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SOFT MEASURES AND THE EXISTENCE OF RENORMALIZED SOLUTIONS TO BOUNDARY-VALUE PROBLEMS FOR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS INVOLVING VARIABLE EXPONENTIAL OPERATOR AND GENERAL MEASURE DATA

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Abstract. We study properties of variable exponential capacity connected with variable exponential Laplacian operator; we prove the representation theorem for measures that vanish on subsets of null capacity. We establish that if bounded measure μ does not charge the subsets of null capacity then the boundary problem $u|_{\partial\Omega} = 0$ for elliptic partial differential equation with the variable exponential operator

$$-div\,(a\,(x,\ \nabla u)) + b\,(x)\,|u|^{\gamma(x)-2}\,u = \mu$$

under modified Leray-Lions conditions has a unique entropy and renormalized solutions.

1. Introduction

In this article, we consider boundary problem for a nonlinear elliptic partial differential equation with the variable exponential operator

$$-div\,(a\,(x,\ \nabla u)) + b\,(x)\,|u|^{\gamma(x)-2}\,u = F - div\,(\Theta) \equiv \mu,$$
$$u|_{\partial\Omega} = 0$$

in the open domain $\Omega \subset R^n$, $\quad n > 2$, where $F \in L^1(\Omega)$ and $\Theta \in \left(L^{q(\cdot)}(\Omega)\right)^n$, $q\,(x) = \frac{p(x)}{p(x)-1}$ and $p \in P^{\log}(\Omega)$, $b \in L^\infty(\Omega)$, $\quad 0 < b\,(x)$, $\quad x \in \Omega$ and function $a\ :\quad \Omega \times R^n \to R^n$ satisfies the modified Leray-Lions conditions.

The boundary problems for partial differential equations of this type often arise in the theory of liquid materials such as electrorheological fluids and highly loading ceramic or polymeric composite systems, also these mathematical tools often use to formalize models of signal processing [1, 3]. The variable exponential conditions can be considered as an important generalization of the standard fractional Laplace operator. We mention some pioneering papers dealing with these kinds of mathematical models [3, 4, 6–8, 10, 12, 16, 17, 19, 21]. If p is constant then we can use Poincare and Sobolev inequalities, in the general situation X. L. Fan, Q. H. Zhang, and D. Zhao have proven that the modular Poincare inequality does not hold even for general continuous variable exponentials so that we employ additional condition $p \in P^{\log}(\Omega)$ under which Poincare and Sobolev inequalities can be applied.

In order to study problems with the general measure data, we need to generalize the concept of the solutions to include less regular functions in the pool of possible

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solutions. Entropy and renormalized solutions present two possible ways of such generalizations [2] authors studied the renormalized solutions in the case when the exponent p is a constant; in [19] C. Zhang and X. Zhang considered the existence and uniqueness of renormalized and entropy solutions for nonlinear parabolic partial differential equations with variable Laplacian; in [18] C. Zhang and X. Zhang proved the existence of unique nonnegative renormalized solution to the fractional Laplacian problem; the regularity of solutions to in variable spaces were considered in [2,4–6,10,11,13–15]; in [17] J. X. Yin, J. K. Li, Y. Y. Ke applied the Krasnoselskii fixed point theorem on the cone to show global regularity of positive solutions to $p\left(x\right)$-Laplace equation with the good right side, authors also proved the Harnack inequality and the Liouville theorem.

In the present article, we extend L. Boccardo, D. Giachetti, J. I. Diaz, F. Murat, C. Zhang, S. Zhou C. Zhang, and S. Zhou results [2, 18, 19] to consider boundary problem for a nonlinear elliptic partial differential equation with the variable exponential operator (3.1), (3.2) under the generalized Leray-Lions conditions, we establish the existence and uniqueness of the renormalized solution in the case $p \in P^{\log}\left(\Omega\right)$. To formulate precise restrictions on the right side of the differential equation (3.1), we studied the properties of the variable exponential variational capacity, which is connected with the minimization of the modular function of the gradient or the norm of variable exponential Sobolev space $W_{1,\,0}^{p(\cdot)}\left(\Omega\right)$ for the functions greater than one on give set. We establish that the bounded Borelian measure $\mathrm{M}_B\left(\Omega\right)$ does not charge zero capacity sets if and only if $\mu \in L^1\left(\Omega\right) + W_{-1}^{q(\cdot)}\left(\Omega\right)$, namely if there are two elements $F \in L^1\left(\Omega\right)$ $\Theta \in \left(L^{q(\cdot)}\left(\Omega\right)\right)^n$ such that $\mu = f - div\left(\Theta\right)$. This paper's main result is the existence and uniqueness of the renormalized solution to a boundary problem for a nonlinear elliptic partial differential equation with the variable exponential operator with general integrable data.

2. Some preliminary information

We assume $\Omega \subset R^n$ is a bounded domain, $p \in P^{\log}\left(\Omega\right)$, and denote $p_m = \inf_{x\in\Omega} p\left(x\right)$ and $p_S = \sup_{x\in\Omega} p\left(x\right)$. We define a modular function by

$$\rho_{p(\cdot)}\left(u\right) = \int_\Omega \left|u\left(x\right)\right|^{p(x)} dx. \tag{2.1}$$

The norm $\left\|\cdot\right\|_{L^{p(\cdot)}(\Omega)}$ of variable exponent Lebesgue space $L^{p(\cdot)}\left(\Omega\right)$ is given by

$$\left\|u\right\|_{L^{p(\cdot)}(\Omega)} = \inf_{\lambda>0}\left\{\rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \le 1\right\}. \tag{2.2}$$

Definition 2.1. A Banach space $W_1^{p(\cdot)}\left(\Omega\right)$ consists of all elements u of $L^{p(\cdot)}\left(Q\right)$ such that $\left|\nabla u\right| \in L^{p(\cdot)}\left(\Omega\right)$ and is equipped with the norm

$$\left\|u\right\|_{W_1^{p(\cdot)}(\Omega)} = \left\|u\right\|_{L^{p(\cdot)}(\Omega)} + \left\|\nabla u\right\|_{L^{p(\cdot)}(\Omega)}. \tag{2.3}$$

The Sobolev space $W_{1,0}^{p(\cdot)}\left(\Omega\right)$ is the closure of $C_0^\infty\left(\Omega\right)$ in the norm of $W_1^{p(\cdot)}\left(\Omega\right)$ such that embedding $W_1^{p(\cdot)}\left(\Omega\right) \to L^{p(\cdot)}\left(\Omega\right)$ is continuous. The dual Sobolev space

$(W_1^{p(\cdot)}(\Omega))^*$ is $W_{-1}^{q(\cdot)}(\Omega)$ equipped with the norm

$$\|f\|_{W_{-1}^{q(\cdot)}(\Omega)} = \sup_{\substack{u \in W_1^{p(\cdot)}(\Omega),\\ u \neq 0}} \frac{|\langle f,\ u\rangle|}{\|u\|_{W_1^{p(\cdot)}(\Omega)}} \tag{2.4}$$

where $q(x) = \frac{p(x)}{p(x)-1}$ for all $x \in \Omega$.

