)$ is nonempty. Hence, there exists $w \in \mathbb{R}^n$ such that

$$\inf\{f(y) - \langle v, y \rangle \mid \langle w, y \rangle \geq \langle w, x \rangle\} \geq f(x) - \langle v, x \rangle .$$

This shows that for each $y \in \mathbb{R}^n$ with $\langle w, y \rangle \geq \langle w, x \rangle$,

$$f(y) \geq f(x) + \langle v, y - x \rangle ,$$

that is, $v \in \partial(f + \delta_{L(w,\geq,\langle w,x \rangle)})(x)$. Therefore,

$$A \subset \bigcap_{x \in \mathbb{R}^n} \bigcup_{w \in \mathbb{R}^n} \partial(f + \delta_{L(w,\geq,\langle w,x \rangle)})(x).$$

The proof of the converse implication is similar and will be omitted. □

4. Discussions and applications

In this section, we discuss about our results and show applications. We show that the characterization (1.1) as a corollary of our results. We study A-monotonicity in [3] and investigate Theorem 3.4 precisely. We show a characterization of quasiconvexity of fractional functions.

4.1. Characterization of convexity in terms of A-quasiconvexity.

At first, we show the characterization (1.1) as a corollary of our result.

Corollary 4.1. *Let f be an extended real-valued proper lsc function on $\mathbb{R}^n$. Then, f is convex if and only if $f - \langle v, \cdot \rangle$ is quasiconvex for each $v \in \mathbb{R}^n$.*

Proof. Assume that f is convex. We can check easily that for each $v \in \mathbb{R}^n$, $f - \langle v, \cdot \rangle$ is convex. Hence f is $\mathbb{R}^n$-quasiconvex.

Conversely, assume that f is $\mathbb{R}^n$-quasiconvex. Then, by Theorem 3.4, for each x, $y \in \mathbb{R}^n$, at least one of the following statements holds:

(i) for each $u \in \partial^C f(x)$ and $w \in \partial^C f(y)$, $\langle u - w, y - x \rangle \leq 0$,
(ii) for each $v \in \mathbb{R}^n$ and $u \in \partial^C f(x)$, $\langle u - v, y - x \rangle \leq 0$,
(iii) for each $v \in \mathbb{R}^n$ and $w \in \partial^C f(y)$, $\langle w - v, x - y \rangle \leq 0$.

Assume that there exists x_0, $y_0 \in \mathbb{R}^n$, $u_0 \in \partial^C f(x_0)$ and $w_0 \in \partial^C f(y_0)$ such that

$$\langle u_0 - w_0, x_0 - y_0 \rangle < 0.$$

Then, one of the above statements (ii), (iii) holds, and $x_0 \neq y_0$. However, $\{v \in \mathbb{R}^n \mid \langle u_0, y_0 - x_0 \rangle \leq \langle v, y - x_0 \rangle\}$ and $\{v \in \mathbb{R}^n \mid \langle w_0, y_0 - x_0 \rangle \geq \langle v, y_0 - x_0 \rangle\}$ are closed half spaces and not equal to $\mathbb{R}^n$. This is a contradiction. Hence, for each x, $y \in \mathbb{R}^n$, $u \in \partial^C f(x)$ and $w \in \partial^C f(y)$,

$$\langle u - w, x - y \rangle \geq 0,$$

that is, $\partial^C f$ is monotone. This shows that f is convex, and completes the proof. □

4.2. ***A*-quasiconvexity and *A*-monotonicity.** In [1–3], Apetrii defines M-convexity and shows some important results for generalized convexity. In this paper, we consider the notion as a generalization of quasiconvexity, and denote by A-quasiconvexity. In this subsection, we study relation between A-quasiconvexity and A-monotonicity in [3].

Definition 4.2 ([3])**.** Let X be a normed space, X^* its topological dual, and A a nonempty subset of X^*. A set-valued operator F on X is said to be A-monotone if for every x, $y \in X$, $x^* \in F(x)$, and $y^* \in F(y)$,

$$\langle x^*, y - x\rangle \leq \inf_{x_0^* \in M_{y,x}(y^*)} \langle x_0^*, y - x\rangle$$

where

$$M_{y,x}(y^*) = \{x_0^* \in A \mid \langle y^* - x_0^*, y - x\rangle < 0\}.$$

Apetrii shows the following characterization of A-quasiconvexity, see Theorem 34 in [3].