For $p_S < \infty$, we have inequalities

$$\min\{\|u\|_{L^{p(\cdot)}}{}^{p_m},\quad \|u\|_{L^{p(\cdot)}}{}^{p_S}\} \leq \rho_{p(\cdot)}(u) \leq \max\{\|u\|_{L^{p(\cdot)}}{}^{p_m},\quad \|u\|_{L^{p(\cdot)}}{}^{p_S}\} \tag{2.5}$$

for all $u \in L^{p(\cdot)}(\Omega)$.

Theorem 2.2 (Poincare inequality)**.** *Let variable exponent p belong to $P^{\log}(\Omega)$. Then, the inequality*

$$\|u\|_{L^{p(\cdot)}} \leq const\,(n,\ p)\, diam\,(\Omega)\, \|\nabla u\|_{L^{p(\cdot)}} \tag{2.6}$$

holds for all elements $u \in W_{1,\,0}^{p(\cdot)}(\Omega)$.

To deal with renormalized solutions to elliptic partial differential equations, we introduce some formal definitions, which are similar to J. Heinonen, T. Kilpelainen, and O. Martio.

Definition 2.3. An elliptic variable exponential capacity of a compact subset K of Ω is defined by

$$Cap_{p(\cdot)}(K) = \inf\left\{\rho_{p(\cdot)}(\nabla u)\ :\ u \in C_0^\infty(\Omega),\ u \geq 1_K\right\} \tag{2.7}$$

and an elliptic variable exponential capacity of a Borelian subset B of Ω is defined by

$$Cap_{p(\cdot)}(B) = \sup\left\{Cap_{p(\cdot)}(K),\ \ K \quad is \quad compact \quad in \quad \Omega, \quad K \subset B\right\}. \tag{2.8}$$

The elliptic variable exponential capacity of a Borelian subset B of Ω can be calculated as

$$Cap_{p(\cdot)}(B) = \inf\left\{\rho_{p(\cdot)}(\nabla u),\ u \in W_{1,0}^{p(\cdot)}(\Omega)\ :\ u \overset{Cap}{=} 1 \ on\ B, \quad u \overset{Cap}{\geq} 0 \ on\ \Omega\right\}, \tag{2.9}$$

where $=^{Cap}$ means equality $Cap_{p(\cdot)}$-quasi everywhere.

Proposition 2.4. *The elliptic variable exponential capacity set function $E \mapsto Cap_{p(\cdot)}(E)$ for all Borelian subsets $E \subset \Omega$ has the properties:*

1) *assume $E_1 \subset E_2$ then $Cap_{p(\cdot)}(E_1) \leq Cap_{p(\cdot)}(E_2)$;*
2) *assume sets K_1 and K_2 are compact then the*

$$Cap_{p(\cdot)}(K_1 \cup K_2) + Cap_{p(\cdot)}(K_1 \cap K_2) \leq Cap_{p(\cdot)}(K_1) + Cap_{p(\cdot)}(K_2); \tag{2.10}$$

3) *assume $E = \bigcup_{j=1,\,\ldots.} E_j$ then*

$$Cap_{p(\cdot)}(E) \leq \sum_{j=1,\,\ldots.} Cap_{p(\cdot)}(E_j). \tag{2.11}$$

Proof. Statement 1) is obvious. □

Let K_1 and K_2 be compact sets. We denote a functional set

$$W(K) = \{u \in C_0^\infty(\Omega) \ : \quad u \geq 1_K\},$$

and assume $u \in W(K_1)$ and $v \in W(K_1)$ then we have

$$\rho_{p(\cdot)}(\nabla \max\{u,\ v\}) + \rho_{p(\cdot)}(\nabla \min\{u,\ v\}) = \rho_{p(\cdot)}(u) + \rho_{p(\cdot)}(v).$$

The functions $\min\{u,\ v\}$ and $\max\{u,\ v\}$ are admissible for condensers $K_1 \cup K_2$ and $K_1 \cap K_2$ so we obtain

$$Cap_{p(\cdot)}(K_1 \cup K_2) + Cap_{p(\cdot)}(K_1 \cap K_2) \leq \rho_{p(\cdot)}(u) + \rho_{p(\cdot)}(v)$$

and

$$\begin{aligned} Cap_{p(\cdot)}(K_1 \cup K_2) + Cap_{p(\cdot)}(K_1 \cap K_2) &\leq \inf_u \left(\rho_{p(\cdot)}(u)\right) + \inf_v \left(\rho_{p(\cdot)}(v)\right) \\ &= Cap_{p(\cdot)}(K_1) + Cap_{p(\cdot)}(K_2). \end{aligned}$$

Let $K,\ D$ and F be a compact subsets of Ω such that $K \subset D$ then

$$\begin{aligned} Cap_{p(\cdot)}(D \cup F) + Cap_{p(\cdot)}(K) &\leq Cap_{p(\cdot)}(D \cup (K \cup F)) + Cap_{p(\cdot)}(D \cap (K \cup F)) \\ &\leq Cap_{p(\cdot)}(D) + Cap_{p(\cdot)}(K \cup F) \end{aligned}$$

so

$$Cap_{p(\cdot)}(D \cup F) - Cap_{p(\cdot)}(K \cup F) \leq Cap_{p(\cdot)}(D) - Cap_{p(\cdot)}(K).$$

By induction, we have

$$Cap_{p(\cdot)}\Big(\bigcup_{j=1,\dots,m} D_j\Big) - Cap_{p(\cdot)}\Big(\bigcup_{j=1,\dots,m} K_j\Big) \leq \sum_{j=1,\dots,m} \left(Cap_{p(\cdot)}(D_j) - Cap_{p(\cdot)}(K_j)\right).$$

Assume E_j and F_j are open sets. Assume $\bigcup_{j=1,\dots,m} E_j \supset D$ and $F_j \supset K_j$ are compact sets such that $\bigcup_{j=1,\dots,m} K_j \supset D$, then compact set $D_j = D \setminus \bigcup_{\substack{k=1,\dots,m \\ k \neq j}} K_k \subset E_j$ contains K_j and $D = \bigcup_{k=1,\dots,m} D_k$. Therefore, for open sets E_j, we have

$$Cap_{p(\cdot)}\Big(\bigcup_{j=1,\dots,m} E_j\Big) - Cap_{p(\cdot)}\Big(\bigcup_{j=1,\dots,m} F_j\Big) \leq \sum_{j=1,\dots,m} \left(Cap_{p(\cdot)}(E_j) - Cap_{p(\cdot)}(F_j)\right)$$

where $F_j \subset E_j, \quad j = 1, \dots, m$ and $Cap_{p(\cdot)}\big(\bigcup_{j=1,\dots,m} F_j\big) < \infty$. Thus, for sets E_j such that $F_j \subset E_j, \quad j = 1, \dots, m$ and $Cap_{p(\cdot)}\big(\bigcup_{j=1,\dots,m} F_j\big) < \infty$, we conclude

$$Cap_{p(\cdot)}\Big(\bigcup_{j=1,\dots,m} E_j\Big) - Cap_{p(\cdot)}\Big(\bigcup_{j=1,\dots,m} F_j\Big) \leq \sum_{j=1,\dots,m} \left(Cap_{p(\cdot)}(E_j) - Cap_{p(\cdot)}(F_j)\right).$$

So, we have

$$Cap_{p(\cdot)}\Big(\bigcup_{j=1,\dots,m} E_j\Big) \leq \sum_{j=1,\dots,m} Cap_{p(\cdot)}(E_j).$$

Proceeding as above, we can pass to the limit as m approaches infinity and obtain

$$Cap_{p(\cdot)}(E) \leq \sum_{j=1,\ \dots} Cap_{p(\cdot)}(E_j).$$

Let $\mathrm{M}_B(\Omega)$ be the space of all bounded measures on the Borelian σ-algebra of subsets of Ω and $\mathrm{M}_B^+(\Omega)$ be the subset of all non-negative measures of $\mathrm{M}_B(\Omega)$.

Definition 2.5. A set $\mathrm{M}_0(\Omega)$ consists of all measures $\mu \in \mathrm{M}_B(\Omega)$ such that $\mu(E) = 0$ for all subsets $E \subset \Omega$ such that $Cap_{p(\cdot)}(E) = 0$. Non-negative measures of $\mathrm{M}_0(\Omega)$ is denoted by $\mathrm{M}_0^+(\Omega)$.

Proposition 2.6. *Let $Cap_{p(\cdot)}(E) = 0$ then $mes(E) = 0$.*

Proof. Assume $Cap_{p(\cdot)}(E) = 0$ then for each $\varepsilon > 0$ there is an open neighborhood O of E such that $Cap_{p(\cdot)}(O) < \varepsilon$. We find a compact set K and a function $\psi \in W(K) = \{u \in C_0^\infty(\Omega) : \quad u \geq 1_K\}$ such that

$$\rho_{p(\cdot)}(\nabla u) \leq Cap_{p(\cdot)}(K) + \varepsilon < 2\varepsilon.$$

Since $p \in P^{\log}(\Omega)$ we can employ the Poincare inequality and obtain

$$mes(K) \leq \rho_{p(\cdot)}(u) \leq c_{\log}(p)\, diam(\Omega)\, \rho_{p(\cdot)}(\nabla u) < c_{\log}(p)\, diam(\Omega)\, \varepsilon$$

since $mes(O) \leq c_{\log}(p)\, diam(\Omega)\, \varepsilon$, we conclude $mes(E) = 0$, which proves the proposition. □

Remark. Some demands of the general variable exponential regularity are necessary. For general variable exponential functions, the statement of Proposition 2.6 does not hold hence the Poincare inequality fails.

Theorem 2.7. *Let $\mu \in \mathrm{M}_B(\Omega)$ then for $\mu \in \mathrm{M}_0(\Omega)$ it necessary and sufficient that $\mu \in L^1(\Omega) + W_{-1}^{q(\cdot)}(\Omega)$.*

Proof. From $\mu \in L^1(\Omega) + W_{-1}^{q(\cdot)}(\Omega)$ straightforwardly follows $\mu \in \mathrm{M}_0(\Omega)$.

Applying arguments of the Hahn decomposition theorem, we can assume that μ is a positive measure. Assume $\mu \in \mathrm{M}_0(\Omega)$ then there are a positive Borel measurable function $\psi \in L^1(\Omega,\, \eta)$ and a positive measure η in $W_{-1}^{q(\cdot)}(\Omega)$. We take a sequence $\{K_j\} \subset 2^\Omega$ of compact sets $K_j \subset \Omega$ such that $\bigcup_j K_j = \Omega$, we denote $\tilde{\mu}_j = T_j(\psi 1_{K_j})\eta$, where the truncation operator $T_m \quad : \qquad R \to R$ given $T_j(s) = \max\{-j, \quad \min\{j,\, s\}\}, \quad j \geq 0$ and for all $s \in R$. We put $\mu_0 = \tilde{\mu}_0$ and $\mu_j = \tilde{\mu}_j - \tilde{\mu}_{j-1}$ so that $\mu = \sum_{j\in N} \mu_j \in \mathrm{M}_B(\Omega)$ since $\sum_{j\in N} \|\mu_j\|_{\mathrm{M}_B(\Omega)} < \infty$.

We denote a sequence of mollifiers $\{\theta_j\}$ such that $\theta_j * \mu_m \xrightarrow[j\to\infty]{W_{-1}^{q(\cdot)}(\Omega)} \mu_m$. We choose j a large enough so that $\theta_j * \mu_m \in C_0^\infty(\Omega)$ and $\|\theta_j * \mu_m - \mu_m\|_{W_{-1}^{q(\cdot)}(\Omega)} \leq 2^{-m}$. So, we can write $\mu_m = (\theta_j * \mu_m) + (\mu_m - \theta_j * \mu_m) = f_m + g_m$ so that $g = \sum_{j\in N} g_j \in W_{-1}^{q(\cdot)}(\Omega)$. We have $f = \sum_{j\in N} f_j \in L^1(\Omega)$ since $\|f_m\|_{L^1(\Omega)} = \|\theta_j * \mu_m\|_{L^1(\Omega)} \leq \|\mu_j\|_{\mathrm{M}_B(\Omega)}$. Therefore, we obtain the decomposition $\mu = f + g$, which proves the theorem. □

Theorem 2.8. *Let μ be a nonnegative Radon measure on Ω. Then, there exist elements $\mu_0 \in W_{-1}^{q(\cdot)}(\Omega)$, $F \in L^1(\Omega)$ and a positive measure μ_2 such that $\mu_2(E) = \mu(E \cap N)$ for all μ-measurable sets E and for some Borel sets N such that $Cap_{p(\cdot)}(N) = 0$.*

Proof. Each σ -finite measure $\tilde{\mu}$ can be presented in the form $\tilde{\mu} = \tilde{\mu}_0 + \tilde{\mu}_1$, where $\tilde{\mu}_0 \in \mathrm{M}_0(\Omega)$ and $\tilde{\mu}_1$ such that $\tilde{\mu}_1(E) = \tilde{\mu}(E \cap N)$ for all μ-measurable sets E and for some Borel sets N such that $Cap_{p(\cdot)}(N) = 0$. Indeed, assuming that the measure μ is finite, then we denote

$$\alpha = \sup\left\{\mu(N) \; : \; Cap_{p(\cdot)}(N) = 0\right\} < \infty.$$

Let $\{E_j\}$ be an increasing sequence of Borelian sets E_j such that $Cap_{p(\cdot)}(E_j) = 0$ and $\lim_{j\to\infty}\tilde{\mu}(E_j) = \alpha$. We obtain $\bigcup_{j\in N} E_j$ is Borelian set such that $Cap_{p(\cdot)}(\bigcup_{j\in N} E_j) = 0$ and $\tilde{\mu}(\bigcup_{j\in N} E_j) = \alpha$. We have $\tilde{\mu}(E\backslash\bigcup_{j\in N} E_j) = 0$ for all Borelian sets E such that $\tilde{\mu}(E) = 0$. We define measures $\tilde{\mu}_0 = 1_{\Omega\backslash\bigcup_{j\in N} E_j}\tilde{\mu}$ and $\tilde{\mu}_1 = 1_{\bigcup_{j\in N} E_j}\tilde{\mu}$, where we can assume $N = \bigcup_{j\in N} E_j$. The application of Theorem 2.2 proves Theorem 2.7. □

3. Formalization of the boundary elliptic problem

We consider an elliptic partial differential equation with variable exponent in the form

$$-div\,(a\,(x,\ \nabla u)) + b\,(x)\,|u|^{\gamma(x)-2}\,u = \mu, \tag{3.1}$$

$$u|_{\partial\Omega} = 0, \tag{3.2}$$

where $x \in \Omega$, $\Omega \subseteq R^n$, $\quad n \geq 3$ is a smooth domain, and the measure μ does not charge sets of null capacity, and $0 \leq \gamma(x) \leq p(x)$. Assume $p \in P^{\log}(\Omega)$, $p_S < \infty$.