Theorem 4.3 ([3])**.** *Let X be a normed space, X^* its topological dual, A a nonempty subset of X^*, and f a lsc function from X to $\mathbb{R} \cup \{\infty\}$. Then, f is A-quasiconvex if and only if the upper Dini subdifferential $\partial^{D+} f$ is A-monotone, where*

$$\partial^{D+} f(x, v) = \left\{ x^* \in X^* \mid \langle x^*, v\rangle \leq \limsup_{t \searrow 0} \frac{f(x + tv) - f(x)}{t} \right\},$$

$$\partial^{D+} f(x) = \bigcap_{v \in X} \partial^{D+} f(x, v).$$

A-monotonicity of subdifferentials are closely related to the following statements in Theorem 3.4:

(i) for each $u \in \partial^C f(x)$ and $w \in \partial^C f(y)$, $\langle u - w, y - x\rangle \leq 0$,
(ii) for each $v \in A$ and $u \in \partial^C f(x)$, $\langle u - v, y - x\rangle \leq 0$,
(iii) for each $v \in A$ and $w \in \partial^C f(y)$, $\langle w - v, x - y\rangle \leq 0$.

Assume that $\partial^C f$ is A-monotone, and (iii) in Theorem 3.4 does not hold. Then there exists $v_0 \in A$ and $w_0 \in \partial^C f(y)$, $\langle w_0 - v_0, x - y\rangle > 0$. Since $v_0 \in M_{y,x}(w_0)$,

$$\langle u, y - x\rangle \leq \inf_{x_0^* \in M_{y,x}(y^*)} \langle x_0^*, y - x\rangle \leq \langle v_0, y - x\rangle$$

for each $u \in \partial^C f(x)$. This shows that $\langle u - v_0, y - x\rangle \leq 0$. However, we can not show that the statement (ii) in Theorem 3.4 holds. By A-monotonicity of $\partial^C f$, we can only show that if the inequality in (iii) does not hold for $v \in A$ and $w \in \partial^C f(y)$, then the inequality in (ii) holds for each $u \in \partial f^C(x)$. Additionally, we use $\partial^C f$ in the paper, though Apetrii uses the upper Dini subdifferential. Hence, we can not show Theorem 4.3 as a corollary of Theorem 3.4, and vice versa.

4.3. **Discussion about Theorem 3.4.** In Theorem 3.4, we show that for each x, $y \in \mathbb{R}^n$, at least one of the statements (i), (ii), (iii) holds. Next, we investigate the set of all x, $y \in \mathbb{R}^n$ satisfying the statement (i) in Theorem 3.4.

Theorem 4.4. *Let f be an extended real-valued proper lsc function on $\mathbb{R}^n$, A is a convex subset of $\mathbb{R}^n$, and $X = \{x \in \mathbb{R}^n \mid \partial^C f(x) \subset A\}$. Then, for each $x, y \in X$, $u \in \partial^C f(x)$ and $w \in \partial^C f(y)$, $\langle u - w, y - x\rangle \leq 0$.*

Proof. Let $x, y \in X$, $u \in \partial^C f(x)$ and $w \in \partial^C f(y)$. Then $\frac{u+w}{2} \in A$ since A is convex. By Theorem 3.4, one of the following statements holds:

(A) $\langle u - w, y - x\rangle \leq 0$,
(B) $\left\langle u - \frac{u+w}{2}, y - x\right\rangle \leq 0$,
(C) $\left\langle w - \frac{u+w}{2}, x - y\right\rangle \leq 0$.

Actually, the above inequalities are equivalent to the following inequality:

$$\langle u - w, y - x\rangle \leq 0.$$

This completes the proof. □

By Theorem 4.4, $\partial^C f$ is monotone on X. In Remark 3.3, we show that $f(x) = \frac{x^4}{4} + \frac{x^3}{3}$ is $\{0, \frac{4}{27}\}$-quasiconvex, and not $[0, \frac{4}{27}]$-quasiconvex. Additionally, we can check that f is $(-\infty, 0] \cup [\frac{4}{27}, \infty)$-quasiconvex. Hence, if A is a convex subset of $(-\infty, 0] \cup [\frac{4}{27}, \infty)$, then Theorem 3.4 and Theorem 4.4 hold. Actually, let $A = [\frac{3}{8}, \infty)$, then $X = [\frac{1}{2}, \infty)$ and $f'(x) = x^3 + x^2$ is monotone on X. If $A = (-\infty, 0]$, then $X = (-\infty, -1] \cup \{0\}$. Although X is not convex, $Y = \{x \in \mathbb{R} \mid \nabla f(x) \subset \mathrm{int} A\} = (-\infty, -1)$ is convex. In more general setting, whether Y is convex or not is an open question.