We assume that the structural coefficients satisfy the Leray-Lions type conditions:

1) a function $a \; : \; \Omega \times R^n \to R^n$ such that $a\,(\cdot,\ \xi)$ is measurable in Ω for each $\xi \in R^n$ and $a\,(x,\ \cdot)$ is continuous on R^n for almost every x in Ω;
2) $a\,(x,\ \xi)\,\xi \geq \nu\,|\xi|^{p(x)}$ for all $\xi \in R^n$ with some constants $\nu > p_m$;
3) $|a\,(x,\ \xi)| \leq \alpha\,|\xi|^{p(x)-1} + \alpha_1\,(x)\,;$
4) $(a\,(x,\ \xi_1) - a\,(x,\ \xi_2))\,(\xi_1 - \xi_2) > 0$
   hold for almost all $x \in \Omega$, and for all $\xi_1,\ \xi_2 \in R^n$, $\quad \xi_1 \neq \xi_2$, with some constants $\nu > p_m,\ \alpha > 0$, and positive function $\alpha_1 \in L^{q(\cdot)}(\Omega)$.

4. Entropy and renormalized solutions

First, we generalized notions of the gradient and weak solutions to partial differential equations.

Definition 4.1. We assume $T_j\,(u) \in W^{p(\cdot)}_{1,\,0}(\Omega)\,,\quad j > 0$ for an almost everywhere finite measurable function u. Then, the gradient $\nabla u \; : \quad \Omega \to R^n$ of u is defined by

$$\nabla T_j\,(u) = 1_{\{|u|<j\}}\nabla u \tag{4.1}$$

almost everywhere in Ω and for each $j > 0$.

Such a gradient operator is uniquely defined if we presume the standard almost everywhere equivalence argument.

Definition 4.2. Let $\mu \in \mathrm{M}_B(\Omega)$. A function u is called an entropy solution to the problem (2.1), (2.2) if

1) $T_j\,(u) \in W^{p(\cdot)}_{1,\,0}(\Omega)\,,\quad j > 0$ and $|\nabla u|^{p(x)-1} \in L^{q(\cdot)}(\Omega)$ for $q\,(x) = \frac{p(x)}{p(x)-1}$;

2) the identity

$$\int_{\Omega} a\left(x,\ \nabla T_j\left(u\right)\right)\nabla T_j\left(u-\phi\right)dx+\int_{\Omega} b\left|u\right|^{\gamma(x)-2}uT_j\left(u-\phi\right)dx$$
$$=\int_{\Omega} FT_j\left(u-\phi\right)dx+\int_{\Omega}\Theta\nabla T_j\left(u-\phi\right)dx \tag{4.2}$$

holds for all $\phi\in C_C^{\infty}\left(\Omega\right)$, where the given measure $\mu\in \mathrm{M}_B\left(\Omega\right)$ is presented in the form $\mu=F-div\left(\Theta\right)$ where $F\in L^1\left(\Omega\right)$ and $\Theta\in\left(L^{q(\cdot)}\left(\Omega\right)\right)^n$.

Definition 4.3. Let $\mu\in \mathrm{M}_B\left(\Omega\right)$. A function u is called a renormalized solution to the problem (2.1), (2.2) if

1) $T_j\left(u\right)\in W_{1,\,0}^{p(\cdot)}\left(\Omega\right),\quad j>0$ and $\left|\nabla u\right|^{p(x)-1}\in L^{q(\cdot)}\left(\Omega\right)$ for $q\left(x\right)=\frac{p(x)}{p(x)-1}$;

$$\lim_{j\to\infty}\int_{\{j\le|u|\le j+1\}} a\left(x,\ \nabla u\right)\nabla u dx=0 \tag{4.3}$$

and

$$\left|b\left(\cdot\right)\left|u\right|^{\gamma(\cdot)-2}u\right|\in {L^1}_{loc}\left(\Omega\right); \tag{4.4}$$

2) for all functions $h\in W^{1,\,\infty}\left(R\right)$ with compact supports, the identity

$$\int_{\Omega} a\left(x,\ \nabla u\right)h\left(u\right)\nabla\phi dx+\int_{\Omega} a\left(x,\ \nabla u\right)h'\left(u\right)\phi dx$$
$$+\int_{\Omega} b\left|u\right|^{\gamma(x)-2}uh\left(u\right)\phi dx \tag{4.5}$$
$$=\int_{\Omega}\phi h\left(u\right)Fdx+\int_{\Omega}\Theta\nabla\left(\phi h\left(u\right)\right)dx$$

holds for all $\phi\in C_C^{\infty}\left(\Omega\right)$.

5. Proof of existence and uniqueness of entropy and renormalized solutions

We are going to show the existence and uniqueness of entropy and renormalized solutions to the elliptic partial differential equations with a variable exponential elliptic operator.

Theorem 5.1. *Let variable exponent p belong to $P^{\log}\left(\Omega\right)$. Let measure $\mu\in \mathrm{M}_B\left(\Omega\right)$ does not charge sets of null capacity. Then, there exists a uniquely defined entropy solution u to the problem (12), (13) under the conditions 1) – 4).*

Proof. The existence of an entropy solution can be proven by considering the approximating problems and obtaining a solution to the problem (3.1), and (3.2) as the limit of solutions of approximate problems, the existence of a solution to approximate Dirichlet problems follows from the variational method.

We consider the sequence of the approximations of (3.1), and (3.2) by problems

$$-div\left(a\left(x,\ \nabla u_k\right)\right)+b\left(x\right)\left|u_k\right|^{\gamma(x)-2}u_k=F_k-div\left(\Theta_k\right),$$
$$u_k|_{\partial\Omega}=0,$$

where sequences $\{F_k\} \subset C_C^{\infty}(\Omega)$ and $\{\Theta_k\} \subset (C_C^{\infty}(\Omega))^n$ such that $F_k \xrightarrow[k\to\infty]{L^1(\Omega)} F$, $\Theta_k \xrightarrow[k\to\infty]{\left(L^{q(\cdot)}(\Omega)\right)^n} \Theta$ and $\|F_k\|_{L^1(\Omega)} \le \|F\|_{L^1(\Omega)}$, $\rho_{q(\cdot)}(\Theta_k) \le \rho_{q(\cdot)}(\Theta)$.

Applying variational methods, we establish the existence of a unique weak solution $u_k \in W_{1,\,0}^{p(\cdot)}(\Omega)$ to each problem indexed by $k \in N$. All these solutions are entropy solutions. We are going to show that a subsequence $\{u_k\} \subset W_{1,\,0}^{p(\cdot)}(\Omega)$ converges to an entropy solution to the problem (3.1), (3.2).

For each fixed j, a sequence $\{T_j(u_k), \quad k \in N\}$ is bounded in $W_{1,\,0}^{p(\cdot)}(\Omega)$-norm since we take the test function $T_j(u_k)$ and apply the definition of the gradient, structural coefficients conditions, and Young's inequality, we estimate

$$\begin{aligned}\nu\rho_{p(\cdot)}(\nabla T_j(u_k)) &\le \int_\Omega F_k T_j(u_k)\,dx + \int_\Omega \Theta_k \nabla T_j(u_k)\,dx\\ &\le j\,\|F_k\|_{L^1(\Omega)} + \rho_{p(\cdot)}\left(\frac{\nabla T_j(u_k)}{p(\cdot)}\right) + \rho_{q(\cdot)}\left(\frac{\Theta_k}{q(\cdot)}\right)\end{aligned}$$

since $\nu > p_m$ we obtain

$$\rho_{p(\cdot)}(\nabla T_j(u_k)) \le \left(\nu - {p_m}^{-1}\right)^{-1}\left(j\,\|F_k\|_{L^1(\Omega)} + \rho_{q(\cdot)}\left(\frac{\Theta_k}{q(\cdot)}\right)\right).$$

So, there exists a subsequence $\left\{{u_k}^j\right\}$ of $\left\{{u_k}^{j-1}\right\}$ and a sequence of functions $\{w_j\} \subset L^{q(\cdot)}(\Omega)$ such that $|w_j| \le j$ and $T_j\left({u_k}^j\right) \xrightarrow[k\to\infty]{L^{q(\cdot)}(\Omega)} w_j$. We can reindex the subsequence $\{u_k\}$ as $u_k = {u_k}^k$ for all $k \in N$. Thus, we obtain $T_j(u_k) \xrightarrow[k\to\infty]{L^{q(\cdot)}(\Omega)} w_j$ for all indices j. Since for all indices $m, \quad j$ such that $j < m$ we have $T_j(T_m u_k) = T_j u_k$, we deduce

$$\lim_{k\to\infty} T_j(T_m(u_k)) = T_j(w_m) = \lim_{k\to\infty} T_j(u_k) = w_j.$$

We define a function u by $u(x) = w_j(x)$ for all $x \in \{j-1 \le |w_j(x)| < j, j \in N\}$ and $u(x) = 0$ on a negligible set. For this function u, the identity $T_j(u) = w_j$ holds for all $j \in N$.

In the definition of an entropy solution, we take a test function ϕ equal 0 and obtain

$$\begin{aligned}&\int_\Omega a(x,\ \nabla T_j(u))\,\nabla T_j(u)\,dx + \int_\Omega b\,|u|^{\gamma(x)-2}\,u T_j(u)\,dx\\ &\qquad = \int_\Omega F T_j(u)\,dx + \int_\Omega \Theta \nabla T_j(u)\,dx,\end{aligned}$$

applying the Young inequality, we obtain

$$\rho_{p(\cdot)}(\nabla T_j(u)) \le \left(\nu - {p_m}^{-1}\right)^{-1}\left(j\,\|F\|_{L^1(\Omega)} + \rho_{q(\cdot)}\left(\frac{\Theta}{q(\cdot)}\right)\right) = c(F,\ \Theta).$$

Employing Sobolev's embedding lemma and set equality $\{j \le |u|\} = \{j \le |T_j(u)|\}$, we conclude the following estimate

$$mes\,\{x \in \Omega\ :\quad j \le |u(x)|\} \le const\,(p,\ F,\ \Theta)\, j^{p_m^*\left(p_m^{-1}-1\right)}.$$

Since $T_j(u_k) \in L^{q(\cdot)}(\Omega), \quad k \in N$ the estimate

$$\lim_{i,\, l\to\infty} supmes\{x \in \Omega \ : \quad |u_i(x) - u_l(x)| > \varepsilon\} \le const(p,\ F,\ \Theta)\, j^{p_m^*\left(p_m^{-1}-1\right)}$$

holds for all $j \in N$ thus passing to the limit as $j \to \infty$ we deduce

$$\lim_{i,\, l\to\infty} supmes\{x \in \Omega \ : \quad |u_i(x) - u_l(x)| > \varepsilon\} = 0.$$

Therefore, we can choose a convergent subsequence $\{u_k\}$ such that $u_k \longrightarrow_{k\to\infty}^{a.e.\ \ in \quad \Omega} u$.

We define

$$v_k = T_{2j}(u_k - T_m(u_k) + T_j(u_k) - T_j(u))$$

for all $j \in N$ and for all $m > j$. Then, we choose $d = 4j + m$ and obtain $\nabla v_k = 0$ for $|u_k| > j$, we take a test function v_k and have

$$\int_\Omega a(x,\ \nabla T_d(u_k))\nabla v_k dx + \int_\Omega b\,|u_k|^{\gamma(x)-2} u_k v_k dx = \int_\Omega F_k v_k dx + \int_\Omega \Theta_k \nabla v_k dx.$$

We split the integral on the left-hand side into two

$$\begin{aligned}\int_\Omega a(x,\ \nabla T_d(u_k))\nabla(T_{2j}(u_k - T_m(u_k) + T_j(u_k) - T_j(u)))\,dx \\ \ge \int_\Omega a(x,\ \nabla T_j(u_k))\nabla(T_j(u_k) - T_j(u))\,dx \\ - \int_{\{|u_k|>j\}} |a(x,\ \nabla T_d(u_k))|\,|\nabla T_j(u)|\,dx\end{aligned}$$

so that we obtain

$$\begin{aligned}&\int_\Omega (a(x,\ \nabla T_j(u_k)) - a(x,\ \nabla T_j(u)))\nabla(T_j(u_k) - T_j(u))\,dx \\ &\le \int_{\{|u_k|>j\}} |a(x,\ \nabla T_d(u_k))|\,|\nabla T_j(u)|\,dx \\ &- \int_\Omega a(x,\ \nabla T_j(u_k))\nabla(T_j(u_k) - T_j(u))\,dx \\ &+ \int_\Omega F_k T_{2j}(u_k - T_m(u_k) + T_j(u_k) - T_j(u))\,dx \\ &+ \int_\Omega \Theta_k \nabla T_{2j}(u_k - T_m(u_k) + T_j(u_k) - T_j(u))\,dx \\ &- \int_\Omega b\,|u_k|^{\gamma(x)-2} u_k T_{2j}(u_k - T_m(u_k) + T_j(u_k) - T_j(u))\,dx.\end{aligned}$$