4.4. **An example of Theorem 3.6.** In this subsection, we investigate characterizations in Theorem 3.6. We show the following example.

Example 4.5. Let $f(x) = \frac{x^4}{4} + \frac{x^3}{3}$, and A is a subset of $\mathbb{R}$. By Theorem 3.6, the following statements are equivalent:

(i) f is A-quasiconvex,
(ii) for each $v \in A$ and $x \in \mathbb{R}$, $\partial^{GP}(f - \langle v, \cdot\rangle)(x)$ is nonempty.
(iii)
$$A \subset \bigcap_{x\in\mathbb{R}} \bigcup_{w\in\mathbb{R}} \partial(f + \delta_{L(w,\geq,wx)})(x).$$

Let $v_0 = \frac{1}{8}$. Then, $x_0 = -\frac{1}{2}$ is a strict local maximizer of $f - v_0$. This shows that $\partial^{GP}(f - \langle v_0, \cdot\rangle)(x_0)$ is empty. By the similar way, we can prove that for each $w \in \mathbb{R}$,

$$v_0 \notin \partial(f + \delta_{L(w,\geq,wx_0)})(x_0).$$

Hence, by (ii) or (iii), if $v_0 \in A$, then f is not A-quasiconvex. Additionally, we can check that

$$\bigcap_{x\in\mathbb{R}} \bigcup_{w\in\mathbb{R}} \partial(f + \delta_{L(w,\geq,wx)})(x) = (-\infty, 0] \cup [\frac{4}{27}, \infty).$$

Hence, if $A \subset (-\infty, 0] \cup [\frac{4}{27}, \infty)$, then f is A-quasiconvex.

4.5. **Quasiconvexity of fractional functions.** In fractional programming, quasiconvexity plays an important role. Especially, in the case of convex fractional programming, an objective function is explicitly quasiconvex. Next, we study quasiconvexity of fractional functions in terms of A-quasiconvexity.

In the following theorem, we show some sufficient conditions for quasiconvexity of $F = \frac{f}{g}$, where D is an open convex subset of $\mathbb{R}^n$, A is a convex subset of D, f is a real-valued quasiconvex function on $\mathbb{R}^n$ such that $f + \delta_D$ is non-negative, g is a real-valued quasiconcave function on $\mathbb{R}^n$ such that $g + \delta_D$ is positive, and F is the following function:

$$F(x) = \begin{cases} \dfrac{f(x)}{g(x)} & x \in D, \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 4.6. *The following statements hold:*

(i) *if f is $-(\mathrm{cone}\ \mathrm{dom}(-g)^*)$-quasiconvex and g is concave, then F is quasiconvex,*

(ii) *if there exists $r > 0$ such that f is $(\mathrm{cone}\ \mathrm{dom}(-g)^*) + B(0, r)$-quasiconvex and g is concave, then F is essentially quasiconvex.*

Proof. We show the statement (i). Assume that f is $(\mathrm{cone}\ \mathrm{dom}(-g)^*)$-quasiconvex, and g is concave. We show that $L(F, \leq, \alpha)$ is convex for each $\alpha \in \mathbb{R}$. If $\alpha < 0$, then $L(F, \leq, \alpha)$ is empty since $f + \delta_D$ is non-negative and $g + \delta_D$ is positive, If $\alpha = 0$, then $L(F, \leq, 0) = L(f, \leq, 0)$ is convex by $0 \in -(\mathrm{cone}\ \mathrm{dom}(-g)^*)$ and Theorem 3.2. Assume that $\alpha > 0$. Then,

$$\begin{aligned} & x \in L(F, \leq, \alpha) \\ \iff \quad & F(x) = \frac{f(x)}{g(x)} \leq \alpha \\ \iff \quad & f(x) - \alpha g(x) \leq 0 \\ \iff \quad & f(x) + \alpha \sup_{w \in \mathrm{dom}(-g)^*} \{\langle w, x\rangle - (-g)^*(w)\} \leq 0 \\ \iff \quad & \forall w \in \mathrm{dom}(-g)^*, f(x) + \langle \alpha w, x\rangle - \alpha(-g)^*(w) \leq 0 \\ \iff \quad & \forall w \in \mathrm{dom}(-g)^*, (f + \alpha w)(x) \leq \alpha(-g)^*(w). \end{aligned}$$

By the assumption, $f + \alpha w$ is quasiconvex for each $w \in \mathrm{dom}(-g)^*$. Hence, $L(F, \leq, \alpha)$ is a convex set, that is, F is quasiconvex.