We take $m = m(\varepsilon)$ sufficiently large, since $|a(x,\ \nabla T_m(u_k))| \in L^{q(\cdot)}(\Omega)$ and $1_{\{|u_k|>j\}}|\nabla T_j(u)| \longrightarrow_{k\to\infty}^{L^{p(\cdot)}(\Omega)} 0 \quad \forall j \in N$ we obtain

$$\lim_{k\to\infty}\int_{\{|u_k|>j\}} |a(x,\ \nabla T_d(u_k))|\,|\nabla T_j(u)|\,dx = 0.$$

Next, we put $v_k = T_{2j}(u_k - T_m(u_k))$ and obtain

$$\int_\Omega a(x,\ \nabla T_d(u_k))\, \nabla (T_{2j}(u_k - T_m(u_k)))\, dx + \int_\Omega b\, |u_k|^{\gamma(x)-2} u_k T_{2j}(u_k - T_m(u_k))\, dx = \int_\Omega F_k T_{2j}(u_k - T_m(u_k))\, dx + \int_\Omega \Theta_k \nabla (T_{2j}(u_k - T_m(u_k)))\, dx$$

repeating the previous calculation, we deduce

$$\int_\Omega |\nabla (T_{2j}(u_k - T_m(u_k)))|^{p(x)}\, dx \le c_1(2j+1)$$

with positive constant c_1 independent on m. Since

$$T_{2j}(u_k - T_m(u_k)) \overset{weakly}{\underset{k\to\infty}{\overset{W_{1,0}^{p(\cdot)}(\Omega)}{\longrightarrow}}} T_{2j}(u - T_m(u))$$

we have

$$\int_\Omega |\nabla (T_{2j}(u - T_m(u)))|^{p(x)}\, dx \le c_1(2j+1)$$

so

$$\int_\Omega |\Theta|\, |\nabla (T_{2j}(u - T_m(u)))|\, dx \le c_3(j) \int_{\{|u|\ge m\}} |\Theta|^{q(x)}\, dx,$$

therefore,

$$\lim_{m\to\infty} \int_\Omega \Theta \nabla (T_{2j}(u - T_m(u)))\, dx = 0.$$

By the Lebesgue theorem, we get

$$\lim_{m\to\infty} \int_\Omega F T_{2j}(u - T_m(u))\, dx = 0$$

and the inequality

$$\int_\Omega F T_{2j}(u - T_m(u))\, dx + \int_\Omega \Theta \nabla (T_{2j}(u - T_m(u)))\, dx \le \varepsilon$$

holds for all large enough $m = m(\varepsilon)$.

Since

$$T_{2j}(u_k - T_m(u_k) + T_j(u_k) - T_j(u)) \overset{weakly}{\underset{k\to\infty}{\overset{W_{1,0}^{p(\cdot)}(\Omega)}{\longrightarrow}}} T_{2j}(u - T_m(u))$$

we conclude

$$\int_\Omega (a(x,\ \nabla T_j(u_k)) - a(x,\ \nabla T_j(u)))\, \nabla (T_j(u_k) - T_j(u))\, dx \le \int_\Omega F T_{2j}(u - T_m(u))\, dx + \int_\Omega \Theta \nabla T_{2j}(u - T_m(u))\, dx \le \varepsilon$$

for all large enough k thus

$$\lim_{k\to\infty} \int_\Omega (a(x,\ \nabla T_j(u_k)) - a(x,\ \nabla T_j(u)))\, \nabla (T_j(u_k) - T_j(u))\, dx = 0.$$

By the Vitali theorem, we conclude $\nabla u_k \xrightarrow[k\to\infty]{\left(L^{p(\cdot)}(\Omega)\right)^n} \nabla u$, thus the limit $T_j\left(u_k\right) \xrightarrow[k\to\infty]{W_{1,0}^{p(\cdot)}(\Omega)} T_j\left(u\right)$ holds for each $j \in N$.

Thus, we have obtained

$$u_k \overset{a.e. \quad in \quad \Omega}{\underset{k\to\infty}{\longrightarrow}} u,$$

$$T_j\left(u_k\right) \overset{W_{1,0}^{p(\cdot)}(\Omega)}{\underset{k\to\infty}{\longrightarrow}} T_j\left(u\right).$$

Finally, in

$$-div\left(a\left(x,\ \nabla u_k\right)\right) + b\left(x\right)\left|u_k\right|^{\gamma(x)-2} u_k = F_k - div\left(\Theta_k\right)$$

we take a test function $w_k = T_j\left(u_k - \phi\right)$ for $j \in N$ with $\phi \in W_{1,0}^{p(\cdot)}\left(\Omega\right) \cap L^\infty\left(\Omega\right)$ then we can choose $s = j + \left\|\phi\right\|_{L^\infty(\Omega)}, \quad k > s$ and obtain

$$\int_\Omega a\left(x,\ \nabla u_k\right) \nabla T_j\left(u_k - \phi\right) dx = \int_\Omega a\left(x,\ \nabla T_s\left(u_k\right)\right) \nabla T_j\left(u_k - \phi\right) dx$$

so that

$$\int_\Omega a\left(x,\ \nabla T_s\left(u_k\right)\right) \nabla\left(T_j\left(u_k - \phi\right)\right) dx + \int_\Omega b\left|u_k\right|^{\gamma(x)-2} u_k T_j\left(u_k - \phi\right) dx$$
$$= \int_\Omega F_k T_j\left(u_k - \phi\right) dx + \int_\Omega \Theta_k \nabla\left(T_j\left(u_k - \phi\right)\right) dx$$

passing to the limit as k goes to infinity, we deduce

$$\int_\Omega a\left(x,\ \nabla u\right) \nabla\left(T_j\left(u - \phi\right)\right) dx + \int_\Omega b\left|u\right|^{\gamma(x)-2} u T_j\left(u - \phi\right) dx$$
$$= \int_\Omega F T_j\left(u - \phi\right) dx + \int_\Omega \Theta \nabla\left(T_j\left(u - \phi\right)\right) dx$$

for all $\phi \in W_{1,0}^{p(\cdot)}\left(\Omega\right) \cap L^\infty\left(\Omega\right)$ and all $j \in N$. □

The uniqueness can be shown from the definition of the entropy solution. Let us assume that there are two entropy solutions u and w then we take test functions $T_m\left(w\right)$ and $T_m\left(u\right)$, respectively, and obtain

$$\int_{\left\{\left|u - T_m(w)\right| \le j\right\}} a\left(x,\ \nabla u\right) \nabla T_j\left(u - T_m\left(w\right)\right) dx$$
$$+ \int_{\left\{\left|w - T_m(u)\right| \le j\right\}} a\left(x,\ \nabla w\right) \nabla T_j\left(w - T_m\left(u\right)\right) dx$$
$$+ \int_\Omega \left(b\left|u\right|^{\gamma(x)-2} u - F\right) T_j\left(u - T_m\left(w\right)\right) dx$$
$$+ \int_\Omega \left(b\left|w\right|^{\gamma(x)-2} w - F\right) T_j\left(w - T_m\left(u\right)\right) dx$$
$$= \int_\Omega \Theta \nabla T_j\left(u - T_m\left(w\right)\right) dx + \int_\Omega \Theta \nabla T_j\left(w - T_m\left(u\right)\right) dx$$