Next, we show the statement (ii). Assume that there exists $r > 0$ such that f is $-(\mathrm{cone}\ \mathrm{dom}(-g)^*) + B(0, r)$-quasiconvex and g is concave. At first, we show that $f - \alpha g$ is explicitly quasiconvex for each $\alpha > 0$. Let $\alpha > 0$ and $v \in B(0, r)$. For each $\beta \in \mathbb{R}$,

$$\begin{aligned} & x \in L(f - \alpha g + v, \leq, \beta) \\ \iff \quad & f(x) - \alpha g(x) + \langle v, x\rangle \leq \beta \\ \iff \quad & f(x) + \alpha \sup_{w \in \mathrm{dom}(-g)^*} \{\langle w, x\rangle - (-g)^*(w)\} + \langle v, x\rangle \leq \beta \\ \iff \quad & \forall w \in \mathrm{dom}(-g)^*, f(x) + \langle \alpha w, x\rangle - \alpha(-g)^*(w) + \langle v, x\rangle \leq \beta \\ \iff \quad & \forall w \in \mathrm{dom}(-g)^*, (f + \alpha w + v)(x) \leq \beta + \alpha(-g)^*(w). \end{aligned}$$

By the assumption, $f + \alpha w + v$ is quasiconvex for each $w \in \mathrm{dom}(-g)^*$. Hence, $L(f - \alpha g + v, \leq, \beta)$ is a convex set, that is, $f - \alpha g + v$ is quasiconvex. Since $f - \alpha g$ is $B(0, r)$-quasiconvex, $f - \alpha g$ is explicitly quasiconvex by Theorem 3.2.

Next, we show that F is essentially quasiconvex. By the similar way in the proof of statement (i), we can show that F is quasiconvex. Let x_0 be a local minimizer of F in D and $\alpha_0 = F(x_0)$. If $\alpha_0 = 0$, then we can check easily that x_0 is a global minimizer of F in D. Hence, we assume that $\alpha_0 > 0$. Then, there exists an open sub set $U \subset D$ such that x_0 is a minimizer of F in U. For each $x \in U$,

$$
\begin{aligned}
& F(x) \geq \alpha_0 = F(x_0) \\
\iff \quad & \frac{f(x)}{g(x)} \geq \alpha_0 = \frac{f(x_0)}{g(x_0)} \\
\iff \quad & f(x) - \alpha_0 g(x) \geq 0 = f(x_0) - \alpha_0 g(x_0).
\end{aligned}
$$

This shows that x_0 is a local minimizer of $f - \alpha_0 g$ in D. Since $f - \alpha_0 g$ is explicitly quasiconvex, $f - \alpha_0 g$ is essentially quasiconvex, that is, x_0 is a global minimizer of $f - \alpha_0 g$ in D. We can check easily that x_0 is a global minimizer of F in D. Hence, F is essentially quasiconvex. This completes the proof. □

By Theorem 4.6, we can show the following corollary.

Corollary 4.7. *The following statements are sufficient conditions for essentially quasiconvexity of F:*

(i) *f is convex and g is concave,*
(ii) *there exists $r > 0$ such that f is* $\mathrm{cone}\{v\} + B(0, r)$*-quasiconvex and $g = v \in \mathbb{R}^n$.*

Proof. We can see that f is convex if and only if $\mathbb{R}^n$-quasiconvex, and $\mathrm{dom}(- \langle v, \cdot \rangle)^* = \{-v\}$. Hence, we can easily show that the statements (i) and (ii) are sufficient conditions for (ii) of Theorem 4.6. □

In [5], it is shown that a ratio of a concave function and a convex function is semistrictly quasiconcave. Additionally, a real-valued continuous quasiconvex function is essentially quasiconvex if and only if it is semistrictly quasiconvex, see [5, 9, 10]. Hence, a convex fractional function is essentially quasiconvex. In this paper, we give another proof of essentially quasiconvexity of a convex fractional function in Theorem 4.6 and Corollary 4.7.

Remark 4.8. In Theorem 4.6, we assume that g is concave. We can prove that the following conditions is also a sufficient condition for quasiconvexity of F:

$$f \text{ is convex and } -g \text{ is } -(\mathrm{cone}\,\mathrm{dom} f^*)\text{-quasiconvex.}$$

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