We have

$$\int_{\{|u-T_m(w)|\le j\}} a(x,\ \nabla u)\nabla T_j(u-T_m(w))\,dx$$
$$=\int_{\{|u|\le m\}\cap\{|u-w|\le j,\quad |w|\le m\}} a(x,\ \nabla u)\nabla(u-w)\,dx$$
$$-\int_{\{|u|>m\}\cap\{|u-w|\le j,\quad |w|\le m\}} a(x,\ \nabla u)\nabla w dx$$

and

$$\int_{\{|w-T_m(u)|\le j\}} a(x,\ \nabla w)\nabla T_j(w-T_m(u))\,dx$$
$$=\int_{\{|w|\le m\}\cap\{|u-w|\le j,\quad |u|\le m\}} a(x,\ \nabla w)\nabla(w-u)\,dx$$
$$-\int_{\{|w|>m\}\cap\{|u-w|\le j,\quad |u|\le m\}} a(x,\ \nabla w)\nabla u dx.$$

By the Holder inequality, we estimate

$$\left|\int_{\{|u|>m\}\cap\{|u-w|\le j,\quad |w|\le m\}} a(x,\ \nabla u)\nabla w dx\right|$$
$$\le\int_{\{|u|>m\}\cap\{|u-w|\le j,\quad |w|\le m\}} \left|\alpha|\nabla u|^{p(x)-1}+\alpha_1(x)\right||\nabla w|\,dx$$
$$\le k_{Hol}\left(\|\alpha_1\|_{L^{q(\cdot)}(\Omega)}+\left\||\nabla u|^{p(x)-1}\right\|_{L^{q(\cdot)}(\{m<|u|\le m+j\})}\right)\|\nabla w\|_{L^{p(\cdot)}(\{m-j<|w|\le m\})}$$

and

$$\left|\int_{\{|w|>m\}\cap\{|u-w|\le j,\quad |u|\le m\}} a(x,\ \nabla w)\nabla u dx\right|$$
$$\le\int_{\{|w|>m\}\cap\{|u-w|\le j,\quad |u|\le m\}} \left|\alpha|\nabla w|^{p(x)-1}+\alpha_1(x)\right||\nabla u|\,dx$$
$$\le k_{Hol}\left(\|\alpha_1\|_{L^{q(\cdot)}(\Omega)}+\left\||\nabla w|^{p(x)-1}\right\|_{L^{q(\cdot)}(\{m<|w|\le m+j\})}\right)\|\nabla u\|_{L^{p(\cdot)}(\{m-j<|u|\le m\})}.$$

We have

$$\int_{\{m<|\psi|\le m+j\}} |\nabla\psi|^{p(x)}\,dx\le\left(\nu-p_m{}^{-1}\right)^{-1}\left(j\|F_k\|_{L^1(\Omega)}+\rho_{q(\cdot)}\left(\frac{\Theta_k}{q(\cdot)}\right)\right)$$

for each entropy solution ψ.

Thus, we obtain

$$\lim_{m\to\infty}\int_{\{|u-T_m(w)|\le j\}} a(x,\ \nabla u)\nabla(u-T_m(w))\,dx$$
$$=\int_{\{|u-w|\le j\}} a(x,\ \nabla u)\nabla(u-w)\,dx$$

and

$$\lim_{m\to\infty}\int_{\{|w-T_m(u)|\le j\}} a\left(x,\ \nabla w\right)\nabla T_j\left(w-T_m\left(u\right)\right)dx$$
$$-\int_{\{|u-w|\le j\}} a\left(x,\ \nabla w\right)\nabla T_j\left(u-w\right)dx.$$

Since

$$T_j\left(u-T_m\left(w\right)\right)+T_j\left(w-T_m\left(u\right)\right)=0$$

in $\{|u|\le m, |w|\le m\}$ we get

$$\left|\int_\Omega F\left(x\right)\left(T_j\left(u-T_m\left(w\right)\right)+T_j\left(w-T_m\left(u\right)\right)\right)dx\right|$$
$$\le k_{Hol} j\left(\int_{\{|u|>m\}}\left|F\left(x\right)\right|dx+\int_{\{|w|>m\}}\left|F\left(x\right)\right|dx\right).$$

The measures $mes\{|u|>m\}$ and $mes\{|w|>m\}$ approach zero as m approaches infinity so that

$$\int_{\{|u-w|\le j\}} a\left(x,\ \nabla u\right)-a\left(x,\ \nabla w\right)\nabla\left(u-w\right)dx=0$$

for all $j\in N$, therefore, we deduce $\nabla u =^{a.e. \quad in \quad \Omega} \nabla w$.

Since $p\in P^{\log}\left(\Omega\right)$ we employ the Poincare inequality and conclude

$$\left\|T_j\left(u-w\right)\right\|_{L^{p(\cdot)}(\Omega)}\le const\left\|\nabla T_j\left(u-w\right)\right\|_{L^{p(\cdot)}(\Omega)}=0$$

for all $j\in N$, therefore, we deduce $u =^{a.e. \quad in \quad \Omega} w$ thus the entropy solutions u and w coincide.

Theorem 5.2. *Let variable exponent* p *belong to* $P^{\log}\left(\Omega\right)$*. Let measure* $\mu\in \mathrm{M}_B\left(\Omega\right)$ *does not charge sets of null capacity. Then, there exists a unique renormalized solution* u *to the problem* (3.1), (3.2) *under the conditions 1) – 4), which coincides with the entropy solution.*

Proof. Let a function u be an entropy solution to (3.1), (3.2) then $T_j\left(u\right)\in W_{1,\,0}^{p(\cdot)}\left(\Omega\right)$ for all $j\in N$ and

$$\lim_{j\to\infty}\int_{\{j\le|u|\le j+1\}}\left|\nabla u\right|^{p(x)}dx=0$$

since

$$\nu\int_{\{j\le|u|\le j+1\}}\left|\nabla u\right|^{p(x)}dx$$
$$\le\int_{\{j<|u|\}}Fdx+\int_{\{j\le|u|\le j+1\}}\frac{\left|\nabla u\right|^{p(x)}}{p\left(\cdot\right)}dx+\int_{\{j\le|u|\le j+1\}}\frac{\left|\Theta\right|^{q(x)}}{q\left(\cdot\right)}dx$$

for $q\left(x\right)=\frac{p(x)}{p(x)-1}$.

Let $\{u_k\} \subset W_{1,\,0}^{p(\cdot)}(\Omega)$ be a proximation sequence as in the proving of Theorem 5.1 then $T_j(u_k) \overset{W_{1,0}^{p(\cdot)}(\Omega)}{\underset{k\to\infty}{\longrightarrow}} T_j(u)$ for all $j \in N$. Let $h \in W^{1,\,\infty}(R)$ with compact support on $[-M, \quad M]$ for some positive number M. For all $\phi \in C_C^{\infty}(\Omega)$, we write

$$\begin{aligned}&\int_\Omega a(x,\ \nabla u_k)\, h(u_k)\, \nabla\phi dx\\ &\quad+\int_\Omega a(x,\ \nabla u_k)\, h'(u_k)\, \phi dx+\int_\Omega b\,|u_k|^{\gamma(x)-2}\, u_k h(u_k)\, \phi dx\\ &\quad=\int_\Omega \phi h(u_k)\, F_k dx+\int_\Omega \Theta_k \nabla(\phi h(u_k))\, dx.\end{aligned}$$

Since

$$\begin{aligned}h(u_k)\, a(x,\ \nabla u_k) &= h(u_k)\, a(x,\ \nabla T_M(u_k)),\\ h(u)\, a(x,\ \nabla T_M(u)) &= h(u)\, a(x,\ \nabla u),\\ h'(u_k)\, a(x,\ \nabla u_k) &= h'(u_k)\, a(x,\ \nabla T_M(u_k)),\\ h'(u)\, a(x,\ \nabla T_M(u)) &= h'(u)\, a(x,\ \nabla u)\end{aligned}$$

and

$$u_k \overset{a.e.\ \ in\ \ \Omega}{\underset{k\to\infty}{\longrightarrow}} u,$$

$$T_j(u_k) \overset{W_{1,0}^{p(\cdot)}(\Omega)}{\underset{k\to\infty}{\longrightarrow}} T_j(u),$$

$$|\nabla T_j(u_k)|^{p(x)-2}\, \nabla T_j(u_k) \overset{\left(L^{q(\cdot)}(\Omega)\right)^n}{\underset{k\to\infty}{\longrightarrow}} |\nabla T_j(u)|^{p(x)-2}\, \nabla T_j(u),$$

we conclude

$$h(u_k)\, a(x,\ \nabla T_M(u_k)) \overset{\left(L^{q(\cdot)}(\Omega)\right)^n}{\underset{k\to\infty}{\longrightarrow}} h(u)\, a(x,\ \nabla T_M(u)),$$

$$h'(u_k)\, a(x,\ \nabla T_M(u_k)) \overset{L^1(\Omega)}{\underset{k\to\infty}{\longrightarrow}} h'(u)\, a(x,\ \nabla T_M(u))$$

and

$$h(u_k)\, a(x,\ \nabla u_k) \overset{\left(L^{q(\cdot)}(\Omega)\right)^n}{\underset{k\to\infty}{\longrightarrow}} h(u)\, a(x,\ \nabla u),$$

$$h'(u_k)\, a(x,\ \nabla u_k) \overset{L^1(\Omega)}{\underset{k\to\infty}{\longrightarrow}} h'(u)\, a(x,\ \nabla u).$$

Therefore, we deduce

$$\begin{aligned}&\int_\Omega a(x,\ \nabla u)\, h(u)\, \nabla\phi dx+\int_\Omega a(x,\ \nabla u)\, h'(u)\, \phi dx+\int_\Omega b\,|u|^{\gamma(x)-2}\, uh(u)\, \phi dx\\ &\quad=\int_\Omega \phi h(u)\, F dx+\int_\Omega \Theta\nabla(\phi h(u))\, dx.\end{aligned}$$

The uniqueness can be proven similar to the previous theorem. Theorem 5.2 is proven. □

References

[1] E. Acerbi and G. Mingione, *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal. **164** (2002), 213–259.

[2] L. Boccardo, D. Giachetti, J. I. Diaz and F. Murat, *Existence and regularity of renormalized solutions for some elliptic problems involving derivations of nonlinear terms*, J. Differential Equations **106** (1993), 215–237.

[3] Y. M. Chen, S. Levine and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. **66** (2006), 1383–1406.

[4] G. Dal Maso, F. Murat, L. Orsina and A. Prignet, *Renormalization solutions of elliptic equations with general measure data*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (IV) **28** (1999), 741–808.

[5] M. Ding, C. Zhang and S. Zhou, *On optimal C1,α estimates for p(x)-Laplace type equations*, Nonlinear Anal. **200** (2020): 112030.

[6] X. Fan and D. Zhao, *A class of De Giorgi type and Holder continuity*, Nonlinear Anal. **36** (1999), 295–318.

[7] X. L. Fan and D. Zhao, *On the spaces Lp(x) and Wm,p(x)*, J. Math. Anal. Appl. **263** (2001), 424–446.

[8] X. L. Fan and Q. H. Zhang, *Existence of solutions for $p(x)$-Laplacian Dirichlet problems*, Nonlinear Anal. **52** (2003), 1843–1852.

[9] X. L. Fan, Q. H. Zhang and D. Zhao, *Eigenvalues of $p(x)$-Laplacian Dirichlet problem*, J. Math. Anal. Appl. **302** (2005), 306–317.

[10] J. Heinonen, T. Kilpelainen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford University Press, 1993.

[11] Y. C. Kim, *Nonlocal Harnack inequalities for nonlocal heat equations*, J. Differ. Equ. **267** (2019), 6691–6757.

[12] G. Li, D. Motreanu, H. Wu and Q. Zhang,*Multiple solutions with the constant sign for a (p, q)-elliptic system Dirichlet problem with product nonlinear term*, Boundary Value Problems **2018** (2018): 67.

[13] D. Liu, B. Wang and P. Zhao, *On the trace regularity results of Musielak–Orlicz–Sobolev spaces in a bounded domain*, Comm. Pure Appl. Anal. **15** (2016), 1643–1659.

[14] D. Liu and P. Zhao, *Solutions for a quasilinear elliptic equation in Musielak–Sobolev spaces*, Nonlinear Anal. Real World Appl. **26** (2016), 315–329.

[15] A. Matsoukas and N. Yannakakis, *The double phase Dirichlet problem when the lowest exponent is equal to 1*, J. Math. Anal. Appl. **526** (2023): 127270.

[16] X. J. Wang, *Neumann problems of semilinear elliptic equations involving critical Sobolev exponents*, J. Differential Equations **93** (1991), 283–310.

[17] J. X. Yin, J. K. Li and Y. Y. Ke, *Existence of positive solutions for the p(x)-Laplacian equation*, Rocky Mt. J. Math. **42** (2012), 1675–1758.

[18] C. Zhang and X. Zhang, *Renormalized solutions for the fractional $p(x)$-Laplacian equation with L 1 data*, Nonlinear Analysis **190**, (2020): 111610.

[19] C. Zhang and S. Zhou, *Entropy and renormalized solutions for the p(x)-Laplacian equation with measure data*, Bull. Aust. Math. Soc. **82**, (2010), 459–479.

[20] D. Z. Zhang, K. H. Shi, Z. C. Guo and B. Y. Wu, *A class of elliptic systems with discontinuous variable exponents and data for image denoising*, Nonlinear Anal. Real World Appl. **50** (2019), 448–468.

[21] Q. H. Zhang,*Existence of solutions for p(x)-Laplacian equations with singular coefficients in RN*, J. Math. Anal. Appl. **348** (2008), 38–50.

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